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Complete Moment Convergence for Weighted Sums of Arrays of Rowwise NA Random Variables

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Abstract In this paper we obtain some new results on complete moment convergence for weighted sums of arrays of rowwise NA random variables. Our results improve and extend some well known results from the literature.

Keywords complete moment convergence; NA random variables; weighted sums.

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1. Introduction

The concept of complete convergence was introduced by Hsu and Robbins [5] as follows: A sequence of random variables $\{X_n, n \geq 1\}$ is said to converge completely to a constant C if $\sum_{n=1}^{\infty} P(|X_n - C| > \epsilon) < \infty$ for all $\epsilon > 0$. From then on, many authors devoted their study to complete convergence [10, 11, 13, 16].

Recall that a finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets A and B of $\{1, 2, \ldots, n\}$,

$$\mathrm{Cov}(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0.$$

Whenever f_1 and f_2 are coordinatewise increasing and such that the covariance exists. An infinite family of random variables $\{X_i, i \geq 1\}$ is NA if for every positive integer $n \geq 2, \{X_i, 1 \leq i \leq n\}$ is NA. An array $\{X_{ni}, i \geq 1,\ n \geq 1\}$ is rowwise NA if for every positive integer n, the sequence of random variables $\{X_{ni}, i \geq 1\}$ is NA. This definition was introduced by Alam and Saxena [1] and carefully studied by Block et al. [2] and Joag-Dev and Proschan [6]. NA sequences have many good properties and extensive applications in multivariate statistical analysis and reliability theory. Qiu [10] obtained the following results which improve and extend the corresponding results obtained by Li et al. [8] and Sung [13] and Wang et al. [16] in independent sequence case, where we omit some insignificant details connected with slowly varying functions.

Theorem A *Let $t \geq -1$, $p > 0$, $\gamma = p(t + \beta + 1) \geq 1$. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array*

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of zero-mean rowwise NA random variables which are stochastically dominated by a random variable X, i.e., $P(|X_{ni}| > x) \le CP(|X| > x)$ for all $x > 0$ and all $i \ge 1, n \ge 1$, where C is a positive constant. Assume that $E|X|^\gamma < \infty$. Let $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of real numbers. Assume that $\sum_{i=1}^\infty a_{ni}X_{ni}$ is finite a.s. for all $n \ge 1$ and

$$\sup_{i\ge1,n\ge1} |a_{ni}| < \infty, \ \sum_{i=1}^\infty |a_{ni}|^\alpha = O(n^\beta) \text{ for some } \alpha < \gamma. \tag{1}$$

If $\gamma \ge 2$, moreover, we assume that $\{a_{ni}, i \ge 1, n \ge 1\}$ satisfies

$$\sum_{i=1}^\infty a_{ni}^2 = O(n^\lambda) \text{ for some } \lambda < 2/p. \tag{2}$$

Then

$$\sum_{n=1}^\infty n^t P(n^{-1/p}|\sum_{i=1}^\infty a_{ni}X_{ni}| > \epsilon) < \infty \text{ for all } \epsilon > 0. \tag{3}$$

If condition (1) on the weights is replaced by a weaker condition

$$\sup_{i\ge1,n\ge1} |a_{ni}| < \infty, \sum_{i=1}^\infty |a_{ni}|^{p(t+\beta+1)} = O(n^\beta), \tag{4}$$

Qiu [10] obtained the following theorem.

Theorem B *Let $t \ge -1$, $p > 0$, $\gamma = p(t+\beta+1) \ge 1$. Let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of zero-mean rowwise NA random variables which are stochastically dominated by a random variable X satisfying $E|X|^\gamma \log(1+|X|) < \infty$. Let $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of real numbers satisfying (4) and $\sum_{i=1}^\infty a_{ni}X_{ni}$ be finite a.s. for all $n \ge 1$. If $\gamma \ge 2$, moreover, we assume that $\{a_{ni}, i \ge 1, n \ge 1\}$ satisfies (2). Then (3) holds.*

When $\{X_n, n \ge 1\}$ is a sequence of i.i.d. random variables with mean zeros, Chow [4] first investigated the complete moment convergence, which is more exact than complete convergence. He obtained the following result:

Theorem C *Let $\{X_n, n \ge 1\}$ be a sequence of i.i.d. random variables with $EX_n = 0$. For $1 \le p < 2$ and $r > 1$, if $E\{|X_1|^{rp} + |X_1|\log(1+|X_1|)\} < \infty$, then*

$$\sum_{n=1}^\infty n^{r-2-1/p} E\{|\sum_{k=1}^\infty X_k| - \epsilon n^{1/p}\}_+ < \infty \text{ for all } \epsilon > 0.$$

Theorem C has been generalized and extended in several directions. Wang and Su [14] extended and generalized Chow's result to a Rademacher type p Banach space, Wang and Zhao [15] for NA random variables, Li and Zhang [9], Chen et al. [3] for moving-average processes based on NA random variables, Kim and Ko [7] for moving-average processes based on ϕ-mixing random variables, etc.

