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The Almost Split Sequences And $\mathcal{D}$-Split Sequences of $T_2(T)$

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Abstract The AR-quiver and derived equivalence are two important subjects in the representation theory of finite dimensional algebras, and for them there are two important research tools-AR-sequences and $\mathcal{D}$-split sequences. So in order to study the representations of triangular matrix algebra $T_2(T)=\begin{pmatrix} T & 0\\ T & T\end{pmatrix}$ where T is a finite dimensional algebra over a field, it is important to determine its AR-sequences and $\mathcal{D}$-split sequences. The aim of this paper is to construct the right(left) almost split morphisms, irreducible morphisms, almost split sequences and $\mathcal{D}$-split sequences of $T_2(T)$ through the corresponding morphisms and sequences of T. Some interesting results are obtained.

Keywords algebras; modules; triangular matrix algebras; AR sequences; approximations.

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1. Introduction and preliminaries

The representation theory of algebras is one of the main branches of mathematics, and the almost split sequence plays an important role in it. A special quiver of a finite dimensional algebra-AR-quiver was constructed through the almost split sequences and irreducible morphisms in [1]. It gives a good description of the finitely generated module category of the algebra, and now becomes a main research tool of the representation theory of algebras. The triangular matrix algebra is a new algebra which is introduced in the study of the decomposition of algebra and direct sum of two rings. The one-point extension is a special case of the triangular matrix algebra. In paper [2], the triangular matrix algebra of rank two was extended to the one of rank n, and it is obtained that there is an equivalent relation between the morphism category and the module category of the corresponding triangular matrix algebra. Furthermore, the relations between its projective modules, injective modules and monomorphism category, epimorphism category are determined. Now, we recall some basic definitions and given results required in the paper. All rings in this paper are artinian algebras and all modules are assumed to be finitely generated.

Let T and U be rings and ${}_UM_T$ a U-T-bimodule. By Λ we denote the triangular matrix algebra $\begin{pmatrix} T & 0\\ {}_UM_T & U\end{pmatrix}$. The module over Λ is in the form of the triple $({}_TA,{}_UB,f)$ with f :

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$M \otimes_T A \to B$ a U-morphism. In particular, let $T_2(T) = \begin{pmatrix} T & 0 \\ T & T \end{pmatrix}$. The $T_2(T)$-modules can be described by triples $({}_T A, {}_T B, f)$ with $f : A \to B$ a T-morphism [1]. We can determine its right(left) almost split and irreducible morphisms and then construct its AR-quiver. The definitions and theorems on the right(left) almost split morphism and irreducible morphism are introduced in [1].

Theorem 1.1 ([1]) *The following are equivalent for a morphism $f : B \to C$.*

(a) The morphism $f : B \to C$ is right almost split.

(b) The morphism f is not a split epimorphism, the module C is indecomposable and if X is an indecomposable module not isomorphic to C, then every morphism $g : X \to C$ factors through f.

Theorem 1.2 ([1]) *The following are equivalent for a morphism $g : A \to B$.*

(a) The morphism $g : A \to B$ is left almost split.

(b) The morphism g is not a split monomorphism, the module A is indecomposable and if Y is an indecomposable module not isomorphic to A, then every nonisomorphism $A \to Y$ factors through g.

Applying the above theorems, the right(left) almost split morphisms and irreducible morphisms of T, we can determine the corresponding morphisms of $T_2(T)$, and its AR-quiver furthermore.

Derived equivalence is another important subject in the modern representation theory. It preserves many significant invariants of groups and algebras such as the number of irreducible representations, Cartan determinants, Hochschild cohomology groups, algebraic K-theory and G-theory and so on. The Morita theory for derived categories was established through the tilting complex by Rickard in [3]. It gives a concrete description of derived equivalence. In [4], the $\mathcal{D}$-split sequence was introduced to study the derived equivalence. Let $\mathcal{C}$ be an additive category, and let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$, and X an object in $\mathcal{C}$. A morphism $f : D \to X$ in $\mathcal{C}$ is called a right $\mathcal{D}$-approximation of X if $D \in \mathcal{D}$ and the induced map $\mathrm{Hom}_{\mathcal{C}}(\_, f) : \mathrm{Hom}(D', D) \to \mathrm{Hom}_{\mathcal{C}}(D', X)$ is surjective for every object $D' \in \mathcal{D}$. Dually, there is the notion of a left $\mathcal{D}$-approximation. We will use them to construct the $\mathcal{D}$-split sequence.

Definition 1.3 ([5]) *Let $\mathcal{C}$ be an additive category and $\mathcal{D}$ a full subcategory of $\mathcal{C}$. A sequence*

$$X \xrightarrow{f} M' \xrightarrow{g} Y$$

in $\mathcal{C}$ is called a $\mathcal{D}$-split sequence if

(1) $M \in \mathcal{D}$;

(2) f is a left $\mathcal{D}$-approximation of X, and g is a right $\mathcal{D}$-approximation of Y;

3) f is a kernel of g, and g is a cokernel of f.

Theorem 1.4 ([4]) *Let $\mathcal{C}$ be an additive category and M an object in $\mathcal{C}$. Suppose*

$$X \xrightarrow{f} M' \xrightarrow{g} Y$$

is an add M*-split sequence in* $\mathcal{C}$*, then the endomorphism ring* $\mathrm{End}_{\mathcal{C}}(M \oplus X)$ *of* $M \oplus X$ *and the endomorphism ring* $\mathrm{End}_{\mathcal{C}}(M \oplus Y)$ *of* $M \oplus Y$ *are derived-equivalent.*

There are some definitons and theorems about the almost split sequence and irreducible morphism in [1]. In this paper, we will use them to construct the right(left) almost split morphisms, irreducible morphisms, almost split sequences and $\mathcal{D}$-split sequences of $T_2(T)$ through the corresponding morphisms and sequences of T.

