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Gradient Estimates for a Nonlinear Heat Equation on Compact Riemannian Manifold

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Abstract In this paper, we study gradient estimates for the nonlinear heat equation $u_t-\triangle u=au\log u$, on compact Riemannian manifold with or without boundary. We get a Hamilton type gradient estimate for the positive smooth solution to the equation on close manifold, and obtain a Li-Yau type gradient estimate for the positive smooth solution to the equation on compact manifold with nonconvex boundary.

Keywords nonlinear heat equation; Riemannian manifold; gradient estimates.

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1. Introduction

Let (M^n,g) be a compact manifold with or without boundary. We consider the gradient estimates for the nonlinear heat equation

$$u_t-\triangle u=au\log u, \tag{1.1}$$

on (M^n,g). Here $a\in\mathbf{R}$ is a constant. This heat equation can be considered as the negative gradient heat flow to W-functional [1], which is closely related to the Log-Sobolev inequalities on Riemannian manifold. Some results have been obtained by many researchers [2–7]. In [5] and [6], Ma obtained several gradient estimates of the positve solution to (1.1). Specially, he got the following two theorems,

Theorem 1.1 *Assume that the closed Riemannian manifold (M^n,g) has the non-negative Ricci curvature condition, i.e., $Rc\geq 0$. Let $u>0$ be a positive smooth solution to (1.1). Assume that $\sup_M u<1$ at the initial time and $a\leq 0$. Let $f=-\log u$. Then we have, for all $t>0$, $\sup_M u<1$ and*

$$t|\nabla f|^2\leq f.$$

The same estimate is true for (1.1) on complete Riemannian manifolds when the maximum principle can be applied.

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Theorem 1.2 *Assume that the compact Riemannian manifold (M,g) has non-negative Ricci curvature. When $a \le 0$ in (1.1), let $u > 0$ be a positive smooth solution to (1.1). Let $f = \log u$. Then we have, for all $t > 0$,*

$$\triangle f - \frac{n}{2t} \le 0,$$

and in other words,

$$f_t - af + |\nabla f|^2 \le \frac{n}{2t}.$$

The same result is also true for (1.1) on complete Riemannian manifold provided the maximum principle is applicable.

One of our purposes in this paper is to get a similar result to Theorem 1.1.

Theorem 1.3 *Let (M^n, g) be closed Riemannian manifold, and $R_{ij} \geqslant -Kg_{ij}$, $K \geqslant 0$. Assume u be a bounded positive smooth solution to (1.1) on $M \times [0, +\infty)$, such that $0 < u(x,t) \le A$, where $A \ge 1$ is a positive constant. Let $f = -\log u, g = f + \log A = \log \frac{A}{u}$. And assume $a \le 0$. Then we have,*

$$\varphi(t)|\nabla g|^2 \le g,$$

that is,

$$\varphi(t)|\nabla f|^2 \le \log \frac{A}{u},$$

where

$$\varphi(t) = \begin{cases} \frac{1-e^{-(a+2K)t}}{a+2K} & \text{if} \quad a + 2K \neq 0; \\ t & \text{if} \quad a + 2K = 0. \end{cases}$$

Remark 1.1 In [5], the author left a question, whether can we derive the Hamilton type gradient estimate for positive solutions to the equation (1.1). Our result partially answers the question of Ma in [5].

Another problem we want to consider in this paper is to establish the Li-Yau type gradient estimates of (1.1) on compact manifold with nonconvex boundary. Let (M^n, g) be an n-dimensional compact Riemannian manifold with boundary ∂M. Let $\frac{\partial}{\partial v}$ be the outward pointing unit normal vector to ∂M, and denote by II the second fundamental form of ∂M with respect to $\frac{\partial}{\partial v}$. Our goal is to derive estimates on the derivatives of positive solutions $u(x,t)$ on $M \times (0, +\infty)$ of the nonlinear equation

$$\begin{cases} u_t - \Delta u = au \log u \\ \frac{\partial u}{\partial v}|_{\partial M} = 0 \end{cases}, \tag{1.2}$$

where $a \in \mathbf{R}$.

Definition 1.1 ([8, 9]) *∂M is said to satisfy the "interior rolling R-ball" condition if for each point $p \in \partial M$ there is a geodesic ball $B_q(\frac{R}{2})$, centered at $q \in M$ with radius $\frac{R}{2}$, such that $\{p\} = B_q(\frac{R}{2}) \cap \partial M$ and $B_q(\frac{R}{2}) \subset M$.*

When $a = 0$ in (1.2), Wang obtained global gradient estimates of the positive solutions in [8].

