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Distortion Theorems and Schlicht Radius for p-Bloch Functions

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Abstract In this paper, we establish distortion theorems for both normalized p-Bloch functions with branch points and normalized locally univalent p-Bloch functions defined on the unit disk, respectively. These distortion theorems give lower bounds on $|f'(z)|$ and $\Re f'(z)$. As applications of these distortion theorems, the lower bounds of the radius of the largest schlicht disk on these Bloch functions are given, respectively. Notice that when $p=1$, our results reduce to that of Liu and Minda.

Keywords p-Bloch functions; distortion theorems; schlicht radius; Julia' lemma.

MR(2010) Subject Classification 30H30; 30C55

1. Introduction

In one complex variable, the famous Bloch [1] theorem shows that there exists an absolute constant $r>0$, such that if $f\in H(D)$ and $f'(0)=1$, then f maps some subdomain of the unit disk D univalently onto some disk with fixed radius r.

The greatest lower bound of the above r is called Bloch constant B. Since de Branges solved the Bieberbach conjecture, finding the exact value of the Bloch constant is the most important problem in the geometric function theory of one complex variable. Though the precise value of the Bloch constant is still unknown, many scholars [2–5] made some improvement of Bloch constant by making use of different methods. For example, Bonk [4] introduced an analytic technique to improve the Bloch constant, which is based on the well-known Bonk Distortion Theorem on the unit disk D as follows.

$$\Re f'(z)\geq \frac{1-\sqrt{3}|z|}{(1-|z|/\sqrt{3})^3}\ \text{for}\ |z|\leq\frac{\sqrt{3}}{3},$$

where f is a normalized Bloch function of Bloch seminorm 1 on the unit disk D.

Motivated by the Bonk's method, many authors have studied the distortion theorems on various subclasses of Bloch function spaces on the unit disk D. In 1992, Liu and Minda [6] established distortion theorems for subclasses $\beta(n)$ of Bloch functions by taking into consideration

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the n-order to which the derivative must vanish, as well as for the limiting case of local schlicht functions. The main approach to solve these problems is the classical form of Julia's Lemma [7, 8]. As applications of these distortion theorems, they obtained some lower bounds for various Bloch constants. In 1999, Graham and Minda [9] introduced the definition of the subclasses $\beta(n,\alpha)$ of non-normalized Bloch functions f, namely $f'(0)=\alpha$. They also gave various results for these Bloch functions, such as distortion theorems and radius of schlichtness. In 2010, Yanagihara and Terada [10] defined the concept of the subclasses of $\beta^p(\alpha)$, and obtained the distortion theorems for the subclasses of these p-Bloch functions on D.

In this article, we will introduce the definition of subclasses $\beta^p(n)$ of p-Bloch functions on the unit disk. The distortion theorems for these subclasses are obtained. As applications, we give the estimates for the radius of the largest univalent disk for these subclasses. Our present results extend the results of the classical Bloch space to the general p-Bloch space.

In the following, we give some notation and definitions. Let D be the unit disk and ∂D be the boundary of the unit disk D in the complex plane $\mathbb{C}$. Denote by $H(D)$ the space of all holomorphic functions from D into $\mathbb{C}$. A function $f\in H(D)$ is called normalized if $f(0)=0$ and $f'(0)=1$. A function f is said to be locally schlicht (or locally univalent) if $f'(z)$ does not vanish for every $z\in D$. A point z_0 is defined to be a branch point of f if $f'(z_0)=0$. For a point $a\in D$ and $f\in H(D)$, we write $r(a,f)$ as the radius of the largest univalent disk on the Riemann surface $f(D)$ centered at $f(a)$ (a univalent disk on $f(D)$ centered at $f(a)$ means that f maps an open subset of D containing the point a univalently onto this disk).

For $s>-1$ and any unimodular constant λ, a horocycle based at λ is defined by

$$\Gamma_\lambda(s)=\{z:\frac{|\lambda-z|^2}{1-|z|^2}=1+s\}=\{z:|z-\frac{\lambda}{2+s}|=\frac{1+s}{2+s}\text{ and } z\neq\lambda\},$$

which is a circle internally tangent to the unit circle at λ. When $\lambda=1$, we write $\Gamma(s)$ rather than $\Gamma_1(s)$. As s ranges over $(-1,+\infty)$, the horocycles $\Gamma(s)$ provide a foliation of D and $0\in\Gamma(0)$.