2. The almost split morphisms and irreducible morphisms of triangular matrix algebras

In this section, we will construct the right(left) almost split morphisms of a triangular matrix algebra from the ones of the corresponding algebras. For the special case $T_2(T)$, we will have more results. According to right(left) almost split morphism for indecomposable projective(injective) modules, it is easy to get the following facts:

Fact 2.1 *For* $\Lambda = \begin{pmatrix} T & 0 \\ {}_UM_T & U \end{pmatrix}$*, we have the following*

(1) If P *is an indecomposable projective* U*-module, and* rP *is the radical of* P*,* $i : rP \to P$ *is the embedding morphism, then* $(0, i) : (0, rP, 0) \to (0, P, 0)$ *is a right almost split morphism in* $\mathrm{mod}\,\Lambda$.

(2) If I *is an indecomposable injective* T*-module, and* $j : I \to I/\mathrm{soc}\, I$ *is the natural epimorphism, then* $(j, 0) : (I, 0, 0) \to (I/\mathrm{soc}\, I, 0, 0)$ *is a left almost split morphism in* $\mathrm{mod}\,\Lambda$.

(3) If P' *is an indecomposable projective* T*-module, then* $(i, Id) : (rP', M \otimes P', i_{M\otimes rP'}) \to (P', M \otimes P', 1_{M\otimes P'})$ *is a right almost split morphism in* $\mathrm{mod}\,\Lambda$.

(4) If I' *is an indecomposable injective* U*-module, then* $(Id, j) : (\mathrm{Hom}_U(M, I'), I', \phi) \to (\mathrm{Hom}_U(M, I'), I'/\mathrm{soc}\, I', \phi)$ *is a left almost split morphism in* $\mathrm{mod}\,\Lambda$*, where* $\phi : M \otimes \mathrm{Hom}_U(M, I') \to I'$ *is given by* $\phi(m \otimes f) = f(m)$ *for* $m \in M$ *and* $f \in \mathrm{Hom}_U(M, I')$.

Corollary 2.2 *For* $T_2(T) = \begin{pmatrix} T & 0 \\ T & T \end{pmatrix}$*, we have the following.*

(1) If P *is an indecomposable projective* T*-module, then morphisms* $(0, i) : (0, rP, 0) \to (0, P, 0)$ *and* $(i, Id) : (rP, P, i) \to (P, P, 1_P)$ *are right almost split in* $\mathrm{mod}\, T_2(T)$.

(2) If I *is an indecomposable injective* T*-module, then morphisms* $(j, 0) : (I, 0, 0) \to (I/\mathrm{soc}\, I, 0, 0)$ *and* $(Id, j) : (I, I, 1_I) \to (I, I/\mathrm{soc}\, I, j)$ *are left almost split in* $\mathrm{mod}\, T_2(T)$.

Fact 2.3 *For* $\Lambda = \begin{pmatrix} T & 0 \\ {}_UM_T & U \end{pmatrix}$*, we have the following*

(1) Let A *and* B *be indecomposable* U*-modules. If the monomorphism* $f : A \to B$ *is a (minimal) right almost split morphism, then* $(0, f) : (0, A, 0) \to (0, B, 0)$ *is a (minimal) right almost split morphism in* $\mathrm{mod}\,\Lambda$.

(2) Let A *and* B *be indecomposable* T*-modules. If the epimorphism* $g : A \to B$ *is a (minimal) left almost split morphism, then* $(g, 0) : (A, 0, 0) \to (B, 0, 0)$ *is a (minimal) left almost split morphism in* $\mathrm{mod}\,\Lambda$.

Corollary 2.4 *For $T_2(T)=\begin{pmatrix} T & 0 \\ T & T \end{pmatrix}$, let A and B be indecomposable T-modules. Then we have the following*

(1) If the monomorphism $f: A \to B$ is a (minimal) right almost split morphism, then $(0,f):(0,A,0)\to(0,B,0)$ is a (minimal) right almost split morphism in $\operatorname{mod} T_2(T)$.

(2) If the epimorphism $g: A \to B$ is a (minimal) left almost split morphism, then $(g,0):(A,0,0)\to(B,0,0)$ is a (minimal) left almost split morphism in $\operatorname{mod} T_2(T)$.

Theorem 2.5 *For $T_2(T)$, we have the following.*

(1) If $f: A\to B$ is not a split monomorphism, and there exists $g: B\to A$ such that $gf=1_A$, then $(1_A,f):(A,A,1_A)\to(A,B,f)$ is a left almost split morphism in $\operatorname{mod} T_2(T)$.

(2) If $g: A\to B$ is not a split epimorphism, and there exists $g: B\to A$ such that $gf=1_A$, then $(g,1_A):(B,A,g)\to(A,A,1_A)$ is a right almost split morphism in $\operatorname{mod} T_2(T)$.

Proof We only prove (1); the proof of (2) is similar.

Since f is not split, we know that $(1_A,f)$ is not a split monomorphism. For any indecomposable $T_2(T)$-module (C_1,C_2,k) and nonisomorphism $(g_1,g_2):(A,A,1_A)\to(C_1,C_2,k)$, it follows that there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "g_1"] \arrow[d, "1_A"'] & C_1 \arrow[d, "k"] \\ A \arrow[r, "g_2"] & C_2 \end{tikzcd}.$$

Thus, $g_2=kg_1$. Now, let $(h_1,h_2)=(g_1,g_2g)$. It is easy to know (h_1,h_2) is a $T_2(T)$-morphism. And also $(h_1,h_2)(1_A,f)=(g_1,g_2gf)=(g_1,g_2)$. Consequently, (g_1,g_2) factors through $(1_A,f)$.

In conclusion, $(1_A,f)$ is a left almost split morphism. $\square$

The irreducible morphism is a main tool when we construct the AR-quiver of a finite dimensional algebra. The arrows between two vertices are determined by the irreducible morphism in an AR-quiver. So, we will study the irreducible morphism of the triangular matrix algebra in the following.

Theorem 2.6 *For Λ, we have the following.*

(1) If the monomorphism $f: A\to B$ is an irreducible morphism in $\operatorname{mod} U$, *then $(0,f):(0,A,0)\to(0,B,0)$ is irreducible in* $\operatorname{mod}\Lambda$.

