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Module-Relative-Hochschild (Co)homology under Ground Ring Extensions

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Abstract In this paper, we consider the module-relative-Hochschild homology and cohomology under the ground ring extensions.

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1. Introduction

Module-relative-Hochschild (co)homology was introduced in [1] by Ardizzoni, Brzeziński and Menini when they studied the formal smoothness. It plays an important role in non-commutative algebraic geometry and provides a natural characterization of the separable bimodules and formally smooth bimodules. One can view the separable bimodules as (non-commutative, relative) "bundles of points", that is, the objects with relative-Hochschild cohomology dimension zero; and the formally smooth bimodules can be understood as (non-commutative, relative) "bundles of curves" or "line bundles", that is, the objects with relative-Hochschild cohomology dimension at most one.

The notions of formal smoothness have attracted much attention in recent literature [1–11]. A convenient description and conceptual interpretation of formal smoothness is provided by $\mathcal{E}$-relative derived functors [6, 12, 13]. Ardizzoni, Menini and Stefan have introduced in [3] the Hochschild cohomology in monoidal abelian categories in this way, instead of generalizing ordinary Hochschild's construction [14] or by using the (co)simplicial approach explained in [15]. This general algebraic approach to formal smoothness in monoidal abelian categories, including the cohomological aspects, was also proposed in [2]. These gave rise to the introduction of module-relative-Hochschild cohomology [1].

Let A and B be k-algebras with k a commutative ring. Given a bimodule ${}_BM_A$ such that ${}_BM$ is a generator in ${}_B\mathcal{M}$, we consider the following projective class of epimorphisms

$$\mathcal{E}_{M,B} := \{f \in {}_B\mathcal{M}_B \mid \mathrm{Hom}_B(M, f) \text{ is split epimorphic in } {}_A\mathcal{M}_B\}.$$

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Here ${}_B\mathcal{M}_B$ and ${}_A\mathcal{M}_B$ denote categories of B-B-bimodules and A-B-bimodules, respectively. Based on the theory of $\mathcal{E}_{M,B}$-relative derived functor, the nth ${}_BM_A$-Hochschild cohomology and homology of B over A are defined by

$$\mathrm{H}^n_{\mathcal{E}_{M,B}}(B) := \mathrm{Ext}^n_{\mathcal{E}_{M,B}}(B, B)$$

and

$$\mathrm{H}_n^{\mathcal{E}_{M,B}}(B) := \mathrm{Tor}_n^{\mathcal{E}_{M,B}}(B, B),$$

respectively. In particular, when k is a field, taking ${}_BM_A = {}_BB_k$, we get the ordinary Hochschild (co)homology of B; moreover, if there is an algebra homomorphism $\mu : A \to B$, by taking ${}_BM_A = {}_BB_A$, we get the relative Hochschild (co)homology of B with respect to μ. Thus the concept of module-relative-Hochschild (co)homology is in fact a generalization of the notion of ordinary (relative) Hochschild (co)homology.

In this paper, we will consider the module-relative-Hochschild homology and cohomology under the ground ring extensions. Let k be a commutative ring. Consider the ground ring extension from k to a commutative k-algebra R. Each k-algebra B yields an R-algebra $B^R = R \otimes_k B$. We will show that the module-relative-Hochschild (co)homology of B^R is entirely determined by that of B if R as a k-module is finitely generated and projective. Moreover, we show that B^R is formally smooth if and only if B is formally smooth, provided k is a field.

2. Module-relative-Hochschild (co)homology

Throughout this paper, for an algebra (a ring) we mean a unital associative algebra (a ring). Let ${}_B\mathcal{M}, \mathcal{M}_A$ and ${}_B\mathcal{M}_A$ denote categories of (unital) left B-modules, right A-modules and B-A-bimodules, respectively. The notation ${}_BM_A$ means that M is a B-A-bimodule.

Let A and B be k-algebras with k a commutative ring. Let $A^e = A \otimes_k A^{op}$ denote the enveloping algebra of A. Given a bimodule ${}_BM_A$, we consider the following adjunction:

$$\mathbb{L}_B := M \otimes_A - : {}_A\mathcal{M}_B \to {}_B\mathcal{M}_B,$$

$$\mathbb{R}_B := \mathrm{Hom}_B(M, -) : {}_B\mathcal{M}_B \to {}_A\mathcal{M}_B.$$

Let

$$\mathcal{E}_{M,B} := \{f \in {}_B\mathcal{M}_B \mid \mathrm{Hom}_B(M, f) \text{ is split epimorphic in } {}_A\mathcal{M}_B\}.$$