Definition 1.1 ([10]) *For any positive number p, a holomorphic function f on the unit disk D is called a p-Bloch function if the p-Bloch seminorm*

$$\|f\|_{\mathcal{B}^p}=\sup\{(1-|z|^2)^p|f'(z)|:z\in D\}$$

is finite. In particular, when $p=1$, the function f reduces to the classical Bloch function on the unit disk D.

Definition 1.2 *For any positive integer n, we define*

$$\begin{aligned}\beta^p(n)=&\{f\in H(D):\|f\|_{\mathcal{B}^p}\le 1,\ f(0)=0,f'(0)=1,\ \text{and if } f'(a)=0 \text{ for some}\\ &a\in D,\ \text{then } f^{(k)}(a)=0 \text{ for } k=1,2,\ldots,n\},\\ \beta^p_\infty=&\{f\in H(D):\|f\|_{\mathcal{B}^p}\le 1,\ f(0)=0,f'(0)=1,\ \text{and } f \text{ is locally schlicht}\}.\end{aligned}$$

It is clear that $\beta^p(n)$ consists of those functions with p-Bloch seminorm 1 and all of whose branch points are of order at least $n+1$. Note that $\beta^p(n)\supset\beta^p(n+1)$ and $\beta^p_\infty=\bigcap_{n=1}^\infty\beta^p(n)$. In particular, when $n=1$, we usually write β^p rather than $\beta^p(1)$, when $p=1$, we write $\beta(n)$ and

β_∞ which were introduced by Liu and Minda [6], respectively. In addition, when $p=1$ and f is non-normalized, more work can be found in [11–14].

2. Extremal functions in the classes

In the following, we will give the extremal functions to enable us to state our results explicitly. For $n\geq 1$, we define

$$F_n(z)=\int_0^z\frac{(1-\sqrt{\frac{n+2p}{n}}\xi)^n}{(1-\sqrt{\frac{n}{n+2p}}\xi)^{n+2p}}\mathrm{d}\xi,$$

then the function F_n is extremal for the classes $\beta^p(n)$. In fact, F_n is holomorphic on the unit disk D, $F_n(0)=0, F_n'(0)=1$ and

$$F_n'(z)=\frac{(1-\sqrt{\frac{n+2p}{n}}z)^n}{(1-\sqrt{\frac{n}{n+2p}}z)^{n+2p}}.$$

Furthermore, it is obvious that the branch points of F_n are of order $n+1$. Some computation shows that

$$\begin{aligned}\|F_n\|_{\mathcal{B}^p}&=\sup\{(1-|z|^2)^p|F_n'(z)|:|z|<1\}\\&=\sup\{(1-|z|^2)^p\frac{|1-\sqrt{\frac{n+2p}{n}}z|^n}{|1-\sqrt{\frac{n}{n+2p}}z|^{n+2p}}:|z|<1\}\\&=\sup\{(1-|z|^2)^p|\frac{\sqrt{\frac{n}{n+2p}}-z}{1-\sqrt{\frac{n}{n+2p}}z}|^n(\sqrt{\frac{n+2p}{n}})^n\frac{1}{|1-\sqrt{\frac{n}{n+2p}}z|^{2p}}:|z|<1\}.\end{aligned}$$

Set

$$T_{n,p}(z)=\frac{\sqrt{\frac{n}{n+2p}}-z}{1-\sqrt{\frac{n}{n+2p}}z},$$

then the function $T_{n,p}(z)$ is a conformal automorphism of D. By some simple computation, we then obtain

$$\begin{aligned}\|F_n\|_{\mathcal{B}^p}&=(\sqrt{\frac{n+2p}{n}})^n\sup\{(\frac{1-|z|^2}{|1-\sqrt{\frac{n}{n+2p}}z|^2})^p|T_{n,p}(z)|^n:|z|<1\}\\&=(\frac{n+2p}{n})^{n/2}(\frac{n+2p}{2p})^p\sup\{(1-|T_{n,p}(z)|^2)^p|T_{n,p}(z)|^n:|z|<1\}.\end{aligned}$$