Theorem 1.4 *Let (M^n, g) be an n-dimensional compact Riemannian manifold with boundary ∂M. Suppose that ∂M satisfies the "interior rolling R-ball" condition. Let K and H be non-negative constants such that the Ricci curvature Ric_M of M is bounded below by $-K$ and the second fundamental form II of ∂M is bounded below by $-H$. By choosing R small, we have for any positive solution $u(x,t)$ of (1.2) on $M \times (0, +\infty)$ and $a = 0$,*

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{C_1}{t} + C_2,$$

on $M \times (0, +\infty)$, for all constant

$$\alpha > (1+H)^2 \ \text{ and } \ 0 < \beta < \frac{1}{2},$$

where

$$C_1 = \frac{n\alpha^2(\alpha-1)^2(1+H)^4}{(2-\beta)(1-\beta)(\alpha-(1+H)^2)^2},$$

$$C_2 = \frac{6n\alpha(\alpha-1)(1+H)^7 K}{(\alpha-(1+H^2)^2)^2} + \frac{309 n^2 \alpha^3(\alpha-1)(1+H)^{10} H}{(\alpha-(1+H)^2)^4 R^2 \beta}.$$

We also have the following result for (1.2), as is similar to Theorem 1.4.

Theorem 1.5 *Let (M^n, g) be an n-dimensional compact Riemannian manifold with boundary ∂M. Suppose that ∂M satisfies the "interior rolling R-ball" condition. Let K and H be non-negative constants such that the Ricci curvature Ric_M of M is bounded below by $-K$ and the second fundamental form II of ∂M is bounded below by $-H$. By choosing R small, we have for any positive solution $u(x,t)$ of (1.2) on $M \times (0, +\infty)$,*

$$\frac{|\nabla u|^2}{u^2} + a\alpha \log u - \alpha \frac{u_t}{u} \leq \frac{C_1}{t} + C_2, \tag{1.3}$$

on $M \times (0, +\infty)$, for all constant

$$a \in \mathbf{R},\ \alpha > (1+H)^2 \text{ and } 0 < \beta < \frac{1}{2},$$

where

$$C_1 = \frac{n\alpha^2(\alpha-1)^2(1+H)^4}{(2-\frac{5}{4}\beta)(1-\beta)(\alpha-(1+H)^2)^2},$$

$$C_2 = \frac{4\sqrt{2}n\alpha^2(\alpha-1)(1+H)^7 K}{(\alpha-(1+H^2)^2)^2} + D_0.$$

And we choose

$$D_0 = \frac{\sqrt{2n}\alpha(\alpha-1)(1+H)^3(|\hat{D}| + 2\alpha|a|)}{(\alpha-(1+H)^2)},$$

where if $a \leq 0$, we choose

$$\hat{D} = \frac{\sqrt{n}\alpha(1+H)^2}{\alpha-(1+H)^2}\Big\{2\alpha a - 2a - (2(1+H)(\frac{2(n-1)H(3H+1)}{R} + \frac{H}{R^2}) + \frac{64}{\beta}H^2(1+H)^2) + \frac{64 n\alpha^2 H^2(1+H)^6}{(\alpha-(1+H)^2)^2}\Big\},$$

and if $a>0$, we choose

$$\hat{D}=\frac{\sqrt{n}\alpha(1+H)^2}{\alpha-(1+H)^2}\Big\{2\alpha a-2a(1+H)^2-(2(1+H)(\frac{2(n-1)H(3H+1)}{R}+\frac{H}{R^2})+\frac{64}{\beta}H^2(1+H)^2)+\frac{64n\alpha^2H^2(1+H)^6}{(\alpha-(1+H)^2)^2}\Big\}.$$

Remark 1.2 If $a=0$, Theorem 1.5 recovers Theorem 1.4.

The rest of the paper is organized as follows. In Section 2, we give the proof of Theorem 1.3. In Section 3, we prove Theorem 1.5, then using the theorem we get a corollary.

2. Hamilton type gradient estimate

We use a modification of the argument of [5], [10] and [11] to prove Theorem 1.3.

Proof of Theorem 1.3 Let $f=-\log u$, $g=f+\log A=\log\frac{A}{u}$. Then we have

$$\frac{\partial g}{\partial t}-\Delta g=\frac{\partial f}{\partial t}-\Delta f=af-|\nabla f|^2. \tag{2.1}$$

Using the Bochner formula,

$$\Delta|\nabla f|^2=2|D^2f|^2+2\langle\nabla f,\nabla\Delta f\rangle+2Rc(\nabla f,\nabla f).$$

Compute,

$$\frac{\partial}{\partial t}|\nabla f|^2-\Delta|\nabla f|^2=2\langle\nabla f,\nabla(\frac{\partial f}{\partial t}-\Delta f)\rangle-2|D^2f|^2-2Rc(\nabla f,\nabla f).$$

Using (2.1), we have

$$\frac{\partial}{\partial t}|\nabla f|^2-\Delta|\nabla f|^2=2a|\nabla f|^2-2\langle\nabla f,\nabla|\nabla f|^2\rangle-2|D^2f|^2-2Rc(\nabla f,\nabla f).$$