In this paper, we obtain new complete moment convergence results for weighted sums of arrays of rowwise NA random variables using Theorems A and B. As a corollary, we obtain com-

plete moment convergence for moving average processes. Our results generalize the corresponding results of Wang and Zhao [15], Li and Zhang [9].

Throughout this paper, we assume that $S_n \equiv \sum_{i=1}^{\infty} a_{ni}X_{ni}$ is finite a.s., C always stands for a positive constant which may differ from one place to another.

2. Main results and proofs

In order to prove our main result, we need some preliminary lemmas. The following lemma comes from Shao [12].

Lemma 1 *Let $p > 0, \{X_n, n \geq 1\}$ be a sequence of NA random variables with $EX_n = 0$ and $E|X_n|^p < \infty, n \geq 1$.*

(i) If $1 \leq p \leq 2$, then

$$E \max_{1\leq j\leq n} |\sum_{i=1}^{j} X_i|^p \leq 2^{3-p} \sum_{i=1}^{n} E|X_i|^p, \ \forall n \geq 2.$$

(ii) If $p > 2$, then

$$E \max_{1\leq j\leq n} |\sum_{i=1}^{j} X_i|^p \leq 2(15p/\ln p)^p \{\sum_{i=1}^{n} E|X_i|^p + (\sum_{i=1}^{n} EX_i^2)^{p/2}\}, \quad \forall n \geq 2.$$

Lemma 2 *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of random variables which is stochastically dominated by a random variable X. Then for any $u > 0$ and $x > 0$*

(i) $E|X_{ni}|^u I(|X_{ni}| \leq x) \leq C\{E|X|^u I(|X| \leq x) + x^u P(|X| > x)\}$,

(ii) $E|X_{ni}|^u I(|X_{ni}| > x) \leq CE|X|^u I(|X| > x)$.

With the preliminary lemmas, we now state and prove one of our main results.

Theorem 1 *Let $t \geq -1, p > 0, \gamma = p(t + \beta + 1) \geq 1, q > 0$. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of zero-mean rowwise NA random variables which are stochastically dominated by a random variable X. Let $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of real numbers satisfying (1). If $\gamma \geq 2$, moreover, we assume that $\{a_{ni}, i \geq 1, n \geq 1\}$ satisfies (2). If*

$$\begin{cases} E|X|^\gamma < \infty, & \text{if } q < \gamma; \\ E|X|^\gamma \log(1 + |X|) < \infty, & \text{if } q = \gamma; \\ E|X|^q < \infty, & \text{if } q > \gamma, \end{cases} \tag{5}$$

then

$$\sum_{n=1}^{\infty} n^{t-q/p} E\{|S_n| - \epsilon n^{1/p}\}_+^q < \infty \text{ for all } \epsilon > 0. \tag{6}$$

Proof For every $\epsilon > 0$

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{t-q/p} E\{|S_n| - \epsilon n^{1/p}\}_+^q \\ &= \sum_{n=1}^{\infty} n^{t-q/p} \int_0^{\infty} P\{|S_n| - \epsilon n^{1/p} > x^{1/q}\} \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} n^{t-q/p} \int_0^{n^{q/p}} P\{|S_n| - \epsilon n^{1/p} > x^{1/q}\} \mathrm{d}x + \\
&\quad \sum_{n=1}^{\infty} n^{t-q/p} \int_{n^{q/p}}^{\infty} P\{|S_n| - \epsilon n^{1/p} > x^{1/q}\} \mathrm{d}x \\
&\le \sum_{n=1}^{\infty} n^t P(|S_n| > \epsilon n^{1/p}) + \sum_{n=1}^{\infty} n^{t-q/p} \int_{n^{q/p}}^{\infty} P\{|S_n| > x^{1/q}\} \mathrm{d}x.
\end{aligned}
$$

Hence by Theorem A, in order to prove Theorem 1, it suffices to show that

$$
\sum_{n=1}^{\infty} n^{t-q/p} \int_{n^{q/p}}^{\infty} P\{|S_n| > x^{1/q}\} \mathrm{d}x < \infty. \tag{7}
$$