Let $|T_{n,p}(z)|=t, M(t)=(1-t^2)^pt^n$. Then $M(t)$ is increasing on $(0,\sqrt{\frac{n}{n+2p}}]$, decreasing on $[\sqrt{\frac{n}{n+2p}},1)$ and

$$M(\sqrt{\frac{n}{n+2p}})=(\frac{2p}{n+2p})^p(\frac{n}{n+2p})^{n/2}.$$

In particular, $(1-|z|^2)^p|F_n'(z)|=1$ if and only if $|T_{n,p}(z)|=\sqrt{\frac{n}{n+2p}}$. Hence $\|F_n\|_{\mathcal{B}^p}=1$. Next, we consider a function that is an extremal function for β^p_∞. We define

$$F(z)=\int_0^z (1-\xi)^{-2p}\exp\{-\frac{2p\xi}{1-\xi}\}\mathrm{d}\xi,$$

then the function F is extremal for locally univalent p-Bloch functions. In fact, F is holomorphic in D, $F(0)=0$, $F'(0)=1$ and

$$F'(z)=(1-z)^{-2p}\exp\{-\frac{2pz}{1-z}\}=\frac{e}{(1-z)^{2p}}\exp\{\frac{-2pz+z-1}{1-z}\}.$$

By the definition of horocycle, for $t>-1$ and

$$z\in\Gamma(\frac{-t}{1+t})=\{z:\frac{1-|z|^2}{|1-z|^2}=1+t\},$$

we then get

$$\Re z-|z|^2=\frac{t}{2(1+t)}(1-|z|^2).$$

Hence

$$\begin{aligned}(1-|z|^2)^p|F'(z)|&=e(\frac{1-|z|^2}{|1-z|^2})^p\exp\{\Re(\frac{-2pz+z-1}{1-z})\}\\&=e(\frac{1-|z|^2}{|1-z|^2})^p\exp\{\frac{(2-2p)(\Re z-|z|^2)+|z|^2-1}{|1-z|^2}\}\\&=(\frac{1-|z|^2}{|1-z|^2})^p\exp\{-pt\}=(1+t)^p\exp\{-pt\}.\end{aligned}$$

Let $N(t)=(1+t)^p\exp\{-pt\}$. Then

$$N'(t)=p(1+t)^p\exp\{-pt\}\frac{-t}{1+t}.$$

And $N(t)$ is increasing on $(-1,0]$, decreasing on $[0,+\infty)$ and attains its maximum value 1 on the interval $(-1,+\infty)$. Moreover, $(1-|z|^2)^p|F'(z)|\le 1$ with equality if and only if $z\in\Gamma(0)$. Hence $\|F\|_{\mathcal{B}^p}=1$.

3. Some lemmas

In order to prove the main results, we need the following three lemmas.

Lemma 3.1 ([6]) *Let f be a holomorphic function on $D\bigcup\{1\}.f(D)\subset D, f(1)=1$ and all zeros of $f(z)$ have multiplicities at least n. If $f'(1)=n$, then*

(1) $|f(x)|\ge x^n$ for $x\in[0,1)$, with equality for some x if and only if $f(x)=x^n$.

(2) $\Re f(x)\ge x^n$ for $\frac{n-1}{n+1}\le x<1$, with equality for some x if and only if $f(x)=x^n$.