$\mathcal{E}_{M,B}$ is always a projective class [2, Theorem 1.4], and if M is a generator in ${}_B\mathcal{M}$, then $\mathcal{E}_{M,B}$ is a projective class of epimorphisms [1, Proposition 3.1]. Then every object in ${}_B\mathcal{M}_B$ has an $\mathcal{E}_{M,B}$-projective resolution, which is unique up to a homotopy. The reader is referred to [6] for further information on relatively projective object and projective class of epimorphisms. Note that ${}_BP_B$ is $\mathcal{E}_{M,B}$-projective if and only if $\mathrm{Hom}_{B^e}(P, -)$ is $\mathcal{E}_{M,B}$-exact. Another condition is that ${}_BP_B$ is $\mathcal{E}_{M,B}$-projective if and only if there is a split epimorphism $\pi : \mathbb{L}_B(X) \to P$ for a suitable $X \in {}_A\mathcal{M}_B$. So it is easy to see that all projective B-B-bimodules and B-B-bimodules of the form $\mathbb{L}_B(X)$, $X \in {}_A\mathcal{M}_B$, are $\mathcal{E}_{M,B}$-projective.

Recall first from [1] some definitions. Let M be a B-A-bimodule which is a generator as a left B-module. The nth ${}_BM_A$-Hochschild cohomology of B over A with coefficients in a B-B-bimodule Y is defined to be

$$\mathrm{H}^n_{\mathcal{E}_{M,B}}(B,Y) := \mathrm{Ext}^n_{\mathcal{E}_{M,B}}(B,Y).$$

In particular, if $Y = B$, then $\mathrm{H}^n_{\mathcal{E}_{M,B}}(B) := \mathrm{H}^n_{\mathcal{E}_{M,B}}(B,B)$ is called the nth ${}_BM_A$-Hochschild cohomology of B over A. The number $\min\{n \in \mathbb{N} \mid \mathrm{H}^{n+1}_{\mathcal{E}_{M,B}}(B,Y) = 0 \text{ for any } Y \in {}_B\mathcal{M}_B\}$ is called the ${}_BM_A$-Hochschild cohomology dimension of B (if it exists), and denoted by $\mathrm{hch.dim}_M(B)$. If such an n does not exist, we will say that ${}_BM_A$-Hochschild cohomology dimension of B is infinite.

Using relative-Tor-functor, we propose the following:

Definition 2.1 ([16]) *Consider a B-A-bimodule M such that ${}_BM$ is a generator in ${}_B\mathcal{M}$. The nth ${}_BM_A$-Hochschild homology of B over A with coefficients in ${}_BY_B$ is defined by*

$$\mathrm{H}_n^{\mathcal{E}_{M,B}}(B,Y) := \mathrm{Tor}_n^{\mathcal{E}_{M,B}}(B,Y).$$

In particular, if $Y = B$, then $\mathrm{H}_n^{\mathcal{E}_{M,B}}(B) := \mathrm{H}_n^{\mathcal{E}_{M,B}}(B,B)$ is called the nth ${}_BM_A$-Hochschild homology of B over A. The number $\min\{n \in \mathbb{N} \mid \mathrm{H}_{n+1}^{\mathcal{E}_{M,B}}(B,Y) = 0$ for any $Y \in {}_B\mathcal{M}_B\}$ is called the ${}_BM_A$-Hochschild homology dimension of B (if it exists), and denoted by $\mathrm{hh.dim}_M(B)$. If such an n does not exist, we will say that ${}_BM_A$-Hochschild homology dimension of B is infinite.

Similarly to the non-relative case, ${}_BM_A$-Hochschild (co)homology can be equivalently described as the (co)homology of a complex associated with the standard resolution. Let $\varepsilon_B : \mathbb{L}_B\mathbb{R}_B \to \mathrm{Id}_{{}_B\mathcal{M}_B}$ be the counit of the adjunction $(\mathbb{L}_B, \mathbb{R}_B)$ and M a B-A-bimodule which is a generator in ${}_B\mathcal{M}$. Then, for every B-B-bimodule X, the associated augmented chain complex $(\mathbb{P}_X, d_*)$ of ${}_BX_B$:

$$\cdots \longrightarrow (\mathbb{L}_B\mathbb{R}_B)^2(X) \xrightarrow{d_1} \mathbb{L}_B\mathbb{R}_B(X) \xrightarrow{d_0} (\mathbb{L}_B\mathbb{R}_B)^0(X) := X \longrightarrow 0$$

where $d_n = \sum_{i=0}^n (-1)^i (\mathbb{L}_B\mathbb{R}_B)^i(\varepsilon_B((\mathbb{L}_B\mathbb{R}_B)^{n-i}(B)))$, is an $\mathcal{E}_{M,B}$-projective resolution of ${}_BX_B$, called the standard $\mathcal{E}_{M,B}$-projective resolution of ${}_BX_B$.