Without loss of generality, we assume that $a_{ni} > 0$ for all $i \ge 1, n \ge 1, \sup_{i\ge1,n\ge1} a_{ni} = 1$, and $\sum_{i=1}^{\infty} a_{ni}^{\alpha} \le n^{\beta}$. Thus, for any $\theta \ge 0$, we have

$$
\sum_{i=1}^{\infty} a_{ni}^{\alpha+\theta} \le n^{\beta}. \tag{8}
$$

For $i \ge 1, n \ge 1, \forall x > 0$, we define $U_{ni}^{(x)} = -x^{1/q} I(a_{ni}X_{ni} < -x^{1/q}) + a_{ni}X_{ni} I(|a_{ni}X_{ni}| \le x^{1/q}) + x^{1/q} I(a_{ni}X_{ni} > x^{1/q})$, and $V_{ni}^{(x)} = a_{ni}X_{ni} - U_{ni}^{(x)}$. Since $EX_{ni} = 0$ and $\gamma \ge 1, \alpha < \gamma, t \ge -1$, by (8) and Lemma 2, we obtain

$$
\begin{aligned}
&\sup_{x\ge n^{q/p}} x^{-1/q} |E \sum_{i=1}^{\infty} U_{ni}^{(x)}| \\
&\le \sup_{x\ge n^{q/p}} x^{-1/q} \sum_{i=1}^{\infty} \{E|a_{ni}X_{ni}| I(|a_{ni}X_{ni}| > x^{1/q}) + x^{1/q} E I(|a_{ni}X_{ni}| > x^{1/q})\} \\
&\le 2 \sup_{x\ge n^{q/p}} x^{-1/q} \sum_{i=1}^{\infty} E|a_{ni}X_{ni}| I(|a_{ni}X_{ni}| > x^{1/q}) \\
&\le 2n^{-1/p} \sum_{i=1}^{\infty} E|a_{ni}X_{ni}| I(|a_{ni}X_{ni}| > n^{1/p}) \\
&\le 2n^{-1/p} \sum_{i=1}^{\infty} n^{-(\gamma-1)/p} E|a_{ni}X_{ni}|^{\gamma} I(|a_{ni}X_{ni}| > n^{1/p}) \\
&\le 2n^{-\gamma/p} \sum_{i=1}^{\infty} |a_{ni}|^{\gamma} E|X|^{\gamma} I(|X| > n^{1/p}) \\
&\le 2n^{\beta-\gamma/p} E|X|^{\gamma} I(|X| > n^{1/p}) \\
&= 2n^{-(t+1)} E|X|^{\gamma} I(|X| > n^{1/p}) \to 0, \quad n \to \infty.
\end{aligned}
$$

Hence for n large enough we have that

$$
\sup_{x\ge n^{q/p}} x^{-1/q} |E \sum_{i=1}^{\infty} U_{ni}^{(x)}| < 1/4.
$$

Therefore, in order to prove (7), it suffices to show that

$$I_1 = \sum_{n=1}^{\infty} n^{t-q/p} \int_{n^{q/p}}^{\infty} P\{|\sum_{i=1}^{\infty}(U_{ni}^{(x)} - EU_{ni}^{(x)})| > x^{1/q}/4\}\mathrm{d}x < \infty, \tag{9}$$

and

$$I_2 = \sum_{n=1}^{\infty} n^{t-q/p} \int_{n^{q/p}}^{\infty} P\{|\sum_{i=1}^{\infty} V_{ni}^{(x)}| > x^{1/q}/2\}\mathrm{d}x < \infty. \tag{10}$$

We first show that $I_2 < \infty$. Taking $\delta > 0$ such that $\alpha < \delta < \gamma$, we get by Lemma 2, (8) and (5) that

$$\begin{aligned}
I_2 &\le \sum_{n=1}^{\infty} n^{t-q/p} \int_{n^{q/p}}^{\infty} \sum_{i=1}^{\infty} P(|a_{ni}X_{ni}| > x^{1/q})\mathrm{d}x \\
&\le \sum_{n=1}^{\infty} n^{t-q/p} \int_{n^{q/p}}^{\infty} \sum_{i=1}^{\infty} x^{-\delta/q} E|a_{ni}X_{ni}|^{\delta} I(|a_{ni}X_{ni}| > x^{1/q})\mathrm{d}x \\
&\le \sum_{n=1}^{\infty} n^{t-q/p} \int_{n^{q/p}}^{\infty} \sum_{i=1}^{\infty} a_{ni}^{\delta} x^{-\delta/q} E|X_{ni}|^{\delta} I(|X_{ni}| > x^{1/q})\mathrm{d}x \\
&\le \sum_{n=1}^{\infty} n^{t-q/p} \int_{n^{q/p}}^{\infty} \sum_{i=1}^{\infty} a_{ni}^{\delta} x^{-\delta/q} E|X|^{\delta} I(|X| > x^{1/q})\mathrm{d}x \\
&\le C \sum_{n=1}^{\infty} n^{t+\beta-q/p} \int_{n^{q/p}}^{\infty} x^{-\delta/q} E|X|^{\delta} I(|X| > x^{1/q})\mathrm{d}x \\
&= C \sum_{n=1}^{\infty} n^{t+\beta-q/p} \sum_{m=n}^{\infty} \int_{m^{q/p}}^{(m+1)^{q/p}} x^{-\delta/q} E|X|^{\delta} I(|X| > x^{1/q})\mathrm{d}x \\
&\le C \sum_{n=1}^{\infty} n^{t+\beta-q/p} \sum_{m=n}^{\infty} m^{-1+(q-\delta)/p} E|X|^{\delta} I(|X| > m^{1/p}) \\
&= C \sum_{m=1}^{\infty} m^{-1+(q-\delta)/p} E|X|^{\delta} I(|X| > m^{1/p}) \sum_{n=1}^{m} n^{t+\beta-q/p} \\
&\le \begin{cases} C \sum_{m=1}^{\infty} m^{t+\beta-\delta/p} E|X|^{\delta} I(|X| > m^{1/p}), & \text{if } q < \gamma, \\ C \sum_{m=1}^{\infty} m^{-1+(q-\delta)/p} \log m E|X|^{\delta} I(|X| > m^{1/p}), & \text{if } q = \gamma, \\ C \sum_{m=1}^{\infty} m^{-1+(q-\delta)/p} E|X|^{\delta} I(|X| > m^{1/p}), & \text{if } q > \gamma \end{cases} \\
&\le \begin{cases} CE|X|^{\gamma}, & \text{if } q < \gamma, \\ CE|X|^{\gamma} \log(1+|X|), & \text{if } q = \gamma, \\ CE|X|^{q}, & \text{if } q > \gamma \end{cases} \\
&< \infty.
\end{aligned}$$