Using $R_{ij}\geq -Kg_{ij}$, we have

$$\frac{\partial}{\partial t}|\nabla f|^2-\Delta|\nabla f|^2\leq 2a|\nabla f|^2-2\langle\nabla f,\nabla|\nabla f|^2\rangle+2K|\nabla f|^2. \tag{2.2}$$

Let

$$\varphi(t)=\begin{cases}\frac{1-e^{-(a+2K)t}}{a+2K} & \text{if } a+2K\neq 0;\\ t & \text{if } a+2K=0.\end{cases}$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)=-(a+2K)\varphi(t)+1. \tag{2.3}$$

Let $L=\varphi(t)|\nabla g|^2-g=\varphi(t)|\nabla f|^2-(f+\log A)$, by (2.2). Then

$$\begin{aligned}\frac{\partial}{\partial t}L-\Delta L\leq&[\varphi'(t)+(2a+2K)\varphi(t)-1]|\nabla f|^2-2\varphi|D^2f|^2-2\langle\nabla f,\nabla L\rangle+\\&a(L-\varphi(t)|\nabla g|^2)+a\log A\\=&[\varphi'(t)+(a+2K)\varphi(t)-1]|\nabla f|^2-2\varphi|D^2f|^2-2\langle\nabla f,\nabla L\rangle+\\&aL+a\log A.\end{aligned}$$

Using $a\leq 0$, $A\geq 1$, $\varphi(t)>0$ and (2.3), we get

$$\frac{\partial}{\partial t}L-\Delta L\leq -2\langle\nabla f,\nabla L\rangle+aL.$$

Let $G = e^{-at}L$. Then

$$\frac{\partial}{\partial t}G - \Delta G \leq -2\langle \nabla f, \nabla G \rangle.$$

Applying the maximum principle to G, we know that $G \leq 0$. That is,

$$\varphi(t)|\nabla g|^2 - g \leq 0,$$

which is the desired gradient estimate of Hamilton type. Then we complete the proof of Theorem 1.3. □

3. Li-Yau type gradient estimate

In this section, let (M^n, g) be an n-dimensional compact Riemannian manifold with boundary ∂M. Let $\frac{\partial}{\partial v}$ be the outward pointing unit normal vector to ∂M, and denote by II the second fundamental form of ∂M with respect to $\frac{\partial}{\partial v}$. Our goal is to derive estimates on the derivatives of positive solutions $u(x,t)$ on $M \times (0, +\infty)$ of the nonlinear equation (1.2).

Modifying the argument of [8] and [11], we give the proof of Theorem 1.5.

Proof of Theorem 1.5 Following [8] (or [9]), we define a function on M by $\phi(x) = \psi(\frac{r(x)}{R})$, where $r(x)$ denotes the distance from $x \in M$ to ∂M and $\psi(r)$ is a nonnegative C^2-function defined on $[0, +\infty)$ such that

$$\begin{cases} \psi(r) \leq H & \text{if} \quad r \in [0, 1/2); \\ \psi(r) = H & \text{if} \quad r \in [1, +\infty), \end{cases}$$

with $\psi(0) = 0$, $0 \leq \psi'(r) \leq 2H$, $\psi'(0) = H$ and $\psi'' \geq -H$. Let $f = \log u$. Then $\frac{\partial f}{\partial t} - \triangle f = af + |\nabla f|^2$. For every $\varepsilon > 0$, consider

$$F(x,t) = t[(1+\phi(x))^2(|\nabla f|^2(x,t) + \varepsilon) + \alpha a f(x,t) - \alpha f_t(x,t)].$$

For any fixed $T < +\infty$, since $F(x,t)$ is continuous on $\overline{M} \times [0,T]$, there exists $(p, t_0) \in \overline{M} \times [0,T]$ at which F achieves its maximum. We may assume that $F(p, t_0) > 0$ as otherwise the inequality (1.3) follows trivially.

Claim 1 $p \in \overline{M} \backslash \partial M$.

In fact, if $p \in \partial M$, then $\frac{\partial F}{\partial v}(p, t_0) \geq 0$. Let $e_1, e_2, \ldots, e_n$ be an orthonormal frame at p with $e_n = v$. Notice that $f_n = f_v = \frac{u_v}{u} = 0$ on ∂M. Therefore, denoting $\chi(x) = (1+\phi(x))^2$, we have

$$0 \leq \frac{\partial F}{\partial v}(p, t_0) = t_0(\frac{\partial \chi}{\partial v}(|\nabla f|^2 + \varepsilon) + 2\chi \sum_{i=1}^{n} f_i f_{iv} + \alpha a f_v - \alpha f_{tv})(p, t_0).$$

Since $f_v = 0$ on ∂M and $t_0 > 0$, we conclude that

$$\frac{\partial \chi}{\partial v} \cdot \frac{1}{\chi} + \frac{2\sum_{i=1}^{n} f_i f_{iv}}{|\nabla f|^2 + \varepsilon} \geq 0$$

at (p, t_0).