3. Ground ring extensions

This section is devoted to the module-relative-Hochschild homology and cohomology under the ground ring extensions.

Let k be a commutative ring. We always write $\otimes$ for $\otimes_k$. Consider the ground ring extension from k to a commutative k-algebra R. Each k-algebra B yields an R-algebra $B^R = R \otimes B$; there are ring homomorphisms $i_k : k \to R$ and $i_B : B \to B^R$ given by $i_k(k_1) = k_1 1_R$ and $i_B(b) = 1_R \otimes b$, so that $(i_k, i_B) : (k, B) \to (R, B^R)$ is a change of algebras. Each B^R-module or bimodule pulls back along i_B to be a B-module or bimodule. Each B-module M determines a B^R-module $M^R = R \otimes M$ and a homomorphism $i_M : M \to M^R$ of B-modules given by $i_M(m) = 1_R \otimes m$. Each B-module homomorphism $\mu : M \to N$ determines a B^R-module homomorphism $\mu^R : M^R \to N^R$ by $\mu^R(r \otimes m) = r \otimes \mu m$, so that $\mu^R i_M = i_N \mu^R$. Thus

$T^R(M) = M^R$, $T^R(\mu) = \mu^R$ is a covariant functor on B-modules to B^R-modules. This functor has some good properties. To see it, we need the following lemma.

Lemma 3.1 *Let R be a commutative k-algebra. Let A and B be two k-algebras. If R as a k-module is finitely generated and projective, then, for any two B-modules M and N, there is an isomorphism*

$$R \otimes \mathrm{Hom}_B(M, N) \simeq \mathrm{Hom}_{R\otimes B}(R \otimes M, R \otimes N)$$

of k-modules.

Proof Let $\varphi : R \otimes \mathrm{Hom}_B(M, N) \longrightarrow \mathrm{Hom}_{R\otimes B}(R \otimes M, R \otimes N)$ be given by $r \otimes g \mapsto g_r$, where $g_r(r' \otimes m) = rr' \otimes g_r(m)$. It is clear that g_r is an $R \otimes B$-map, and it is an isomorphism when $M = B$. There is an exact sequence of left B-modules

$$\coprod_J B \longrightarrow \coprod_I B \longrightarrow M \longrightarrow 0.$$

Applying the functors $R \otimes -$ and $\mathrm{Hom}_B(M, -)$ to it, respectively, we get the following two exact sequences

$$R \otimes \coprod_J B \to R \otimes \coprod_I B \to R \otimes M \to 0$$

and

$$0 \to \mathrm{Hom}_B(M, N) \to \mathrm{Hom}_B(\coprod_I B, N) \to \mathrm{Hom}_B(\coprod_J B, N).$$

Note that we have

$$\begin{aligned}\mathrm{Hom}_{R\otimes B}(R \otimes \coprod_I B, R \otimes N) &\simeq \mathrm{Hom}_{R\otimes B}(\coprod_I (R \otimes B), R \otimes N)\\ &\simeq \prod_I \mathrm{Hom}_{R\otimes B}(R \otimes B, R \otimes N)\\ &\simeq \prod_I (R \otimes N)\end{aligned}$$

and

$$R \otimes \mathrm{Hom}_B(\coprod_I B, N) \simeq R \otimes \prod_I \mathrm{Hom}_B(B, N) \simeq R \otimes \prod_I N.$$

By [12, Theorem 3.2.22], $R\otimes\prod_I B \simeq \prod_I (R\otimes B)$ for any index set I since R is a finitely generated projective k-module. Hence there is a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_{R\otimes B}(R \otimes M, R \otimes N) & \longrightarrow & \prod_I (R \otimes N) & \longrightarrow & \prod_J (R \otimes N) \\ & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} \\ 0 \longrightarrow & R \otimes \mathrm{Hom}_B(M, N) & \longrightarrow & R \otimes \prod_I N & \longrightarrow & R \otimes \prod_J N. \end{array}$$

Note that the second two vertical maps are isomorphisms, so is the first. Then the result follows. □

Let A and B be two k-algebras. Then we have two R-algebras A^R and B^R. By Lemma 3.1, we get the following proposition.