We now prove that $I_1 < \infty$. Fix any $v \ge 2$ and $v > \max\{q, \gamma\}$ (to be specified later). By

Markov inequality, Lemma 1, C_r inequality and Jensen inequality, we have

$$\begin{aligned}
I_1 \leq & C\sum_{n=1}^{\infty} n^{t-q/p}\int_{n^{q/p}}^{\infty} x^{-v/q}\{\sum_{i=1}^{\infty} E|U_{ni}^{(x)}|^v + (\sum_{i=1}^{\infty} E|U_{ni}^{(x)}|^2)^{v/2}\}\mathrm{d}x \\
\leq & C\sum_{n=1}^{\infty} n^{t-\frac{q}{p}}\int_{n^{\frac{q}{p}}}^{\infty} x^{\frac{-v}{q}}\{\sum_{i=1}^{\infty}(E|a_{ni}X_{ni}|^v I(|a_{ni}X_{ni}|\leq x^{\frac{1}{q}}) + x^{\frac{v}{q}}P(|a_{ni}X_{ni}|>x^{\frac{1}{q}}))\}\mathrm{d}x + \\
& C\sum_{n=1}^{\infty} n^{t-\frac{q}{p}}\int_{n^{\frac{q}{p}}}^{\infty} x^{\frac{-v}{q}}\{\sum_{i=1}^{\infty}(E|a_{ni}X_{ni}|^2 I(|a_{ni}X_{ni}|\leq x^{\frac{1}{q}}) + x^{2/q}P(|a_{ni}X_{ni}|>x^{\frac{1}{q}}))\}^{\frac{v}{2}}\mathrm{d}x \\
\stackrel{\text{def}}{=} & I_3 + I_4. \qquad (11)
\end{aligned}$$

Let $I_{nk} = \{i : (k+1)^{-1/p} < |a_{ni}| \leq k^{-1/p}\}, k \geq 1, n \geq 1$. Then $\cup_{k=1}^{\infty} I_{nk} = \mathbb{N}$ for all $n \geq 1$, where $\mathbb{N}$ is the set of positive integers. Since $v > \gamma > \alpha$, we have $k^{(v-\alpha)/p} > j^{(v-\alpha)/p}$ for all $k > j, j, k \geq 1$. For $\alpha > 0$, we have

$$\begin{aligned}
n^{\beta} \geq & \sum_{i=1}^{\infty}|a_{ni}|^{\alpha} = \sum_{k=1}^{\infty}\sum_{i\in I_{nk}}|a_{ni}|^{\alpha} \geq \sum_{k=1}^{\infty}(\sharp I_{nk})(k+1)^{-\alpha/p} \\
\geq & \sum_{k=j}^{\infty}(\sharp I_{nk})(k+1)^{-v/p}(j+1)^{(v-\alpha)/p} \\
> & 2^{-\alpha/p}\sum_{k=j}^{\infty}(\sharp I_{nk})k^{-v/p}j^{(v-\alpha)/p}.
\end{aligned}$$

For $\alpha < 0$, we also have

$$\begin{aligned}
n^{\beta} \geq & \sum_{i=1}^{\infty}|a_{ni}|^{\alpha} = \sum_{k=1}^{\infty}\sum_{i\in I_{nk}}|a_{ni}|^{\alpha} \geq \sum_{k=1}^{\infty}(\sharp I_{nk})k^{-\alpha/p} \\
> & \sum_{k=j}^{\infty}(\sharp I_{nk})k^{-v/p}j^{(v-\alpha)/p}.
\end{aligned}$$