By a direct computation, one shows that

$$\sum_{i=1}^{n} f_i f_{iv} = -II(\nabla f, \nabla f) \leq H|\nabla f|^2.$$

Thus at (p,t_0), if we choose $R<1$, then

$$\frac{\partial\chi}{\partial v}\cdot\frac{1}{\chi}+\frac{2\sum_{i=1}^n f_i f_{iv}}{|\nabla f|^2+\varepsilon}\le -\frac{2H}{R}+\frac{2H|\nabla f|^2}{|\nabla f|^2+\varepsilon}<0.$$

This is a contradiction and the claim follows.

Thus, $F(x,t)$ achieves its maximum at $(p,t_0)\in(\overline{M}\backslash\partial M)\times(0,T]$.

Hence at (p,t_0), $\nabla F=0$, $\frac{\partial F}{\partial t}\ge 0$ and $\triangle F\le 0$. In the following, all the computations are performed at the point (p,t_0) and the summation convention is used with i and j both moving between 1 and n. Direct computation gives us

$$\begin{aligned}\triangle F =&t\Big(\triangle\chi\cdot(|\nabla f|^2+\varepsilon)+\chi\triangle|\nabla f|^2+2\langle\nabla\chi,\nabla|\nabla f|^2\rangle+\\ &\alpha a\triangle f-\alpha(\triangle f)_t\Big)\\ =&t\Big(\triangle\chi\cdot(|\nabla f|^2+\varepsilon)+2\chi(f_{ij}^2+f_i f_{ijj})+2\langle\nabla\chi,\nabla|\nabla f|^2\rangle+\\ &\alpha a\triangle f-\alpha(\triangle f)_t\Big).\end{aligned}$$

Since

$$f_i f_{ijj}=f_i f_{jji}+\mathrm{Ric}(\nabla f,\nabla f)\ge\langle\nabla f,\nabla\triangle f\rangle-K|\nabla f|^2$$

and $\triangle f=f_t-|\nabla f|^2-af$, we obtain

$$\begin{aligned}\triangle F\ge&t\Big(\triangle\chi\cdot(|\nabla f|^2+\varepsilon)+2\chi(f_{ij}^2+\langle\nabla f,\nabla\triangle f\rangle-K|\nabla f|^2)+\\ &2\langle\nabla\chi,\nabla|\nabla f|^2\rangle+\alpha a\triangle f-\alpha(f_t-|\nabla f|^2-af)_t\Big).\end{aligned}$$

And,

$$F_t=\chi(|\nabla f|^2+\varepsilon)+\alpha af-\alpha f_t+t(\chi|\nabla f|_t^2+\alpha af_t-\alpha f_{tt}).$$

Thus $0\ge\triangle F-\frac{\partial F}{\partial t}$ implies,

$$\begin{aligned}0\ge&t\Big[\triangle\chi(|\nabla f|^2+\varepsilon)+2\chi f_{ij}^2-2K\chi|\nabla f|^2+2\chi\langle\nabla f,\nabla(f_t-af-|\nabla f|^2)\rangle+\\ &2\langle\nabla\chi,\nabla|\nabla f|^2\rangle+\alpha a\triangle f+(\alpha-\chi)|\nabla f|_t^2\Big]-\chi(|\nabla f|^2+\varepsilon)-\alpha af+\alpha f_t\\ =&t[\varepsilon\triangle\chi+(\triangle\chi-2K\chi-2a\chi)|\nabla f|^2+2\chi f_{ij}^2+\alpha a\triangle f]-\frac{F}{t}+\\ &t[2\langle\nabla\chi,\nabla|\nabla f|^2\rangle-2\chi\langle\nabla f,\nabla|\nabla f|^2\rangle+2\alpha\langle\nabla f,\nabla f_t\rangle].\end{aligned}$$

Using

$$\nabla F=t[(|\nabla f|^2+\varepsilon)\nabla\chi+\chi\nabla|\nabla f|^2+\alpha a\nabla f-\alpha\nabla f_t]=0,$$

so

$$-2\chi\langle\nabla f,\nabla|\nabla f|^2\rangle+2\alpha\langle\nabla f,\nabla f_t\rangle=2(|\nabla f|^2+\varepsilon)\nabla f\cdot\nabla\chi+2\alpha a|\nabla f|^2,$$

combining Young inequality $2a_0b_0\le\beta a_0^2+\frac{1}{\beta}b_0^2$ $(\beta>0)$, we get

$$\begin{aligned}0\ge&t[\varepsilon\triangle\chi+(\triangle\chi-2K\chi-2a\chi+2\alpha a)|\nabla f|^2+2\chi f_{ij}^2+\alpha a\triangle f]-\frac{F}{t}+\\ &2t\langle\nabla\chi,\nabla|\nabla f|^2\rangle+2t(|\nabla f|^2+\varepsilon)\langle\nabla\chi,\nabla f\rangle\end{aligned}$$