(2) If the epimorphism $f: A\to B$ is an irreducible morphism in $\operatorname{mod} T$, *then $(f,0):(A,0,0)\to(B,0,0)$ is irreducible in* $\operatorname{mod}\Lambda$.

Proof (1) We will prove the theorem in two steps by the definition of irreducible morphism.

1°. As f is neither a split monomorphism nor a split epimorphism, it is easy to see that the same conclusion is true for $(0,f)$.

2°. If there exist a Λ-module (X_1,X_2,h) and a Λ-morphism $(0,s):(0,A,0)\to(X_1,X_2,h)$ as well as $(0,t):(X_1,X_2,h)\to(0,B,0)$ such that $(0,f)=(0,t)(0,s)$, then we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r, "0"] \arrow[d, "0"'] & M \otimes X_1 \arrow[r, "0"] \arrow[d, "h"] & 0 \arrow[d, "0"] \\ A \arrow[r, "s"] & X_2 \arrow[r, "t"] & B \end{tikzcd} .$$

Thus, $ts = f$ and $th = 0$. Since f is irreducible, we know that t is a split epimorphism or s is a split monomorphism.

(a) If t is a split epimorphism, then there exists a U-module B' such that $X_2 \cong B \oplus B'$ where $B' \cong \ker t$. We have that $\operatorname{Im} h \subseteq \ker t \cong B'$ since $th = 0$. And so $(X_1, X_2, h) \cong (X_1, B', h) \oplus (0, B, 0)$. Hence, $(0, t)$ is a split epimorphism.

(b) If s is a split monomorphism, then there exists a U-module A' such that $X_2 \cong A \oplus A'$ where $A \cong \operatorname{Im} s$. Since $f = st$ is a monomorphism and $th = 0$, we have that $\operatorname{Im} h \subseteq \ker t \subseteq A'$. This means that $(X_1, X_2, h) \cong (X_1, A', h) \oplus (0, A, 0)$. Hence, $(0, s)$ is a split monomorphism.

In conclusion, $(0, f)$ is irreducible.

(2) Similarly to the proof of (1), there is a commutative diagram

$$\begin{tikzcd} M \otimes A \arrow[r, "1 \otimes s"] \arrow[d, "0"'] & M \otimes X_1 \arrow[r, "1 \otimes t"] \arrow[d, "h"] & M \otimes B \arrow[d, "0"] \\ 0 \arrow[r, "0"] & X_2 \arrow[r, "0"] & 0 \end{tikzcd} \tag{1}$$

Hence, $h(1 \otimes s) = 0$ and $(1 \otimes t)(1 \otimes s) = 1 \otimes f$. Thus, $f = ts$. Since f is irreducible, t is a split epimorphism or s is a split monomorphism.

(a) If t is a split epimorphism, then there exists a T-module B' such that $X_1 \cong B \oplus B'$ where $B' \cong \ker t$. Since f is an epimorphism and $h(1 \otimes s) = 0$, we have that $M \otimes B \subseteq \operatorname{Im}(1 \otimes s) \subseteq \ker h$. Thus, we have that $(X_1, X_2, h) \cong (B, 0, 0) \oplus (B', X_2, h)$. This means that $(0, t)$ is a split epimorphism.

(b) If s is a split monomorphism, then there exists a T-module A' such that $X_1 \cong A \oplus A'$ where $A \cong \operatorname{Im} s$. We have that $M \otimes A \subseteq \ker h$ since $\operatorname{Im}(1 \otimes s) \subseteq \ker h$. Thus, $(X_1, X_2, h) \cong (A, 0, 0) \oplus (A', X_2, h)$. This means that $(0, s)$ is a split monomorphism.

In conclusion, $(f, 0)$ is irreducible. □

Corollary 2.7 *For $T_2(T)$, we have the following.*

(1) If a monomorphism $f : A \to B$ is irreducible, then $(0, f) : (0, A, 0) \to (0, B, 0)$ is irreducible in $\operatorname{mod} T_2(T)$.

(2) If an epimorphism $f : A \to B$ is irreducible, then $(f, 0) : (A, 0, 0) \to (B, 0, 0)$ is irreducible in $\operatorname{mod} T_2(T)$.

Theorem 2.8 *For the algebra $T_2(T)$, we have the following.*

(1) If a monomorphism $f : A \to B$ is irreducible, then $(1_A, f) : (A, A, 1_A) \to (A, B, f)$ is irreducible in $\operatorname{mod} T_2(T)$.

(2) If $f : A \to B$ and $g : B \to A$ are irreducible in $\operatorname{mod} T$*, then $(f, 0) : (A, 0, 0) \to (B, A, g)$ and $(0, g) : (A, B, f) \to (0, A, 0)$ are irreducible in* $\operatorname{mod} T_2(T)$.

Proof (1) Similarly to the proof of Theorem 2.6, there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "s_1"] \arrow[d, "1_A"'] & X_1 \arrow[r, "t_1"] \arrow[d, "h"] & A \arrow[d, "f"] \\ A \arrow[r, "s_2"] & X_2 \arrow[r, "t_2"] & B \end{tikzcd} \tag{2}$$

Thus, $hs_1 = s_2$, $t_1s_1 = 1_A$ and $t_2s_2 = f$. Hence, s_1 is a split monomorphism and t_1 is a split epimorphism. So, there exists a T-module A' such that $X_1 \cong A \oplus A'$ where $A' \cong \ker t_1$. Since f is irreducible, we know that t_2 is a split epimorphism or s_2 is a split monomorphism.

(a) If s_2 is a split monomorphism, then there exists a T-module A'' such that $X_2 \cong A \oplus A''$. From the above commutative diagram we know that $h(A) = A$ and $h|_A = 1_A$. Since $ft_1(A') = t_2h(A')$ and $f = t_2s_2$ is a monomorphism, we have that $h(A') \subseteq \ker t_2 \subseteq A''$. Now, we know that $(X_1, X_2, h) \cong (A, A, 1_A) \oplus (A', A'', h)$. This means that (s_1, s_2) is a split monomorphism.