Therefore,

$$\sum_{k=j}^{\infty}(\sharp I_{nk})k^{-v/p} \leq Cn^{\beta}j^{-(v-\alpha)/p}, \quad \forall j \geq 1. \qquad (12)$$

By Lemma 2 and the proof of (10), we have

$$\begin{aligned}
I_3 \leq & C\sum_{n=1}^{\infty} n^{t-q/p}\int_{n^{q/p}}^{\infty} x^{-v/q}\sum_{i=1}^{\infty} E|a_{ni}X_{ni}|^v I(|a_{ni}X_{ni}|\leq x^{1/q})\mathrm{d}x + C \\
\leq & C\sum_{n=1}^{\infty} n^{t-q/p}\int_{n^{q/p}}^{\infty} x^{-v/q}\sum_{i=1}^{\infty} E|a_{ni}X|^v I(|a_{ni}X|\leq x^{1/q})\mathrm{d}x + \\
& C\sum_{n=1}^{\infty} n^{t-q/p}\int_{n^{q/p}}^{\infty}\sum_{i=1}^{\infty} P\{|a_{ni}X|>x^{1/q}\}\mathrm{d}x + C \\
\leq & C\sum_{n=1}^{\infty} n^{t-q/p}\int_{n^{q/p}}^{\infty} x^{-v/q}\sum_{j=1}^{\infty}\sum_{i\in I_{nj}} E|a_{ni}X|^v I(|a_{ni}X|\leq x^{1/q})\mathrm{d}x + C
\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{n=1}^{\infty} n^{t-q/p}\int_{n^{q/p}}^{\infty} x^{-v/q}\sum_{j=1}^{\infty}(\sharp I_{nj})j^{-v/p}E|X|^v I(|X|^q \leq x(j+1)^{q/p})\mathrm{d}x + C\\
&\leq C\sum_{n=1}^{\infty} n^{t-q/p}\int_{n^{q/p}}^{\infty} x^{-v/q}\sum_{j=1}^{\infty}(\sharp I_{nj})j^{-v/p}\sum_{k=0}^{[x(j+1)^{q/p}]} E|X|^v I(k < |X|^q \leq k+1)\mathrm{d}x + C\\
&= C\sum_{n=1}^{\infty} n^{t-q/p}\int_{n^{q/p}}^{\infty} x^{-v/q}\sum_{j=1}^{\infty}(\sharp I_{nj})j^{-v/p}\sum_{k=0}^{[2x]} E|X|^v I(k < |X|^q \leq k+1)\mathrm{d}x +\\
&\quad C\sum_{n=1}^{\infty} n^{t-q/p}\int_{n^{q/p}}^{\infty} x^{-v/q}\sum_{j=1}^{\infty}(\sharp I_{nj})j^{-v/p}\sum_{k=[2x]+1}^{[x(j+1)^{q/p}]} E|X|^v I(k < |X|^q \leq k+1)\mathrm{d}x + C\\
&\stackrel{\text{def}}{=} I_5 + I_6 + C. \qquad (13)
\end{aligned}
$$