$$
\begin{aligned}
&\geq 2\chi t f_{ij}^2 + \alpha a t\triangle f + t(\triangle\chi - 2K\chi - 2a\chi + 2\alpha a)|\nabla f|^2 + \varepsilon t\triangle\chi - \frac{F}{t} - \\
&\frac{4t}{\beta}|\nabla\chi|^2|\nabla f|^2 - \beta t f_{ij}^2 - 2t|\nabla\chi||\nabla f|^3 - t\varepsilon|\nabla\chi|^2 - t\varepsilon|\nabla f|^2 \\
&\geq (2\chi - \beta)t f_{ij}^2 + \alpha a t\triangle f + t\left(\triangle\chi - 2K\chi - 2a\chi + 2\alpha a - \frac{4}{\beta}|\nabla\chi|^2 - \varepsilon\right)|\nabla f|^2 - \\
&2t|\nabla\chi||\nabla f|^3 + \varepsilon t(\triangle\chi - |\nabla\chi|^2) - \frac{F}{t}.
\end{aligned}
$$

Since

$$\sum f_{ij}^2 \geq \sum f_{ii}^2 \geq \frac{(\sum f_{ii})^2}{n} = \frac{(\triangle f)^2}{n} = \frac{(|\nabla f|^2 + af - f_t)^2}{n},$$

and

$$\alpha a t\triangle f \leq \frac{(\alpha a)^2}{\beta}t + \frac{\beta}{4}t(\triangle f)^2,$$

so

$$
\begin{aligned}
0 \geq &\frac{(2\chi - \frac{5}{4}\beta)t^2}{n}(|\nabla f|^2 + af - f_t)^2 - 2t^2|\nabla\chi||\nabla f|^3 + \\
&t^2(\triangle\chi - 2K\chi - 2a\chi + 2\alpha a - \frac{4}{\beta}|\nabla\chi|^2 - \varepsilon)|\nabla f|^2 + \\
&\varepsilon t^2(\triangle\chi - |\nabla\chi|^2) - F - \frac{(\alpha a)^2}{\beta}t^2.
\end{aligned}
\tag{3.1}
$$

Claim 2

$$(|\nabla f|^2 + af - f_t)^2 \geq \frac{(1-\beta)(\alpha - (1+H)^2)^2}{(\alpha - 1)^2(1+H)^4}(\chi(|\nabla f|^2 + \varepsilon) + af - f_t) - \frac{2\varepsilon^2}{\beta}.$$

In fact, as in [8], using the inequality $a_0^2 \geq (1-\beta)(a_0 + b_0)^2 - \frac{2}{\beta}b_0^2$, we conclude that

$$(|\nabla f|^2 + af - f_t)^2 \geq (1-\beta)(|\nabla f|^2 + \varepsilon + af - f_t)^2 - \frac{2\varepsilon^2}{\beta}.$$

On the other hand, $F \geq 0$ at (p, t_0), hence $\chi(|\nabla f|^2 + \varepsilon) + \alpha a f - \alpha f_t \geq 0$, that is,

$$f_t - af \leq \frac{\chi}{\alpha}(|\nabla f|^2 + \varepsilon). \tag{3.2}$$

Therefore

$$
\begin{aligned}
&(1-\beta)(|\nabla f|^2 + \varepsilon + af - f_t)^2 - \frac{(1-\beta)(\alpha - (1+H)^2)^2}{(\alpha - 1)^2(1+H)^4}(\chi(|\nabla f|^2 + \varepsilon) + af - f_t)^2 \\
&= (1-\beta)\Big[(|\nabla f|^2 + \varepsilon + af - f_t) + \frac{(\alpha - (1+H)^2)}{(\alpha - 1)(1+H)^2}(\chi(|\nabla f|^2 + \varepsilon) + af - f_t)\Big] \times \\
&\Big[(|\nabla f|^2 + \varepsilon + af - f_t) - \frac{(\alpha - (1+H)^2)}{(\alpha - 1)(1+H)^2}(\chi(|\nabla f|^2 + \varepsilon) + af - f_t)\Big].
\end{aligned}
$$