(b) If t_2 is a split epimorphism, then there exists a T-module B' such that $X_2 \cong B \oplus B'$ where $B' \cong \ker t_2$. From the above commutative diagram we learn that $h(A) \subseteq B$ and $h|_A = f$, and so $h(A') \subseteq \ker t_2 \subseteq B'$. Now, we have that $(X_1, X_2, h) \cong (A, B, f) \oplus (A', B', h)$. This means that (t_1, t_2) is a split epimorphism.

In conclusion, $(1_A, f)$ is irreducible.

(2) Similarly to the proof of Theorem 2.6, there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "s_1"] \arrow[d, "0"'] & X_1 \arrow[r, "t_1"] \arrow[d, "h"] & B \arrow[d, "g"] \\ 0 \arrow[r, "0"] & X_2 \arrow[r, "t_2"] & A \end{tikzcd} \tag{3}$$

Thus, $hs_1 = 0$ and $t_1s_1 = f$. As f is irreducible, we have that t_2 is a split epimorphism or s_2 is a split monomorphism.

(a) If s_1 is a split monomorphism, then there exists a T-module A' such that $X_1 \cong A \oplus A'$. Now, we have that $h(A) = 0$ since $hs_1 = 0$. Thus, $(X_1, X_2, h) \cong (A, 0, 0) \oplus (A', X_2, h)$. This means that $(s_1, 0)$ is a split monomorphism.

(b) If t_2 is a split epimorphism, then there exists t_1' which is the right inverse of t_2, such that $t_1t_1' = 1$. So we have that $g = t_2ht_1'$. Since g is irreducible, we know that ht_1' is a split monomorphism or t_2 is a split epimorphism. If ht_1' is a split monomorphism, then there exist T-modules B' and B'' such that $X_1 \cong B \oplus B'$ and $X_2 \cong B \oplus B''$. We have that $hs_1(A) = 0$ because $hs_1 = 0$. Since ht_1' is a split monomorphism, it is easy to know that $s_1(A) \subseteq B'$. Now $t_1s_1 = 0$. This means that $f = 0$. It is a contradiction. Therefore, ht_1' is not a split monomorphism, and so t_2 is a split epimorphism. It follows that there exists a T-module A'' such that $X_2 \cong A \oplus A''$ where $A'' \cong \ker t_2$. From the above commutative diagram we learn that $h|_B = g$ and $t_2h(B') = 0$, i.e., $h(B') \subseteq A''$. Hence $(X_1, X_2, h) \cong (B, A, g) \oplus (B', A'', h)$. This means that (t_1, t_2) is a split epimorphism.

In conclusion, $(f, 0)$ is irreducible.

Similarly, one can prove that $(0, g)$ is irreducible. $\square$

3. The $\mathcal{D}$-split sequences

We can establish the derived equivalence of algebras from the $\operatorname{add} M$-split sequence [2]. So we try to construct the $\mathcal{D}$-split sequence from the corresponding sequence of T in this section. We denote $T_2(T)$ by Δ for the convenience of the statement in the following. It is easy to get the following fact from Theorem 1.4.

Fact 3.1 *If $0 \to X \xrightarrow{f} M' \xrightarrow{g} Y \to 0$ is an $\operatorname{add} M$-split sequence in $\operatorname{mod} T$, then $\operatorname{End}_\Delta((X \oplus M, 0, 0))$ and $\operatorname{End}_\Delta((Y \oplus M, 0, 0))$ are derived-equivalent.*

Theorem 3.2 *If $0 \to X \xrightarrow{f} M' \xrightarrow{g} Y \to 0$ is an $\operatorname{add} M$-split sequence in $\operatorname{mod} T$, then $\operatorname{End}_\Delta((M, M \oplus X, (1_M, 0)))$ and $\operatorname{End}_\Delta((M \oplus M', M \oplus Y, \begin{pmatrix} 1_M & 0 \\ 0 & g \end{pmatrix}))$ are derived-equivalent. Meanwhile, $\operatorname{End}_\Delta((M \oplus X, M \oplus M', \begin{pmatrix} 1_M & 0 \\ 0 & f \end{pmatrix}))$ and $\operatorname{End}_\Delta((M \oplus Y, M, (1_M, 0)))$ are derived-equivalent.*

Proof It follows from the definition of a submodule of the module over triangular matrix algebra $\Delta = T_2(T)$, we can choose $(N, N, 1_N) \in \operatorname{add}(M, M, 1_M)$ and Δ-morphism $(0, s) : (0, X, 0) \to (N, N, 1_N)$ without loss of generality. Then there exists $h : M' \to N$ such that $s = hf$ according to the fact that f is a left $\operatorname{add} M$-approximation of X. So, $(0, s) = (h, h)(0, f)$. This means that $(0, f)$ is a left $\operatorname{add}(M, M, 1_M)$-approximation of $(0, X, 0)$. Now, for any $(L, L, 1_L) \in \operatorname{add}(M, M, 1_M)$ and Δ-morphism $(t_1, t_2) : (L, L, 1_L) \to (M', Y, g)$ we learn that $t_2 = gt_1$ from the commutative diagram

$$\begin{tikzcd} L \arrow[r, "t_1"] \arrow[d, "1_L"'] & M' \arrow[d, "g"] \\ L \arrow[r, "t_2"'] & Y \end{tikzcd} .$$

Thus, $(1'_M, g)(t_1, t_1) = (t_1, gt_1) = (t_1, t_2)$. So, $(1_{M'}, g)$ is a right $\operatorname{add}(M, M, 1_M)$-approximation of (M', Y, g). Hence, $0 \to (0, X, 0) \xrightarrow{(0,f)} (M', M', 1_{M'}) \xrightarrow{(1_{M'},g)} (M', Y, g) \to 0$ is an $\operatorname{add}(M, M, 1_M)$-split sequence. By Theorem 1.4 $\operatorname{End}_\Delta((M, M\oplus X, (1_M, 0)))$ and $\operatorname{End}_\Delta((M\oplus M', M\oplus Y, \begin{pmatrix} 1_M & 0 \\ 0 & g \end{pmatrix}))$ are derived-equivalent.