Lemma 3.2 ([6]) *Let w be a holomorphic function on $D\bigcup\{1\}$. w maps D into the right half plane $\mathbb{H}=\{z:\Re z>0\}$ and $w(1)=0$, then $d=-w'(1)>0$, and*

$$\Re(w(x))\le 2d\frac{1-x}{1+x}$$

for all $x\in(-1,1)$, with equality for some $x\in(-1,1)$ if and only if

$$w(x)=2d\frac{1-x}{1+x}.$$

Lemma 3.3 *Suppose $f \in \beta^p, \|f\|_{\mathcal{B}^p} \le 1$. Then $f''(0) = 0$.*

Proof Because $\|f\|_{\mathcal{B}^p} \le 1$, we have the following inequality

$$|f'(z)| = |1 + f''(0)z + o(|z|)| \le \frac{1}{(1-|z|^2)^p} = 1 + p|z|^2 + o(|z|^2),$$

which implies the desired result $f''(0) = 0$. □

4. Main results

In this section, we are ready to present distortion theorems and schlicht fadius for subclasses of p-Bloch functions in the unit disk, respectively.

Theorem 4.1 *If $f \in \beta^p(n)$, then*

(1) $|f'(z)| \ge F_n'(|z|) = \frac{(1-\sqrt{\frac{n+2p}{n}}|z|)^n}{(1-\sqrt{\frac{n}{n+2p}}|z|)^{n+2p}}$ for $|z| \le \sqrt{\frac{n}{n+2p}}$, with equality for some $z = re^{i\theta} \neq 0$ if and only if $f(z) = e^{i\theta}F_n(e^{-i\theta}z)$ for some real θ.

(2) $\Re f'(z) \ge F_n'(|z|) = \frac{(1-\sqrt{\frac{n+2p}{n}}|z|)^n}{(1-\sqrt{\frac{n}{n+2p}}|z|)^{n+2p}}$ for $|z| \le \frac{\sqrt{n(n+2p)}}{n+p(n+1)}$, with equality for some $z = re^{i\theta} \neq 0$ if and only if $f(z) = e^{i\theta}F_n(e^{-i\theta}z)$ for some real θ.

Proof (1) Set $g(u) = (\frac{1-a^2}{1-a^2u})^{2p} f'(\frac{a-au}{1-a^2u}\xi)$, where $\xi \in \partial D$, i.e., $|\xi| = 1$, $u \in D$ and $a = \sqrt{\frac{n}{n+2p}}$, then g is a holomorphic function in the closure of D. Since $g(1) = 1, f'(0) = 1$, we have $g'(1) = n$. Hence, all zeros of $g(z)$ have multiplicities at least n. Notice that for any $u \in D$, we have

$$\begin{aligned}|g(u)| &= |\frac{1-a^2}{1-a^2u}|^{2p}|f'(\frac{a-au}{1-a^2u}\xi)| \le |\frac{1-a^2}{1-a^2u}|^{2p}\frac{1}{(1-|\frac{a-au}{1-a^2u}\xi|^2)^p}\\ &= |\frac{1-a^2}{1-a^2u}|^{2p}\frac{1}{[\frac{(1-a^2)(1-|au|^2)}{|1-a^2u|^2}]^p} = |\frac{1-a^2}{1-a^2u}|^{2p}\frac{|1-a^2u|^{2p}}{(1-a^2)^p(1-|au|^2)^p}\\ &= (\frac{1-a^2}{1-|au|^2})^p < 1.\end{aligned}$$

Thus $g(D) \subset D$.

By making use of the inequality $|g(u)| \ge u^n$ with $u \in [0, 1)$, from Lemma 3.1(1), we obtain

$$|(\frac{1-a^2}{1-a^2u})^{2p} f'(\frac{a-au}{1-a^2u}\xi)| \ge u^n$$

with equality for some u if and only if $g(u) = u^n$. Set $v = \frac{a-au}{1-a^2u}$, then $u = \frac{1}{a}\frac{a-v}{1-av}$. And $v \in (0, \sqrt{\frac{n}{n+2p}}]$ when $u \in [0, 1)$. Hence

$$|f'(v\xi)| \ge (\frac{1}{a}\frac{a-v}{1-av})^n \frac{1}{(1-av)^{2p}} = \frac{(1-\frac{1}{a}v)^n}{(1-av)^{n+2p}}$$

for all $v \in (0, a]$. For any $z \in D$ with $|z| \le \sqrt{\frac{n}{n+2p}}$, we take $v = |z|$, $\xi = \frac{z}{|z|}$, then the above inequality reduces to

$$|f'(z)| \ge \frac{(1-\frac{1}{a}|z|)^n}{(1-a|z|)^{n+2p}} = F_n'(|z|),$$

which completes the proof of the desired inequality.