Using (3.2) and $\alpha > (1+H)^2 \geq \chi \geq 1$, we get,

$$(|\nabla f|^2 + \varepsilon + af - f_t) + \frac{(\alpha - (1+H)^2)}{(\alpha - 1)(1+H)^2}(\chi(|\nabla f|^2 + \varepsilon) + af - f_t) \geq 0,$$

and

$$(|\nabla f|^2 + \varepsilon + af - f_t) - \frac{(\alpha - (1+H)^2)}{(\alpha - 1)(1+H)^2}(\chi(|\nabla f|^2 + \varepsilon) + af - f_t)$$

$$\begin{aligned}
&= \frac{1}{(\alpha-1)(1+H)^2}\big[(|\nabla f|^2+\varepsilon+af-f_t)(\alpha-1)(1+H)^2-(\alpha-(1+H)^2)\times \\
&\quad (\chi(|\nabla f|^2+\varepsilon)+af-f_t)\big] \\
&= \frac{1}{(\alpha-1)(1+H)^2}\Big\{[(\alpha-1)(1+H)^2-(1-(1+H)^2)\chi](|\nabla f|^2+\varepsilon)+ \\
&\quad [(\alpha-1)(1+H)^2-(\alpha-(1+H)^2)](af-f_t)\Big\} \\
&= \frac{1}{(\alpha-1)(1+H)^2}\Big\{[(\chi-1)(1+H)^2+\alpha(1+H)^2-\chi](|\nabla f|^2+\varepsilon)+ \\
&\quad \alpha((1+H)^2-1)(af-f_t)\Big\} \\
&\geq \frac{1}{(\alpha-1)(1+H)^2}\Big\{[(\chi-1)(1+H)^2+\alpha(1+H)^2-\chi]- \\
&\quad ((1+H)^2-1)\chi\Big\}(|\nabla f|^2+\varepsilon) \\
&= |\nabla f|^2+\varepsilon \geq 0.
\end{aligned}$$

From above two inequalities, Claim 2 is verified.

Using Claim 2 and (3.1), we obtain

$$\begin{aligned}
0 \geq & \frac{(2\chi-\frac{5}{4}\beta)t^2}{n}\frac{(1-\beta)(\alpha-(1+H)^2)^2}{(\alpha-1)^2(1+H)^4}\big(\chi(|\nabla f|^2+\varepsilon)+af-f_t\big)- \\
& 2t^2|\nabla\chi||\nabla f|^3+t^2(\triangle\chi-2K\chi-2a\chi+2\alpha a-\frac{4}{\beta}|\nabla\chi|^2-\varepsilon)|\nabla f|^2+ \\
& \varepsilon t^2(\triangle\chi-|\nabla\chi|^2-\frac{2\varepsilon(2\chi-\frac{5}{4}\beta)}{n\beta})-F-\frac{(\alpha a)^2}{\beta}t^2. \qquad (3.3)
\end{aligned}$$

Let $y=\chi(|\nabla f|^2+\varepsilon)$ and $z=f_t-af$. Then

$$\begin{aligned}
(y-z) =&[\frac{1}{\alpha}(y-\alpha z)+\frac{\alpha-1}{\alpha}y]=\frac{1}{\alpha^2}(y-\alpha z)^2+(\frac{\alpha-1}{\alpha})^2y^2+\frac{2(\alpha-1)}{\alpha^2}y(y-\alpha z) \\
\geq&\frac{1}{\alpha^2t^2}F^2+(\frac{\alpha-1}{\alpha})^2y^2, \qquad (3.4)
\end{aligned}$$

where in the last line, we have used $y-\alpha z=\frac{F}{t}>0$.

Combining (3.3) and (3.4), we get

$$\begin{aligned}
0 \geq & \frac{(2\chi-\frac{5}{4}\beta)(1-\beta)(\alpha-(1+H)^2)^2}{n\alpha^2(\alpha-1)^2(1+H)^4}F^2-F+ \\
& \frac{(2\chi-\frac{5}{4}\beta)(1-\beta)t^2(\alpha-(1+H)^2)^2}{n\alpha^2(1+H)^4}y^2-2t^2|\nabla\chi||\nabla f|^3+ \\
& t^2(\triangle\chi-2K\chi-2a\chi+2\alpha a-\frac{4}{\beta}|\nabla\chi|^2-\varepsilon)|\nabla f|^2+ \\
& \varepsilon t^2(\triangle\chi-|\nabla\chi|^2-\frac{2\varepsilon(2\chi-\frac{5}{4}\beta)}{n\beta})-\frac{(\alpha a)^2}{\beta}t^2. \qquad (3.5)
\end{aligned}$$

To compute $\triangle\phi$, as in [9] (or [8]), we let $\partial M(R)=\{x\in M|\ r(x)<R\}$ and K_R be the upper bound of the sectional curvature in $\partial M(R)$. We choose R satisfying the following two inequalities,