Similarly, one can prove that $0 \to (X, M', f) \xrightarrow{(f,1_{M'})} (M', M', 1_{M'}) \xrightarrow{(g,0)} (Y, , 0, 0) \to 0$ is an $\operatorname{add}(M, M, 1_M)$-split sequence. By Theorem 1.4 we know that $\operatorname{End}_\Delta((M \oplus X, M \oplus M', \begin{pmatrix} 1_M & 0 \\ 0 & f \end{pmatrix}))$ and $\operatorname{End}_\Delta((M \oplus Y, M, (1_M, 0)))$ are derived-equivalent. $\square$

Theorem 3.3 *If $0 \to X \xrightarrow{f} M' \xrightarrow{g} Y \to 0$ is an $\operatorname{add} M$-split sequence in $\operatorname{mod} T$, then $\operatorname{End}_\Delta((X \oplus M, X \oplus M, \begin{pmatrix} 1_X & 0 \\ 0 & 1_M \end{pmatrix}))$ and $\operatorname{End}_\Delta((Y \oplus M, Y \oplus M, \begin{pmatrix} 1_Y & 0 \\ 0 & 1_M \end{pmatrix}))$ are derived-equivalent.*

Proof It is easy to know that $0 \to (X, X, 1_X) \xrightarrow{(f,f)} (M', M', 1_{M'}) \xrightarrow{(g,g)} (Y, Y, 1_Y) \to 0$ is an

$\mathrm{add}(M, M, 1_M)$-split sequence. Then by Theorem 1.4 we can get the derived equivalence in the theorem. □

By the easy calculation we know that $\mathrm{End}_\Delta((X\oplus M, X\oplus M, \begin{pmatrix} 1_X & 0 \\ 0 & 1_M \end{pmatrix})) \simeq \mathrm{Hom}_T(X, M)\oplus \mathrm{Hom}_T(M, X)$ and $\mathrm{End}_\Delta((Y \oplus M, Y \oplus M, \begin{pmatrix} 1_Y & 0 \\ 0 & 1_M \end{pmatrix})) \simeq \mathrm{Hom}_T(M, Y) \oplus \mathrm{Hom}_T(Y, M)$. Thus by Theorem 3.3, we know that $\mathrm{Hom}_T(X, M) \oplus \mathrm{Hom}_T(M, X)$ and $\mathrm{Hom}_T(M, Y) \oplus \mathrm{Hom}_T(Y, M)$ are derived equivalent.

Theorem 3.4 *(1) If $0 \to X \xrightarrow{f} M' \xrightarrow{g} Y \to 0$ is an $\mathrm{add}\, M$-split sequence in $\mathrm{mod}\, T$, then we have that $\mathrm{End}_\Delta((X \oplus M, A, (h, 0)))$ and $\mathrm{End}_\Delta((Y \oplus M, 0, 0))$ are derived-equivalent. Meanwhile, $\mathrm{End}_\Delta((X \oplus M, 0, 0))$ and $\mathrm{End}_\Delta((A, Y \oplus M, (k, 0)))$ are derived-equivalent for any T-module A, T-morphism $h : X \to A$ and $k : A \to Y$.*

(2) If $0 \to X_1 \xrightarrow{f_1} M' \xrightarrow{g_1} Y_1 \to 0$ and $0 \to X_2 \xrightarrow{f_2} M' \xrightarrow{g_2} Y_2 \to 0$ are $\mathrm{add}\, M$-split sequences in $\mathrm{mod}\, T$. $h : X_1 \to X_2$ and $k : Y_1 \to Y_2$ in $\mathrm{mod}\, T$ satisfy $f_1 = f_2h$ and $g_2 = kg_1$. Then $\mathrm{End}_\Delta((X_1 \oplus M, X_2 \oplus M, \begin{pmatrix} h & 0 \\ 0 & 1_M \end{pmatrix}))$ and $\mathrm{End}_\Delta((Y_1 \oplus M, Y_2 \oplus M, \begin{pmatrix} k & 0 \\ 0 & 1_M \end{pmatrix}))$ are derived-equivalent.

Proof (1) Similarly to the proof of Theorem 3.2, it is easy to know that $0 \to (X, A, h) \xrightarrow{(f,0)} (M', 0, 0) \xrightarrow{(g,0)} (Y, 0, 0) \to 0$ is an $\mathrm{add}(M, 0, 0)$-split sequence. At the same time $0 \to (0, X, 0) \xrightarrow{(0,f)} (0, M', 0) \xrightarrow{(0,g)} (A, Y, k) \to 0$ is an $\mathrm{add}(0, M, 0)$-split sequence. By Theorem 1.4 we can get the corresponding derived equivalences.

(2) Similarly to the proof of Theorem 3.2, we know that $0 \to (X_1, X_2, h) \xrightarrow{(f_1,f_2)} (M', M', 1_{M'}) \xrightarrow{(g_1,g_2)} (Y_1, Y_2, k) \to 0$ is the $\mathrm{add}(M, M, 1_M)$-split sequence. By Theorem 1.4 we can get the corresponding derived equivalence. □

Remark (1) Let $A = 0$ in Theorem 3.4(1). It is clear that Fact 3.1 is the corollary of Theorem 3.4(1).

(2) Let $h = 1_X$, $k = 1_Y$ in Theorem 3.4(2). Then Theorem 3.3 can be viewed as the corollary of Theorem 3.4(2).

Up to now we have constructed some irreducible morphisms, $\mathcal{D}$-split sequences in $\mathrm{mod}\, T_2(T)$. In the next section we will study the almost split sequence in $\mathrm{mod}\, T_2(T)$.