By (5) and (12), we have

$$
\begin{aligned}
I_5 \leq& C\sum_{n=1}^{\infty} n^{t-q/p}\int_{n^{q/p}}^{\infty} x^{-v/q} n^{\beta}\sum_{k=0}^{[2x]} E|X|^v I(k < |X|^q \leq k+1)\mathrm{d}x\\
\leq& C\int_1^{\infty} y^{t+\beta-q/p}\mathrm{d}y\int_{y^{q/p}}^{\infty} x^{-v/q}\sum_{k=0}^{[2x]} E|X|^v I(k < |X|^q \leq k+1)\mathrm{d}x\\
=& C\int_1^{\infty} u^{-2+\gamma/p}\mathrm{d}u\int_u^{\infty} x^{-v/q}\sum_{k=0}^{[2x]} E|X|^v I(k < |X|^q \leq k+1)\mathrm{d}x\\
&(\text{ letting } u = y^{q/p})\\
=& C\int_1^{\infty} x^{-v/q}\sum_{k=0}^{[2x]} E|X|^v I(k < |X|^q \leq k+1)\mathrm{d}x\int_1^{x} u^{-2+\gamma/q}\mathrm{d}u\\
\leq& \begin{cases}
C\displaystyle\int_1^{\infty} x^{-1+\gamma/q-v/q}\sum_{k=0}^{[2x]} E|X|^v I(k < |X|^q \leq k+1)\mathrm{d}x, & \text{if } q < \gamma,\\
C\displaystyle\int_1^{\infty} x^{-v/q}\log x\sum_{k=0}^{[2x]} E|X|^v I(k < |X|^q \leq k+1)\mathrm{d}x, & \text{if } q = \gamma,\\
C\displaystyle\int_1^{\infty} x^{-v/q}\sum_{k=0}^{[2x]} E|X|^v I(k < |X|^q \leq k+1)\mathrm{d}x, & \text{if } q > \gamma
\end{cases}\\
\leq& \begin{cases}
C\displaystyle\sum_{k=1}^{\infty} E|X|^v I(k < |X|^q \leq k+1)\int_{k/2}^{\infty} x^{-1+\gamma/q-v/q}\mathrm{d}x + C, & \text{if } q < \gamma,\\
C\displaystyle\sum_{k=2}^{\infty} E|X|^v I(k < |X|^q \leq k+1)\int_{k/2}^{\infty} x^{-v/q}\log x\mathrm{d}x + C, & \text{if } q = \gamma,\\
C\displaystyle\sum_{k=1}^{\infty} E|X|^v I(k < |X|^q \leq k+1)\int_{k/2}^{\infty} x^{-v/q}\mathrm{d}x + C, & \text{if } q > \gamma
\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \begin{cases} C\sum_{k=1}^{\infty} k^{\gamma/q-v/q}E|X|^v I(k<|X|^q\leq k+1)+C, & \text{if } q<\gamma,\\ C\sum_{k=2}^{\infty} k^{1-v/q}\log k E|X|^v I(k<|X|^q\leq k+1)+C, & \text{if } q=\gamma,\\ C\sum_{k=1}^{\infty} k^{1-v/q}E|X|^v I(k<|X|^q\leq k+1)+C, & \text{if } q>\gamma \end{cases}\\
&\leq \begin{cases} CE|X|^\gamma+C, & \text{if } q<\gamma,\\ CE|X|^\gamma\log(1+|X|)+C, & \text{if } q=\gamma,\\ CE|X|^q+C, & \text{if } q>\gamma \end{cases}\\
&<\infty. \qquad (14)
\end{aligned}
$$

and

$$
\begin{aligned}
I_6 \leq & C\sum_{n=1}^{\infty} n^{t-q/p}\int_{n^{q/p}}^{\infty} x^{-v/q}\sum_{k=[2x]+1}^{\infty}\sum_{j\geq[(k/x)^{p/q}]-1}(\sharp I_{nj})j^{-v/p}E|X|^v I(k<|X|^q\leq k+1)\mathrm{d}x\\
\leq & C\sum_{n=1}^{\infty} n^{t-q/p}\int_{n^{q/p}}^{\infty} x^{-v/q}\sum_{k=[2x]+1}^{\infty} n^{\beta}(k/x)^{-(v-\alpha)/q}E|X|^v I(k<|X|^q\leq k+1)\mathrm{d}x\\
\leq & C\int_1^{\infty} y^{t+\beta-q/p}\mathrm{d}y\int_{y^{q/p}}^{\infty} x^{-\alpha/q}\sum_{k=[2x]+1}^{\infty} k^{-(v-\alpha)/q}E|X|^v I(k<|X|^q\leq k+1)\mathrm{d}x\\
= & C\int_1^{\infty} u^{-2+\gamma/q}\mathrm{d}u\int_u^{\infty} x^{-\alpha/q}\sum_{k=[2x]+1}^{\infty} k^{-(v-\alpha)/q}E|X|^v I(k<|X|^q\leq k+1)\mathrm{d}x\\
& (\text{ letting } u=y^{q/p})\\
= & C\int_1^{\infty} x^{-\alpha/q}\sum_{k=[2x]+1}^{\infty} k^{-(v-\alpha)/q}E|X|^v I(k<|X|^q\leq k+1)\mathrm{d}x\int_1^{x} u^{-2+\gamma/q}\mathrm{d}u\\
\leq & \begin{cases} C\int_1^{\infty} x^{-1+\gamma/q-\alpha/q}\sum_{k=[2x]+1}^{\infty} k^{-(v-\alpha)/q}E|X|^v I(k<|X|^q\leq k+1)\mathrm{d}x, & \text{if } q<\gamma,\\ C\int_1^{\infty} x^{-\alpha/q}\log x\sum_{k=[2x]+1}^{\infty} k^{-(v-\alpha)/q}E|X|^v I(k<|X|^q\leq k+1)\mathrm{d}x, & \text{if } q=\gamma,\\ C\int_1^{\infty} x^{-\alpha/q}\sum_{k=[2x]+1}^{\infty} k^{-(v-\alpha)/q}E|X|^v I(k<|X|^q\leq k+1)\mathrm{d}x, & \text{if } q>\gamma \end{cases}\\
\leq & \begin{cases} C\sum_{k=3}^{\infty} k^{-(v-\alpha)/q}E|X|^v I(k<|X|^q\leq k+1)\int_1^{k/2} x^{-1+\gamma/q-\alpha/q}\mathrm{d}x, & \text{if } q<\gamma,\\ C\sum_{k=3}^{\infty} k^{-(v-\alpha)/q}E|X|^v I(k<|X|^q\leq k+1)\int_1^{k/2} x^{-\alpha/q}\log x\mathrm{d}x, & \text{if } q=\gamma,\\ C\sum_{k=3}^{\infty} k^{-(v-\alpha)/q}E|X|^v I(k<|X|^q\leq k+1)\int_1^{k/2} x^{-\alpha/q}\mathrm{d}x, & \text{if } q>\gamma \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \begin{cases} C\sum_{k=3}^{\infty} k^{\gamma/q-v/q} E|X|^v I(k<|X|^q \leq k+1), & \text{if } q<\gamma, \\ C\sum_{k=3}^{\infty} k^{1-v/q} \log k E|X|^v I(k<|X|^q \leq k+1), & \text{if } q=\gamma, \\ C\sum_{k=3}^{\infty} k^{1-v/q} E|X|^v I(k<|X|^q \leq k+1), & \text{if } q>\gamma \end{cases} \\
&\leq \begin{cases} CE|X|^{\gamma}, & \text{if } q<\gamma, \\ CE|X|^{\gamma}\log(1+|X|), & \text{if } q=\gamma, \\ CE|X|^{q}, & \text{if } q>\gamma \end{cases} \\
&<\infty. \qquad (15)
\end{aligned}
$$