$$\sqrt{K_R}\tan(R\sqrt{K_R})\leq\frac{H}{2}+\frac{1}{2},$$

and

$$\frac{H}{\sqrt{K_R}}\tan(R\sqrt{K_R}) \le \frac{1}{2},$$

then

$$\triangle\phi \ge -\frac{2(n-1)H(3H+1)}{R} - \frac{H}{R^2}.$$

Therefore,

$$\begin{aligned}\triangle\chi &= 2(1+\phi)\triangle\phi + 2|\nabla\phi|^2 \\ &\ge 2(1+H)(-\frac{2(n-1)H(3H+1)}{R} - \frac{H}{R^2}) = -C_3.\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{(2\chi-\frac{5}{4}\beta)(1-\beta)(\alpha-(1+H)^2)^2}{n\alpha^2(1+H)^4}y^2 - 2|\nabla\chi||\nabla f|^3 + \\ &\quad(\triangle\chi - 2K\chi - 2a\chi + 2\alpha a - \frac{4}{\beta}|\nabla\chi|^2 - \varepsilon)|\nabla f|^2 \\ &\ge \frac{(\alpha-(1+H)^2)^2}{2n\alpha^2(1+H)^4}y^2 - 8H(1+H)y^{\frac{3}{2}} + \Big[2\alpha a - 2a\chi - \big(C_3 + 2K(1+H)^2 + \\ &\quad\frac{64}{\beta}H^2(1+H)^2 + \varepsilon\big)\Big]y. \qquad (3.6)\end{aligned}$$

As in [8], consider $Ay^2 - By^{\frac{3}{2}} - Cy$, where A is positive constant. Using the inequality $2a_0b_0 \le a_0^2 + b_0^2$, clearly we get

$$\begin{aligned}Ay^2 - By^{\frac{3}{2}} - Cy &= \frac{A}{2}y^2 + \frac{A}{2}y^2 - By^{\frac{3}{2}} + \frac{B^2}{2A}y - (C + \frac{B^2}{2A})y \\ &\ge \frac{A}{2}y^2 - (C+\frac{B^2}{2A})y = \frac{A}{2}y^2 - (C+\frac{B^2}{2A})y + D^2 - D^2 \\ &\ge -D^2, \qquad (3.7)\end{aligned}$$

where $D^2 = \frac{(C+\frac{B^2}{2A})^2}{2A}$.

Applying (3.7) to (3.6), we conclude from (3.5) that

$$\begin{aligned}0 \ge &\frac{(2\chi-\frac{5}{4}\beta)(1-\beta)(\alpha-(1+H)^2)^2}{n\alpha^2(\alpha-1)^2(1+H)^4}F^2 - F + \\ &\varepsilon t^2(\triangle\chi - |\nabla\chi|^2 - \frac{2\varepsilon(2\chi-\frac{5}{4}\beta)}{n\beta}) - (D^2 + \frac{(\alpha a)^2}{\beta})t^2, \qquad (3.8)\end{aligned}$$

where $D^2 = \frac{(C+\frac{B^2}{2A})^2}{2A}$, and $A = \frac{(\alpha-(1+H)^2)^2}{2n\alpha^2(1+H)^4}$, $B = 8H(1+H)$ and

$$C = \begin{cases} 2\alpha a - 2a - (C_3 + 2K(1+H)^2 + \frac{64}{\beta}H^2(1+H)^2 + \varepsilon) & \text{if} \quad a \le 0; \\ 2\alpha a - 2a(1+H)^2 - (C_3 + 2K(1+H)^2 + \frac{64}{\beta}H^2(1+H)^2 + \varepsilon) & \text{if} \quad a > 0. \end{cases}$$

That is

$$D = \frac{\sqrt{n}\alpha(1+H)^2}{\alpha-(1+H)^2}(C + \frac{64n\alpha^2H^2(1+H)^6}{(\alpha-(1+H)^2)^2}).$$

From (3.8), we get

$$F \le \frac{1+\sqrt{1+4PQ}}{2P}, \qquad (3.9)$$

where

$$P=\frac{(2\chi-\frac{5}{4}\beta)(1-\beta)(\alpha-(1+H)^2)^2}{n\alpha^2(\alpha-1)^2(1+H)^4},$$

and

$$Q=(D^2+\frac{(\alpha a)^2}{\beta})t^2-\varepsilon t^2(\triangle\chi-|\nabla\chi|^2-\frac{2\varepsilon(2\chi-\frac{5}{4}\beta)}{n\beta}).$$

But

$$|\nabla f|^2+\alpha af-\alpha f_t\le\chi(|\nabla f|^2+\varepsilon)+\alpha af-\alpha f_t=\frac{F}{t}. \tag{3.10}$$