4. The almost split sequences in $\mathrm{mod}\, T_2(T)$

We constructed the right(left) almost split morphisms and irreducible morphisms in $\mathrm{mod}\, T_2(T)$ from the ones of T in section 1. We can construct the almost split sequence from them in this section.

Firstly we will introduce some properties of the module over $T_2(T)$. Let D be the duality $\mathrm{Hom}_\Delta(\_, k)$ and (A, B, f) in $\mathrm{mod}\, T_2(T)$. Then $D(A, B, f) = (DB, DA, D\phi(f)\varphi)$, with ϕ the adjoint isomorphism $\mathrm{Hom}_T(M \otimes_T A, B) \to \mathrm{Hom}_T(A, \mathrm{Hom}_T(M, B))$, and $\varphi : M \otimes_{T^{op}} DB \to D\mathrm{Hom}_T(M, B)$, given by $\psi(m \otimes g)(f) = gf(m)$. By $()^*$ we denote the morphism $\mathrm{Hom}_\Delta(\_, \Delta)$.

For an indecomposable projective $T_2(T)$-module $(P, P, 1_P)$, we have that $(P, P, 1_P)^* = (0, P^*, 0)$. When $X = (0, P, 0)$, we have that $(0, P, 0)^* = (P^*, P^*, 1_{P^*})$. In order to compute the AR-translation, we will investigate the minimal projective resolution in $\operatorname{mod} T_2(T)$.

Lemma 4.1 *Let A in $\operatorname{mod} T$, $\cdots \to P_1 \xrightarrow{d_0} P_0 \xrightarrow{\epsilon} A \to 0$ be a minimal projective resolution of A. Then*

(1) $\cdots \to (0, P_1, 0) \xrightarrow{(0,d_0)} (0, P_0, 0) \xrightarrow{(0,\epsilon)} (0, A, 0) \to 0$ *is a minimal projective resolution of* $(0, A, 0)$.

(2) $\cdots \to (P_1, P_1 \oplus P_0, (1, 0)) \xrightarrow{(d_0,\begin{pmatrix} d_0 \\ 1 \end{pmatrix})} (P_0, P_0, 1) \xrightarrow{(\epsilon,0)} (A, 0, 0) \to 0$ *is a minimal projective resolution of* $(A, 0, 0)$.

Proof (1) is obvious. We will prove (2).

Firstly, $(\epsilon, 0)(d_0, \begin{pmatrix} d_0 \\ 1 \end{pmatrix}) = 0$. Secondly, $\ker(\epsilon, 0) = (\ker\epsilon, P_0)$ and $\operatorname{Im}(d_0, \begin{pmatrix} d_0 \\ 1 \end{pmatrix}) = (\operatorname{Im}d_0, \operatorname{Im}d_0 + P_0) = (\ker\epsilon, P_0)$. It means that $(P_1, P_1 \oplus P_0, (1, 0)) \xrightarrow{(d_0,\begin{pmatrix} d_0 \\ 1 \end{pmatrix})} (P_0, P_0, 1) \xrightarrow{(\epsilon,0)} (A, 0, 0) \to 0$ is an exact sequence.

At the same time, $r(P_1, P_1 \oplus P_0, (1, 0)) = \begin{pmatrix} \operatorname{rad} T & 0 \\ T & \operatorname{rad} T \end{pmatrix} \begin{pmatrix} P_1 \\ P_1 \oplus P_0 \end{pmatrix} = (rP_1, P_1 \oplus rP_0, (i, 0))$. Also we know that $\ker(d_0, \begin{pmatrix} d_0 \\ 1 \end{pmatrix}) \subseteq (\ker d_0, P_1 \oplus \operatorname{Im} d_0, (1, 0)) \subseteq r(P_1, P_1 \oplus P_0, (1, 0))$.

In conclusion, the exact sequence in (2) is a minimal projective resolution of $(A, 0, 0)$. $\square$

Theorem 4.2 *(1) If A is an indecomposable nonprojective T-module, then $0 \to (DTrA, DTrA, 1) \to E \to (0, A, 0) \to 0$ is an almost split sequence in $\operatorname{mod} T_2(T)$, where E is a $T_2(T)$-module.*

(2) If A is an indecomposable noninjective T-module, then $0 \to (A, 0, 0) \to E \to (TrDA, TrDA, 1) \to 0$, is an almost split sequence, where E is a $T_2(T)$-module.

Proof Suppose A is an indecomposable nonprojective T-module, then $(0, A, 0)$ is an indecomposable nonprojective $T_2(T)$-module. According to Proposition 1.12 in [1, p. 142] there exists an almost split sequence $0 \to DTr(0, A, 0) \to E \to (0, A, 0) \to 0$. We compute the $DTr(0, A, 0)$ as follows.

(a) Suppose $P_1 \xrightarrow{d_0} P_0 \xrightarrow{\epsilon} A \to 0$ is a minimal projective presentation of A. Then $(0, P_1, 0) \xrightarrow{(0,d_0)} (0, P_0, 0) \xrightarrow{(0,\epsilon)} (0, A, 0) \to 0$ is a minimal projective presentation of $(0, A, 0)$.

(b) $(0, d_0)^* : (0, P_0, 0)^* \to (0, P_1, 0)^*$. By the introduction in the front of this section we know that $(0, P_0, 0)^* = (P_0^*, P_0^*, 1)$ and $(0, P_1, 0)^* = (P_1^*, P_1^*, 1)$. Hence $(0, d_0)^* = (d_0^*, d_0^*) : (P_1^*, P_1^*, 1) \to (P_0^*, P_0^*, 1)$. So, $Tr(0, A, 0) = \operatorname{Coker}(0, d_0)^* = \operatorname{Coker}(d_0^*, d_0^*) = (TrA, TrA, 1)$.

(c) From the foregoing discussion, it is clear that $DTr(0, A, 0) = D(TrA, TrA, 1) = (DTrA, DTrA, 1)$.

Therefore, by the above calculation and Proposition 1.12 in [1, p. 142] we have that $0 \to (DTrA, DTrA, 1) \to E \to (0, A, 0) \to 0$ is an almost split sequence in $\operatorname{mod} T_2(T)$.