Proposition 3.2 *Let R be a commutative k-algebra. Let A and B be two k-algebras.*

(1) If R as a k-module is finitely generated and projective, then, for any ${}_BM_A$ and ${}_BN$, we have

$$\mathrm{Hom}_{B^R}((M^R)_{A^R}, N^R) \simeq \mathrm{Hom}_B(M_A, N)^R$$

as left A^R-modules.

(2) For ${}_BM_A$, ${}_AN$, we have

$${}_{B^R}M^R \otimes_{A^R} N^R \simeq (M \otimes_A N)^R$$

as left B^R-modules.

Given a bimodule ${}_BM_A$, we get a bimodule ${}_{B^R}(M^R)_{A^R}$. Consider the following adjunctions

$$\mathbb{L}_B := M \otimes_A - : {}_A\mathcal{M}_B \to {}_B\mathcal{M}_B,$$

$$\mathbb{R}_B := \mathrm{Hom}_B(M, -) : {}_B\mathcal{M}_B \to {}_A\mathcal{M}_B$$

and

$$\mathbb{L}_{B^R} := M^R \otimes_{A^R} - : {}_{A^R}\mathcal{M}_{B^R} \to {}_{B^R}\mathcal{M}_{B^R},$$

$$\mathbb{R}_{B^R} := \mathrm{Hom}_{B^R}(M^R, -) : {}_{B^R}\mathcal{M}_{B^R} \to {}_{A^R}\mathcal{M}_{B^R}.$$

Let

$$\mathcal{E}_{M,B} := \{f \in {}_B\mathcal{M}_B \mid \mathrm{Hom}_B(M, f) \text{ is split epimorphic in } {}_A\mathcal{M}_B\}$$

and

$$\mathcal{E}_{M^R,B^R} := \{f \in {}_{B^R}\mathcal{M}_{B^R} \mid \mathrm{Hom}_{B^R}(M^R, f) \text{ is split epimorphic in } {}_{A^R}\mathcal{M}_{B^R}\}.$$

Lemma 3.3 *If ${}_BM_A$ is a generator in ${}_B\mathcal{M}$, then ${}_{B^R}(M^R)_{A^R}$ is a generator in ${}_{B^R}\mathcal{M}$.*

Proof Since ${}_BM_A$ is a generator in ${}_B\mathcal{M}$, B is a direct summand of a sum of copies of ${}_BM$ as a left B-module. Then $B^R = R \otimes B$ is a direct summand of a sum of copies of $M^R = R \otimes M$ as a left B^R-module. The result follows. □

Suppose that ${}_BM_A$ is a generator in ${}_B\mathcal{M}$. Then ${}_{B^R}(M^R)_{A^R}$ is a generator in ${}_{B^R}\mathcal{M}$. Thus both $\mathcal{E}_{M,B}$ and $\mathcal{E}_{M^R,B^R}$ are projective classes of epimorphisms. The module-relative-Hochschild (co)homology of an extended algebra B^R over A^R, with coefficients in any B^R-bimodule Y^R where Y is a B-B-bimodule, is entirely determined by that of B over A with coefficients in ${}_BY_B$.

Theorem 3.4 *Let R be a commutative k-algebra such that R is finitely generated and projective over k. Let A and B be two k-algebras. Consider a bimodule ${}_BM_A$ such that ${}_BM$ is a generator in ${}_B\mathcal{M}$. Then, for all B-B-bimodules Y and $n \geq 0$,*

$$H^n_{\mathcal{E}_{M^R,B^R}}(B^R, Y^R) \simeq R \otimes H^n_{\mathcal{E}_{M,B}}(B, Y), \quad H_n^{\mathcal{E}_{M^R,B^R}}(B^R, Y^R) \simeq R \otimes H_n^{\mathcal{E}_{M,B}}(B, Y).$$

Moreover, when k is a field,

$$\mathrm{hch.dim}_{M^R}(B^R) = \mathrm{hch.dim}_M(B), \quad \mathrm{hh.dim}_{M^R}(B^R) = \mathrm{hh.dim}_M(B).$$

Proof Let $\mathbb{P}_B$ be the standard $\mathcal{E}_{M,B}$-projective resolution of B in ${}_B\mathcal{M}_B$. Let $\mathbb{P}_{B^R}$ be the standard $\mathcal{E}_{M^R,B^R}$-projective resolution of B^R in ${}_{B^R}\mathcal{M}_{B^R}$. Note that

$$\begin{aligned}(\mathbb{L}_{B^R}\mathbb{R}_{B^R})(X^R) &= {}_{B^R}(M^R)\otimes_{A^R}\mathrm{Hom}_{B^R}((M^R)_{A^R}, X^R)\\ &\simeq {}_{B^R}(M^R)\otimes_{A^R}\mathrm{Hom}_B(M_A, X)^R\\ &\simeq (M\otimes_A \mathrm{Hom}_B(M_A, X))^R\\ &= ((\mathbb{L}_B\mathbb{R}_B)(X))^R\end{aligned}$$

for all B-B-bimodules X. In particular, we have

$$(\mathbb{L}_{B^R}\mathbb{R}_{B^R})^n(B^R)\simeq((\mathbb{L}_B\mathbb{R}_B)^n(B))^R.$$