Thus by (3.9), (3.10) and letting $\varepsilon\to 0$, we have

$$\begin{aligned}|\nabla f|^2+\alpha af-\alpha f_t\le& n\alpha^2(\alpha-1)^2(1+H)^4\times\\ &\frac{1+\sqrt{1+8(\tilde{D}^2+\frac{(\alpha a)^2}{\beta})t^2(\alpha-(1+H)^2)^2/n\alpha^2(\alpha-1)^2(1+H)^2}}{2(2\chi-\frac{5}{4}\beta)(1-\beta)(\alpha-(1+H)^2)^2t}\\ \le&\frac{n\alpha^2(\alpha-1)^2(1+H)^4(2+\frac{2\sqrt{2}(|\tilde{D}|+\frac{\alpha|a|}{\beta})t(\alpha-(1+H)^2)}{\sqrt{n}\alpha(\alpha-1)(1+H)})}{2(2\chi-\frac{5}{4}\beta)(1-\beta)(\alpha-(1+H)^2)^2t},\end{aligned}$$

where if $a\le 0$, we choose

$$\tilde{D}=\frac{\sqrt{n}\alpha(1+H)^2}{\alpha-(1+H)^2}\{2\alpha a-2a-(C_3+2K(1+H)^2+\frac{64}{\beta}H^2(1+H)^2)+\frac{64n\alpha^2H^2(1+H)^6}{(\alpha-(1+H)^2)^2}\},$$

and if $a>0$, we choose

$$\begin{aligned}\tilde{D}=&\frac{\sqrt{n}\alpha(1+H)^2}{\alpha-(1+H)^2}\{2\alpha a-2a(1+H)^2-(C_3+2K(1+H)^2+\frac{64}{\beta}H^2(1+H)^2)+\\ &\frac{64n\alpha^2H^2(1+H)^6}{(\alpha-(1+H)^2)^2}\}.\end{aligned}$$

In conclusion, $|\nabla f|^2+\alpha af-\alpha f_t\le\frac{C_1}{t}+C_2$, where

$$\begin{aligned}C_1=&\frac{n\alpha^2(\alpha-1)^2(1+H)^4}{(2\chi-\frac{5}{4}\beta)(1-\beta)(\alpha-(1+H)^2)^2}\\ \le&\frac{n\alpha^2(\alpha-1)^2(1+H)^4}{(2-\frac{5}{4}\beta)(1-\beta)(\alpha-(1+H)^2)^2},\end{aligned}$$

and

$$C_2\le\frac{4\sqrt{2}n\alpha^2(\alpha-1)(1+H)^7K}{(\alpha-(1+H^2)^2)^2}+D_0.$$

We choose

$$D_0=\frac{\sqrt{2n}\alpha(\alpha-1)(1+H)^3(|\hat{D}|+2\alpha|a|)}{(\alpha-(1+H)^2)},$$

where if $a\le 0$, we choose

$$\hat{D}=\frac{\sqrt{n}\alpha(1+H)^2}{\alpha-(1+H)^2}\{2\alpha a-2a-(C_3+\frac{64}{\beta}H^2(1+H)^2)+\frac{64n\alpha^2H^2(1+H)^6}{(\alpha-(1+H)^2)^2}\},$$

and if $a>0$, we choose

$$\hat{D}=\frac{\sqrt{n}\alpha(1+H)^2}{\alpha-(1+H)^2}\{2\alpha a-2a(1+H)^2-(C_3+\frac{64}{\beta}H^2(1+H)^2)+$$

$$\frac{64n\alpha^2H^2(1+H)^6}{(\alpha-(1+H)^2)^2}\},$$

where

$$C_3 = 2(1+H)(\frac{2(n-1)H(3H+1)}{R} + \frac{H}{R^2}).$$

Then we complete the proof of Theorem 1.5. □

Using Theorem 1.5, we get a corollary,

Corollary 3.1 *Let M and u be as in Theorem 1.5. If the boundary ∂M of M is convex, i.e., $H = 0$, then for $\alpha > 1$ we have*

$$|\nabla f|^2 + \alpha a f - \alpha f_t \le \frac{C_4}{t} + C_5,$$

where $C_4 = \frac{n\alpha^2}{2}$ and $C_5 = \frac{2\sqrt{2}n\alpha^2K}{\alpha-1} + 2\sqrt{2}\alpha^2|a|(n+\sqrt{n})$. If furthermore the Ricci curvature of M is also nonnegative, then

$$|\nabla f|^2 + af - f_t \le \frac{n}{2t} + 2\sqrt{2}|a|(n+\sqrt{n}),$$

where $a \in \mathbf{R}$.

Remark 3.1 Comparing Corollary 3.1 with Theorem 1.2, Corollary 3.1 may be improved. But now we have not found the method to improve it.

In fact, choose $H = 0$ and let β approach to 0 in Theorem 1.5, we can get

$$|\nabla f|^2 + \alpha a f - \alpha f_t \le \frac{C_4}{t} + C_5,$$

where $C_4 = \frac{n\alpha^2}{2}$ and $C_5 = \frac{2\sqrt{2}n\alpha^2K}{\alpha-1} + 2\sqrt{2}\alpha^2|a|(n+\sqrt{n})$.

Furthermore, letting $K = 0$ and $\alpha \to 1$, we can get

$$|\nabla f|^2 + af - f_t \le \frac{n}{2t} + 2\sqrt{2}|a|(n+\sqrt{n}),$$

where $a \in \mathbf{R}$.

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