(2) is the duality of (1). □

Theorem 4.3 *(1) If $f : A \to B$ is an irreducible monomorphism in* $\operatorname{mod} T$*, with A indecomposable, then $0 \to (0, A, 0) \to (0, B, 0) \oplus E \to (P_0^*, P_1^*, f) \to 0$ is an almost split sequence in* $\operatorname{mod} T_2(T)$*, where $\operatorname{Im} f = TrDA$, E is a $T_2(T)$-module, and $P_1 \to P_0 \to DA \to 0$ is a minimal projective presentation of DA.*

(2) If $g : B \to A$ is an irreducible epimorphism, with A indecomposable, then $0 \to (DP_1^*, DP_0^*, D\phi(f)\varphi) \to (B, 0, 0) \oplus E \to (A, 0, 0) \to 0$ is an almost split sequence in $\operatorname{mod} T_2(T)$*, where E is a $T_2(T)$-module, ϕ and φ are as the same isomorphisms as ones in the front of this section, and $P_1 \to P_0 \to A \to 0$ is a minimal projective presentation of A.*

Proof (1) By Corollary 2.7(1) we know that $(0, f) : (0, A, 0) \to (0, B, 0)$ is an irreducible morphism. Thus, there exists E in $\operatorname{mod} T_2(T)$ such that $(0, A, 0) \to (0, B, 0) \oplus E$ is a left almost split monomorphism. So, $0 \to (0, A, 0) \to (0, B, 0) \oplus E \to TrD(0, A, 0) \to 0$ is an almost split sequence in $\operatorname{mod} T_2(T)$. Next, we will compute $TrD(0, A, 0)$.

(a) We know that $D(0, A, 0) = (DA, 0, 0)$.

(b) Suppose $P_1 \xrightarrow{d_0} P_0 \xrightarrow{\epsilon} DA \to 0$ is a minimal projective presentation of DA, then according to Lemma 4.1 we have that $(P_1, P_1 \oplus P_0, (1, 0)) \xrightarrow{(d_0, \begin{pmatrix} d_0 \\ 1 \end{pmatrix})} (P_0, P_0, 1) \xrightarrow{(\epsilon, 0)} (DA, 0, 0) \to 0$ is a minimal projective presentation of $(DA, 0, 0)$.

(c) $(d_0, \begin{pmatrix} d_0 \\ 1 \end{pmatrix})^* : (P_0, P_0, 1)^* \to (P_1, P_1 \oplus P_0, (1, 0))^*$. For any $(h, h) \in (P_0, P_0, 1)^*$ we know that $(d_0, \begin{pmatrix} d_0 \\ 1 \end{pmatrix})^*(h, h) = (d_0 h, \begin{pmatrix} d_0 h & h \\ 0 & 0 \end{pmatrix})$. Hence, $TrD(0, A, 0) = Tr(DA, 0, 0) = \operatorname{Coker}(d_0, \begin{pmatrix} d_0 \\ 1 \end{pmatrix})^* \cong \{(h, k) \mid h \in P_0^*, k \in P_1^*, d_0 h \notin \operatorname{Im} d_0^*\} = (P_0^*, P_1^*, f)$, where $f(P_0^*) = TrDA$.

Therefore, by the computation above and Proposition 1.12 in [1, p. 142] we have that $0 \to (0, A, 0) \to (0, B, 0) \oplus E \to (P_0^*, P_1^*, f) \to 0$ is an almost split sequence in $\operatorname{mod} T_2(T)$.

(2) From Lemma 4.1 we learn that $(P_1, P_1 \oplus P_0, (1, 0)) \xrightarrow{(d_0, \begin{pmatrix} d_0 \\ 1 \end{pmatrix})} (P_0, P_0, 1) \xrightarrow{(\epsilon, 0)} (A, 0, 0) \to 0$ is a minimal projective presentation of $(A, 0, 0)$ if $P_1 \to P_0 \to A \to 0$ is a minimal projective presentation of A. The rest of the proof is similar to (1).□

According to the above theorems we can construct the almost split sequences in $\operatorname{mod} T_2(T)$ from the corresponding sequences in $\operatorname{mod} T$.

Corollary 4.4 *(1) If A is an indecomposable injective T-module, then $0 \to (0, A, 0) \to E \to ((DA)^*, 0, 0) \to 0$ is an almost split sequence in* $\operatorname{mod} T_2(T)$*, where E is a $T_2(T)$-module.*

(2) If A is an indecomposable projective T-module, then $0 \to (0, D(A^*), 0) \to E \to (A, 0, 0) \to 0$ is an almost split sequence in $\operatorname{mod} T_2(T)$*, where E is a $T_2(T)$-module.*

Proof Suppose A is an indecomposable injective T-module, then DA is an indecomposable projective T^{op}-module, and $(0, A, 0)$ is a noninjective $T_2(T)$-module. Hence, $TrDA = 0$ and $0 \to DA \to DA \to 0$ is a minimal projective presentation of DA. From the proof of Theorem

4.3, we know that $TrD(A,0,0) = ((DA)^*,0,0)$. One can get the almost split sequence in (1) by Proposition 1.13 in [1, p. 143].