Moreover, it is not difficult to know that the above equality holds for some $z = re^{i\theta} \neq 0$ if and only if $f(z) = e^{i\theta}F_n(e^{-i\theta}z)$ for some real θ according to Lemma 3.1.

(2) The proof is similar to that of part (1). Define g as in the part (1), then Lemma 3.1(2) says that $\Re g(u) \geq u^n$ for $\frac{n-1}{n+1} \leq u < 1$, with equality for some u if and only if $g(u) = u^n$. This is equivalent to

$$\Re f'(\frac{a-au}{1-a^2u}\xi) \geq \frac{u^n}{(\frac{1-a^2}{1-a^2u})^{2p}}, \quad \frac{n-1}{n+1} \leq u < 1.$$

Let $v = \frac{a-au}{1-a^2u}$. Then $u = \frac{1}{a}\frac{a-v}{1-av}$. Since $u \in [\frac{n-1}{n+1}, 1)$, we then obtain $v \in (0, \frac{2a}{(n+1)-(n-1)a^2}]$. For $|z| \leq \frac{\sqrt{n(n+2p)}}{n+p(n+1)}$, we set $v = |z|$, $\xi = \frac{z}{|z|}$, then the above inequality reduces to

$$\begin{aligned}\Re f'(z) &\geq \frac{u^n}{(\frac{1-a^2}{1-a^2u})^{2p}} = (\frac{1}{a}\frac{a-v}{1-av})^n\frac{1}{(1-av)^{2p}} = \frac{(1-\frac{1}{a}v)^n}{(1-av)^{n+2p}}\\ &= \frac{(1-\sqrt{\frac{n+2p}{n}}|z|)^n}{(1-\sqrt{\frac{n}{n+2p}}|z|)^{n+2p}} = F_n'(|z|),\end{aligned}$$

which completes the proof of the desired inequality.

Similarly, it is clear that equality holds for some $z = re^{i\theta} \neq 0$ if and only if $f(z) = e^{i\theta}F_n(e^{-i\theta}z)$ for some real θ.

Remark When $p = 1$, Theorem 4.1 reduces to Theorem 4 in [6]. When $n = 1$, Theorem 4.1 reduces to the special case of $\alpha = 1$ in Theorem 1 in [10]. When $n = 1$ and $p = 1$, Theorem 4.1 reduces to the Bonk's distortion theorem in [11].

As an application of the theorem 4.1, the lower bound of the radius of the largest schlicht disk on $\beta^p(n)$ is given as follows.

Corollary 4.1 *If $f \in \beta^p(n)$, then*

$$r(0,f) \geq \int_0^{\sqrt{\frac{n}{n+2p}}} F_n'(|z|)\mathrm{d}|z| = \int_0^{\sqrt{\frac{n}{n+2p}}} \frac{(1-\sqrt{\frac{n+2p}{n}}t)^n}{(1-\sqrt{\frac{n}{n+2p}}t)^{n+2p}}\mathrm{d}t$$

with equality for some $z = re^{i\theta} \neq 0$ if and only if $f(z) = e^{i\theta}F_n(e^{-i\theta}z)$ for some real θ.

Proof By the definition of $r(0,f)$, there is a simply connected region $E \subset D$ containing 0 such that f maps E conformally onto a disk centered at $f(0) = 0$ with radius $r(0,f)$. This disk must meet the boundary of the Riemann surface $f(D)$. If not, then the boundary of E would be a Jordan curve inside D. Because f is locally schlicht, we could find an open subset of D containing the closure of E such that f is schlicht on this larger set. Then the image of this larger set would contain a disk centered at $f(0)$ with radius larger than $r(0,f)$, which contradicts the definition of $r(0,f)$. So there is a radial segment Γ in the above disk joining $f(0)$ to a boundary point of $f(D)$. Let γ be the inverse image of Γ under the mapping f. Then γ joins 0 to the boundary of