Firstly, we apply the functor $-\otimes_{(B^R)^e} Y$ to $\mathbb{P}_{B^R}$. Note that

$$(B^R)^e = B^R\otimes_R (B^R)^{op}\simeq (B^e)^R.$$

We have

$$\begin{aligned}(\mathbb{L}_{B^R}\mathbb{R}_{B^R})^n(B^R)\otimes_{(B^R)^e} Y^R &\simeq ((\mathbb{L}_B\mathbb{R}_B)^n(B))^R\otimes_{(B^R)^e} Y^R\\ &\simeq ((\mathbb{L}_B\mathbb{R}_B)^n(B))^R\otimes_{(B^e)^R} Y^R\\ &\simeq R\otimes((\mathbb{L}_B\mathbb{R}_B)^n(B)\otimes_{B^e} Y)\end{aligned}$$

where the third isomorphism follows from Lemma 3.2(2). Since R is projective over k, $R\otimes -$ preserves monomorphisms and kernels. Thus one can easily get

$$\mathrm{H}_n^{\mathcal{E}_{M^R,B^R}}(B^R, Y^R)\simeq R\otimes \mathrm{H}_n^{\mathcal{E}_{M,B}}(B, Y)$$

for all $n\geq 0$.

Secondly, applying the functor $\mathrm{Hom}_{(B^R)^e}(-, Y^R)$ to $\mathbb{P}_{B^R}$, we have

$$\begin{aligned}\mathrm{Hom}_{(B^R)^e}((\mathbb{L}_{B^R}\mathbb{R}_{B^R})^n(B^R), Y^R) &\simeq \mathrm{Hom}_{(B^e)^R}(((\mathbb{L}_B\mathbb{R}_B)^n(B))^R, Y^R)\\ &\simeq R\otimes \mathrm{Hom}_{B^e}((\mathbb{L}_B\mathbb{R}_B)^n(B), Y)\end{aligned}$$

where the second isomorphism follows from Lemma 3.2(1). By the same arguments as above, we conclude that

$$\mathrm{H}^n_{\mathcal{E}_{M^R,B^R}}(B^R, Y^R)\simeq R\otimes \mathrm{H}^n_{\mathcal{E}_{M,B}}(B, Y)$$

for all $n\geq 0$.

It remains to prove the last statement. Suppose that k is a field. Since R is finitely generated over k, one can easily check that

$$\mathrm{H}^n_{\mathcal{E}_{M^R,B^R}}(B^R, Y^R)\simeq R\otimes \mathrm{H}^n_{\mathcal{E}_{M,B}}(B, Y)=0 \Longleftrightarrow \mathrm{H}^n_{\mathcal{E}_{M,B}}(B, Y)=0,$$

$$\mathrm{H}_n^{\mathcal{E}_{M^R,B^R}}(B^R, Y^R)\simeq R\otimes \mathrm{H}_n^{\mathcal{E}_{M,B}}(B, Y)=0 \Longleftrightarrow \mathrm{H}_n^{\mathcal{E}_{M,B}}(B, Y)=0.$$

This completes the proof. □

Corollary 3.5 *Let R be a commutative k-algebra such that R is finitely generated and projective* *over k. Let A and B be two arbitrary k-algebras. Consider a bimodule ${}_BM_A$ such that ${}_BM$ is a generator in ${}_B\mathcal{M}$. Then for all $n \geq 0$, we have*

$$H^n_{\mathcal{E}_{M^R,B^R}}(B^R) \simeq R \otimes H^n_{\mathcal{E}_{M,B}}(B)$$

and

$$H_n^{\mathcal{E}_{M^R,B^R}}(B^R) \simeq R \otimes H_n^{\mathcal{E}_{M,B}}(B).$$

Recall in [1] that B is ${}_BM_A$-separable if and only if

$$\text{hch.dim}_M(B) = 0;$$

and that B is M-smooth if and only if

$$\text{hch.dim}_M(B) \leqslant 1.$$

By Theorem 3.4, we directly obtain the following corollary.

Corollary 3.6 *When k is a field, B^R is M^R-smooth (resp. M^R-separable) if and only if B is M-smooth (resp. M-separable).*

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