Thus $I_3 < \infty$ by (13), (14) and (15).

We now prove that $I_4 < \infty$. Consider three separate cases. If $\gamma \geq 2$, note that in this case $E|X_{ni}|^2 \leq CE|X|^2 < \infty$. Taking v such that $v > \max\{q, \gamma, (t+1)/(1/p-\lambda/2)$, we get by (2) that

$$
\begin{aligned}
I_4 \leq & C\sum_{n=1}^{\infty} n^{t-q/p} \int_{n^{q/p}}^{\infty} x^{-v/q} \{\sum_{i=1}^{\infty} E|a_{ni}X_{ni}|^2\}^{v/2} \mathrm{d}x \\
\leq & C\sum_{n=1}^{\infty} n^{t-q/p} \int_{n^{q/p}}^{\infty} x^{-v/q} (n^{\lambda})^{v/2} \mathrm{d}x \\
= & C\sum_{n=1}^{\infty} n^{t-(1/p-\lambda/2)v} < \infty. \qquad (16)
\end{aligned}
$$

If $1 \leq \gamma < 2, q \geq 2$, note that in this case $1/p - \beta/2 > 0$ when $\beta \leq 0, 1/p - \beta/2 = \{2(t+1) + (2-\gamma)\beta\}/(2\gamma) > 0$ when $\beta > 0$, and $E|X_{ni}|^2 \leq CE|X|^2 < \infty$. Taking v such that $v > \max\{q, (t+1)/(1/p-\beta/2)$, we get by (1) and (8) that

$$
\begin{aligned}
I_4 \leq & C\sum_{n=1}^{\infty} n^{t-q/p} \int_{n^{q/p}}^{\infty} x^{-v/q} \{\sum_{i=1}^{\infty} E|a_{ni}X_{ni}|^2\}^{v/2} \mathrm{d}x \\
\leq & C\sum_{n=1}^{\infty} n^{t-q/p} \int_{n^{q/p}}^{\infty} x^{-v/q} (n^{\beta})^{v/2} \mathrm{d}x \\
= & C\sum_{n=1}^{\infty} n^{t-(1/p-\beta/2)v} < \infty. \qquad (17)
\end{aligned}
$$

If $1 \leq \gamma < 2, q < 2$, let $v = 2$. Then

$$
I_4 = I_3 < \infty. \qquad (18)
$$

Thus $I_4 < \infty$ by (16), (17) and (18). Therefore $I_1 < \infty$ by (11).

Theorem 2 *Let $t \geq -1, p > 0, \gamma = p(t+\beta+1) \geq 1, q > 0$. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of zero-mean rowwise NA random variables which are stochastically dominated by a random variable X. Let $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of real numbers satisfying (4). If $\gamma \geq 2$,*

Moreover, we assume that $\{a_{ni}, i \geq 1, n \geq 1\}$ *satisfies (2). If*

$$\begin{cases} E|X|^{\gamma}\log(1+|X|) < \infty, & \text{if } q < \gamma, \\ E|X|^{\gamma}\log^2(1+|X|) < \infty, & \text{if } q = \gamma, \\ E|X|^{q} < \infty, & \text{if } q > \gamma, \end{cases}$$

then (6) holds.