D. Hence, from Theorem 4.1, if $f \in \beta^p(n)$ and $|z| \le \sqrt{\frac{n}{n+2p}}$, then

$$r(0,f) = \int_\Gamma |\mathrm{d}\omega| = \int_\gamma |f'(z)\mathrm{d}z| \ge \int_0^{\sqrt{\frac{n}{n+2p}}} F_n'(|z|)\mathrm{d}|z|$$

$$= \int_0^{\sqrt{\frac{n}{n+2p}}} \frac{(1-\sqrt{\frac{n+2p}{n}}|z|)^n}{(1-\sqrt{\frac{n}{n+2p}}|z|)^{n+2p}}\mathrm{d}|z| = \int_0^{\sqrt{\frac{n}{n+2p}}} \frac{(1-\sqrt{\frac{n+2p}{n}}t)^n}{(1-\sqrt{\frac{n}{n+2p}}t)^{n+2p}}\mathrm{d}t,$$

which completes the proof of the desired inequality.

Moreover, it is clear that equality holds for some $z = re^{i\theta} \ne 0$ if and only if $f(z) = e^{i\theta}F_n(e^{-i\theta}z)$ for some real θ.

Remark When $p = 1$, Corollary 4.1 reduces to the result of Liu and Minda [6]. When $n = 1$, Corollary 4.1 reduces to the special case of $\alpha = 1$ in Theorem 1 in [10].

Theorem 4.2 *If $f \in \beta^p_\infty$, then*

(1) $|f'(z)| \ge F'(|z|) = (1-|z|)^{-2p}\exp\{-\frac{2p|z|}{1-|z|}\}$ for all $z \in D$, with equality for some $z = re^{i\theta} \ne 0$ if and only if $f(z) = e^{i\theta}F(e^{-i\theta}z)$ for some real θ.

(2) $\Re f'(z) \ge F'(|z|) = (1-|z|)^{-2p}\exp\{-\frac{2p|z|}{1-|z|}\}$ for all $|z| \le \dfrac{1}{2}$, with equality for some $z = re^{i\theta} \ne 0$ if and only if $f(z) = e^{i\theta}F(e^{-i\theta}z)$ for some real θ.

Proof (1) Let $g(u) = (\frac{1+u}{2})^{2p}f'(\frac{1-u}{2}\xi)$, where $\xi \in \partial D$, i.e., $|\xi| = 1$. Then g is holomorphic in the closure of D except at the point -1. Further, $g(1) = 1$ and $g \ne 0$. By making use of Lemma 3.3, we get $f''(0) = 0$. Hence, by some simple calculation, we have $g'(1) = p$. Note that for all $u \in D$, we also have

$$|g(u)| = |\frac{1+u}{2}|^{2p}|f'(\frac{1-u}{2}\xi)| \le |\frac{1+u}{2}|^{2p}\frac{1}{(1-|\frac{1-u}{2}|^2)^p} < 1.$$

Hence $\|f\|_{\mathcal{B}^p} = 1$. Notice that g maps D into $D\backslash\{0\}$. Because g never vanishes, there exists a holomorphic function ω which maps D into $\{z : \Re z > 0\}$, such that $\omega(1) = 0$ and $g(u) = \exp\{-\omega(u)\}$. By direct calculation, we get $d = -\omega'(1) = g'(1) = p$. Hence by Lemma 3.2, we then have

$$\Re\omega(u) \le 2d\frac{1-u}{1+u} = 2p\frac{1-u}{1+u}$$

for all $u \in (-1, 1)$, with equality for some u if and only if

$$w(u) = 2d\frac{1-u}{1+u} = 2p\frac{1-u}{1+u}.$$

Consequently,

$$|g(u)| = \exp\{-\Re\omega(u)\} \ge \exp\{-2d\frac{1-u}{1+u}\} = \exp\{-2p\frac{1-u}{1+u}\}.$$