One can prove (2) by Theorem 4.3(2). □

Corollary 4.5 *Suppose* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is an almost split sequence in* $\mathrm{mod}\, T$*. Then*

(1) $0 \to (0,A,0) \to (0,B,0) \oplus E \to (P_0^*, P_1^*, f) \to 0$ *is an almost split sequence in* $\mathrm{mod}\, T_2(T)$*, where* E *is a* $T_2(T)$*-module, and* $P_1 \to P_0 \to DA \to 0$ *is a minimal projective presentation of* DA*.*

(2) $0 \to (DP_1^*, DP_0^*, D\phi(f)\varphi) \to (B,0,0) \oplus E \to (C,0,0) \to 0$ *is an almost split sequence in* $\mathrm{mod}\, T_2(T)$*, where* E *is a* $T_2(T)$*-module, and* $P_1 \to P_0 \to C \to 0$ *is a minimal projective presentation of* C*.*

(3) If A *is an indecomposable noninjective* T*-module, then* $0 \to (A,0,0) \to E \to (C,C,1) \to 0$ *is an almost split sequence in* $\mathrm{mod}\, T_2(T)$*, where* E *is a* $T_2(T)$*-module.*

(4) If C *is an indecomposable nonprojective* T*-module, then* $0 \to (A,A,1) \to (A,B,f) \oplus E \to (0,C,0) \to 0$ *is an almost split sequence in* $\mathrm{mod}\, T_2(T)$*, where* E *is a* $T_2(T)$*-module.*

Proof We learn that $A \cong DTrC$ and $C \cong TrDA$ from the definition of the almost split sequence. By Theorem 2.8(1) we have that $(A,A,1) \to (A,B,f)$ is an irreducible morphism. Then by Theorems 4.2 and 4.3 we can get the conclusions in the Corollary 4.5. □

Now, some almost split sequences in $\mathrm{mod}\, T_2(T)$ can be constructed and the others can be given from the properties of AR-quivers. Next, we will give an example to illustrate it.

Example Let k be a field and A be the k-algebra given by the quiver $\bullet \longrightarrow \bullet$. Then the AR-quiver of A is

$$\begin{tikzcd} & P(1) \arrow[dr, "p"] & \\ S(2) \arrow[ur, "\sigma"] & & S(1) \end{tikzcd} \tag{4}$$

Now, we will give the AR-quiver of $T_2(A)$ as showed in this paper as follows.

For the noninjective indecomposable module $(0,S(2),0)$, there exists an irreducible morphism $(0,S(2),0) \to (0,P(1),0)$ by Corollary 2.7(1). From Corollary 2.2 we learn that $(0,S(2),0) \to (S(2),S(2),1)$ is irreducible. It is easy to see that $DS(1) \to DP(1) \to DS(2) \to 0$ is a minimal projective presentation of $DS(2)$. By computation we know that $(DS(1))^* = P(1)$ and $(DP(1))^* = S(2)$. Thus, we know that $TrD(0,S(2),0) = (S(2),P(1),\sigma)$ by Theorem 4.3(1). In conclusion, we have the following almost split sequence

$$0 \to (0,S(2),0) \to (0,P(1),0) \oplus (S(2),S(2),1) \to (S(2),P(1),\sigma) \to 0.$$

For $(0,P(1),0)$, it is easy to see that $0 \to DP(1) \to DP(1) \to 0$ is a minimal projective presentation of $DP(1)$. By Theorem 4.3(1) we have the following almost split sequence

$$0 \to (0,P(1),0) \to (S(2),P(1),\sigma) \to (S(2),0,0) \to 0.$$

For $(S(2),S(2),1)$, we know that $DTrS(1) = S(2)$. By Theorem 4.2 we have the following almost split sequence

$$0 \to (S(2), S(2), 1) \to (S(2), P(1), \sigma) \to (0, S(1), 0) \to 0.$$

Since $(P(1), P(1), 1)$ is an indecomposable projective module, there exists a right almost split morphism $(S(2), P(1), \sigma) \to (P(1), P(1), 1)$ by Corollary 2.2. So it is irreducible. From the above two almost split sequences, there exists an almost split sequence

$$0 \to (S(2), P(1), \sigma) \to (S(2), 0, 0) \oplus (P(1), P(1), 1) \oplus (0, S(1), 0) \to (P(1), S(1), p) \to 0.$$

Similarly, we have the following almost split sequences

$$0 \to (S(2), 0, 0) \to (P(1), S(1), p) \to (S(1), S(1), 1) \to 0,$$

$$0 \to (0, S(1), 0) \to (P(1), S(1), p) \to (P(1), 0, 0) \to 0.$$

Since $P(1) \to S(1)$ is irreducible, there exists an irreducible morphism $(P(1), 0, 0) \to (S(1), 0, 0)$ by Corollary 2.7(2). Combining the above two sequences, we have the following almost split sequence

$$0 \to (P(1), S(1), p) \to (S(1), S(1), 1) \oplus (P(1), 0, 0) \to (S(1), 0, 0) \to 0.$$

Collecting all the above information, we have that the entire AR-quiver of $T_2(A)$ is as follows.

$$\begin{tikzcd}
 & (0, P(1), 0) \arrow[dr] & & (S(2), 0, 0) \arrow[dr] & & (S(1), S(1), 1) \arrow[dr] & \\
(0, S(2), 0) \arrow[ur] \arrow[dr] & & (S(2), P(1), \sigma) \arrow[ur] \arrow[r] \arrow[dr] & (P(1), P(1), 1) \arrow[r] & (P(1), S(1), p) \arrow[ur] \arrow[dr] & & (S(1), 0, 0) \\
 & (S(2), S(2), 1) \arrow[ur] & & (0, S(1), 0) \arrow[ur] & & (P(1), 0, 0) \arrow[ur] &
\end{tikzcd} \tag{5}$$

References

[1] M. AUSLANDER, I. REITEN, S. O. SMALØ. *Representation Theory of Artin Algebras*. Cambridge University Press, Cambridge, 1995.

[2] Pu ZHANG. *Monomorphism categories, cotilting theory, and Gorenstein-projective modules*. J. Algebra, 2011, **339**: 181–202.

[3] J. RICKARD. *Morita theory for derived categories*. J. London Math. Soc., 1989, **39**(2): 436–456.

[4] Wei HU, Changchang XI. *$\mathcal{D}$-split sequences and derived equivalences*. Adv. Math., 2011, **227**(1): 292–318.

[5] M. AUSLANDER, S. O. SMALØ. *Preprojective modules over Artin algebras*. J. Algebra, 1980, **66**(1): 61–122.

[6] I. ASSEM, D. SIMSON, A. SKOWRONSKI. *Elements of the Representation Theory of Associative Algebras*. Cambridge University Press, Cambridge, 2006.