Proof The proof is similar to that of Theorem 1, so it is omitted here. □

Corollary 1 *Let* $\{X_{ni}, i \geq 1,\ n \geq 1\}$ *be an array of zero-mean rowwise NA random variables which are stochastically dominated by a random variable* X*. Let* $\{a_{ni}, i \geq 1, n \geq 1\}$ *be an array of real numbers satisfying (2) and*

$$\sum_{i=1}^{\infty} |a_{ni}|^{\theta} = O(1) \text{ for some } 2 \leq \theta < p.$$

If

$$\begin{cases} E|X|^{p} < \infty, & \text{if } q < p, \\ E|X|^{p}\log(1+|X|) < \infty, & \text{if } q = p, \\ E|X|^{q} < \infty, & \text{if } q > p, \end{cases}$$

then

$$\sum_{n=1}^{\infty} n^{-q/p} E\{|S_n| - \epsilon n^{1/p}\}_{+}^{q} < \infty \text{ for all } \epsilon > 0.$$

Proof Let $t = 0$ and $\beta = 0$. Clearly, $\sup_{i\geq 1, n\geq 1} |a_{ni}| < \infty$. Thus the result follows from Theorem 1.

Corollary 2 *Let* $\{X_{ni}, i \geq 1,\ n \geq 1\}$ *be an array of zero-mean rowwise NA random variables which are stochastically dominated by a random variable* X*. Let* $\{a_{ni}, i \geq 1, n \geq 1\}$ *be an array of real numbers satisfying*

$$\sum_{i=1}^{\infty} |a_{ni}|^{2} = O(1).$$

If

$$\begin{cases} E|X|^{2}\log(1+|X|) < \infty, & \text{if } q < 2, \\ E|X|^{2}\log^2(1+|X|) < \infty, & \text{if } q = 2, \\ E|X|^{q} < \infty, & \text{if } q > 2, \end{cases}$$

then

$$\sum_{n=1}^{\infty} n^{-q/p} E\{|S_n| - \epsilon n^{1/2}\}_{+}^{q} < \infty \text{ for all } \epsilon > 0.$$

Proof Let $t = 0$, $\beta = 0$ and $p = 2$. Clearly, $\sup_{i\geq 1, n\geq 1} |a_{ni}| < \infty$. Thus the result follows from Theorem 2.

Remark 1 When $X_{ni} = X_i$, $\forall n \geq 1$, $i \geq 1$, let $\{X_i, i \geq 1\}$ be a sequence of i.i.d. random variables. Corollaries 1 and 2 extend and improve the corresponding results of Li et al. [8], since complete moment convergence is more exact than complete convergence.

From Theorem 1, we can obtain a result on the rate of convergence of moving average processes.

Corollary 3 *Let $\{X_n, -\infty < n < \infty\}$ be a sequence of zero-mean NA random variables which are stochastically dominated by a random variable X. Let $\{a_n, -\infty < n < \infty\}$ be a sequence of real numbers such that $\sum_{n=-\infty}^{\infty} |a_n| < \infty$. Set $a_{ni} = \sum_{j=i+1}^{i+n} a_j$ for each i and n. If $0 < p < 2$, $t \geq -1$, $p(t+2) > 1$ and*

$$\begin{cases} E|X|^{p(t+2)} < \infty, & \text{if } 0 < q < p(t+2), \\ E|X|^{p(t+2)} \log(1+|X|) < \infty, & \text{if } q = p(t+2), \\ E|X|^q < \infty, & \text{if } q > p(t+2), \end{cases}$$

then

$$\sum_{n=1}^{\infty} n^{t-q/p} E\{|\sum_{i=-\infty}^{\infty} a_{ni} X_i| - \epsilon n^{1/p}\}_+^q < \infty \text{ for all } \epsilon > 0.$$

Proof Since $\sum_{n=-\infty}^{\infty} |a_n| < \infty$, we have that

$$\sup_{i\geq 1, n\geq 1} |a_{ni}| \leq \sup_{i\geq 1, n\geq 1} \sum_{j=i+1}^{i+n} |a_j| \leq \sum_{n=-\infty}^{\infty} |a_n| < \infty,$$

$$\sum_{i=-\infty}^{\infty} |a_{ni}| \leq \sum_{i=-\infty}^{\infty} \sum_{j=i+1}^{i+n} |a_j| \leq n \sum_{j=-\infty}^{\infty} |a_j| = O(n),$$

$$\sum_{i=-\infty}^{\infty} a_{ni}^2 \leq C \sum_{i=-\infty}^{\infty} |a_{ni}| = O(n).$$

Taking $\beta = 1$ and $\lambda = 1$, all conditions of Theorem 1 are satisfied. Thus the result follows from Theorem 1.

Remark 2 Let $\{X_n, -\infty < n < \infty\}$ be a sequence of zero-mean identically distributed NA random variables, $r = p(t+2), q = 1$. Theorem 1.1 of Li and Zhang [9] is obtained from Corollary 3, but the restrictions of p and $r = p(t+2)$ in Corollary 3 are weaker than those in Theorem 1.1 of Li and Zhang [9].

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