In other words, for all $u \in D$, we have

$$|(\frac{1+u}{2})^{2p}f'(\frac{1-u}{2}\xi)| \ge \exp\{-2p\frac{1-u}{1+u}\},$$

with equality for some u if and only if $\omega(u) = 2p\frac{1-u}{1+u}$. Suppose $\frac{1-u}{2} = v$. Then $u = 1-2v$. For all $z\in D$, let $\xi=\frac{z}{|z|}, |z|=v$. We get

$$|f'(z)|=|f'(v\xi)|\ge \exp\{-2p\frac{1-u}{1+u}\}\frac{1}{(\frac{1+u}{2})^{2p}}=\exp\{-2p\frac{|z|}{1-|z|}\}\frac{1}{(1-|z|)^{2p}}$$
$$=(1-|z|)^{-2p}\exp\{-2p\frac{|z|}{1-|z|}\}=F'(|z|),$$

which completes the proof of the desired inequality.

Moreover, it is easy to know that equality holds for some $z=re^{i\theta}\neq 0$ if and only if $f(z)=e^{i\theta}F(e^{-i\theta}z)$ for some real θ according to Lemma 3.2.

(2) Define g as in (1), by Lemma 3.2, we have

$$\Re\omega(u)\le 2d\frac{1-u}{1+u}=2p\frac{1-u}{1+u}.$$

Let $g(u)=\exp\{-\omega(u)\}$. Then $\Re g(u)=\Re\exp\{-\omega(u)\}\ge \exp\{-2p\frac{1-u}{1+u}\}$ for all u in $[0,1)$. This inequality is equivalent to

$$\Re\{(\frac{1+u}{2})^{2p}f'(\frac{1-u}{2}\xi)\}\ge \exp\{-2p\frac{1-u}{1+u}\}.$$

Set $\frac{1-u}{2}=v$. Then $u=1-2v$. For all $|z|\in(0,\frac{1}{2}]$, let $\xi=\frac{z}{|z|}, |z|=v$. We get

$$\Re f'(z)=\Re f'(v\xi)\ge (\frac{1+u}{2})^{-2p}\exp\{-2p\frac{1-u}{1+u}\}$$
$$=(1-v)^{-2p}\exp\{-2p\frac{v}{1-v}\}=(1-|z|)^{-2p}\exp\{-2p\frac{|z|}{1-|z|}\}=F'(|z|),$$

which completes the proof of the desired inequality.

Similarly, it is clear that equality holds for some $z=re^{i\theta}\neq 0$ if and only if $f(z)=e^{i\theta}F(e^{-i\theta}z)$ for some real θ.

Remark When $p=1$, Theorem 4.2 reduces to Theorem 1 in [6]. As an application, we obtain the following Corollary 4.2.

Corollary 4.2 *If $f\in\beta^p_\infty$, then*

$$r(0,f)\ge\int_0^1 F'(|z|)\mathrm{d}|z|=\int_0^1(1-t)^{-2p}\exp\{-2p\frac{t}{1-t}\}\mathrm{d}t,$$

with equality for some $z=re^{i\theta}\neq 0$ if and only if $f(z)=e^{i\theta}F(e^{-i\theta}z)$ for some real θ.

Proof The proof is similar to that of Corollary 4.1. Since $f\in\beta^p_\infty$, by making use of Theorem 4.2, for all $z\in D$, we then obtain

$$|f'(z)|\ge F'(|z|)=(1-|z|)^{-2p}\exp\{-2p\frac{|z|}{1-|z|}\}.$$

Hence

$$r(0,f)=\int_\Gamma|\mathrm{d}\omega|=\int_\gamma|f'(z)\mathrm{d}z|\ge\int_0^1F'(|z|)\mathrm{d}|z|$$
$$=\int_0^1(1-|z|)^{-2p}\exp\{-2p\frac{|z|}{1-|z|}\}\mathrm{d}|z|=\int_0^1(1-t)^{-2p}\exp\{-2p\frac{t}{1-t}\}\mathrm{d}t,$$

which completes the proof of the desired inequality.

Moreover, it is easy to know that equality holds for some $z = re^{i\theta} \neq 0$ if and only if $f(z) = e^{i\theta}F(e^{-i\theta}z)$ for some real θ.

Remark When $p = 1$, Corollary 4.2 reduces to the result of Liu and Minda [6].

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