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Multiple Blow-up Rates to a Coupling Heat System

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Abstract This paper deals with a heat system coupled via local and localized sources subject to null Dirichlet boundary conditions. Based on a complete classification for all the four nonlinear parameters, we establish multiple blow-up rates for the system under various dominations. We also determine uniform blow-up profiles for the three cases where localized source couplings dominate the system.

Keywords coupled localized sources; coupled local sources; uniform blow-up profile; blow-up rate.

MR(2010) Subject Classification 35K55; 35B33

1. Introduction

In this paper, we consider the following heat system coupled via local and localized sources

$$\begin{cases} u_t = \Delta u + v^{p_1} + v^{q_1}(0,t), & (x,t) \in \Omega \times (0,T), \\ v_t = \Delta v + u^{p_2} + u^{q_2}(0,t), & (x,t) \in \Omega \times (0,T), \\ u = v = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

where $\Omega = B = \{x \in R^N : |x| < 1\}$, $p_1, p_2 > 1$, $q_1, q_2 > 0$; $u_0, v_0 \in C^2(\Omega) \cap C(\bar{\Omega})$ are radial and satisfy

$$\text{(A)} \qquad \begin{cases} u_0 = u_0(r), v_0 = v_0(r),\ u_0, v_0 \ge 0,\ u_0(0), v_0(0) > 1; \\ u_0(1) = v_0(1) = 0,\ u_{0r}, v_{0r} < 0 \text{ for } r \in (0,1], \end{cases}$$

and

$$\text{(B)} \qquad \begin{cases} \Delta u_0 + v_0^{p_1} + v_0^{q_1}(0) \ge \eta\varphi_0(v_0^{p_1} + v_0^{q_1}(0), & x \in \bar{B}; \\ \Delta v_0 + u_0^{p_2} + u_0^{q_2}(0) \ge \eta\varphi_0(u_0^{p_2} + u_0^{q_2}(0)), & x \in \bar{B}, \end{cases}$$

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where $\eta \in (0, \frac{1}{2}]$, $\varphi_0 \in C^2(B) \cap C(\bar{B})$ is the first eigenfunction of

$$\Delta\varphi + \lambda\varphi = 0 \text{ in } B, \quad \varphi = 0 \text{ on } \partial B, \tag{1.2}$$

with the first eigenvalue λ_0, normalized by $\varphi_0 > 0$ in B and $\|\varphi_0\|_\infty = 1$. It is easy to see that φ_0 is a radially symmetric with $\varphi_0' < 0$ for $r \in (0, 1]$. Such u_0 and v_0 do exist indeed [7, 14].

It is well known that there exists a unique local solution to (1.1), which blows up in finite time for large initial data [1–3]. Denote by T the maximum existence time of the solution.

System (1.1) can be viewed as a combination of the following two coupled problems: the system with local coupling

$$u_t = \Delta u + v^{p_1},\ v_t = \Delta v + u^{p_2}, \quad (x, t) \in \Omega \times (0, T), \tag{1.3}$$

and the system with localized coupling

$$u_t = \Delta u + v^{q_1}(0, t),\ v_t = \Delta v + u^{q_2}(0, t), \quad (x, t) \in \Omega \times (0, T), \tag{1.4}$$

subject to null Dirichlet boundary conditions. It was known that the blow-up solutions of (1.3) with $p_1p_2 > 1$ must be single point blow-up [3, 8]. While for (1.4) with $q_1q_2 > 1$, the blow-up occurs everywhere in $\Omega = B$ (see [6]), where the uniform blow-up profile was observed. It is easy to understand the system (1.1) may admit both single point blow-up and uniform blow-up profiles.

In this paper, we will study the multiple blow-up rates for (1.1), by using the scaling technique [5], under various dominations. To get a complete classification for the discussion, introduce the following characteristic algebraic system [12, 15] associate with (1.1):

$$\begin{pmatrix} -1 & \theta_1 p_1 + (1-\theta_1) q_1 \\ \theta_2 p_2 + (1-\theta_2) q_2 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.5}$$

with $\theta_1, \theta_2 \in \{0, 1\}$, namely,

$$(\alpha, \beta) = \begin{cases} (\alpha_1, \beta_1) = \Big(\dfrac{p_1+1}{p_1p_2-1}, \dfrac{p_2+1}{p_1p_2-1}\Big) & \text{for } \theta_1 = 1,\ \theta_2 = 1; \\ (\alpha_2, \beta_2) = \Big(\dfrac{p_1+1}{p_1q_2-1}, \dfrac{q_2+1}{p_1q_2-1}\Big) & \text{for } \theta_1 = 1,\ \theta_2 = 0; \\ (\alpha_3, \beta_3) = \Big(\dfrac{q_1+1}{p_2q_1-1}, \dfrac{p_2+1}{p_2q_1-1}\Big) & \text{for } \theta_1 = 0,\ \theta_2 = 1; \\ (\alpha_4, \beta_4) = \Big(\dfrac{q_1+1}{q_1q_2-1}, \dfrac{q_2+1}{q_1q_2-1}\Big) & \text{for } \theta_1 = 0,\ \theta_2 = 0. \end{cases} \tag{1.6}$$

It will be shown that all possible blow-up rates can be described via such (α_i, β_i), $i = 1, \dots, 4$.

We need the auxiliary function ϕ solving heat equation

$$\phi_t = \Delta\phi \text{ in } B \times R^+, \quad \phi = 0 \text{ on } \partial B, \quad \phi(x, 0) = \varphi_0(x) \text{ on } \bar{B}. \tag{1.7}$$

The maximum principle yields

$$\sup_{B \times R^+} |\phi| \le 1. \tag{1.8}$$

Next, we will deal with the multiple blow-up rates in Section 2, and then consider the uniform blow-up profiles in Section 3.

2. Multiple blow-up rates

The maximum principle with the assumptions (A) and (B) implies that u, v are radial, and $\max_{[0,1]} u(\cdot,t) = u(0,t)$, $\max_{[0,1]} v(\cdot,t) = v(0,t)$ for $t \in (0,T)$, $u_t, v_t \geq 0$ for $(x,t) \in \bar{B} \times [0,T)$. We have furthermore:

Lemma 2.1 *The solution (u,v) of (1.1) satisfies*

$$u_t \geq \eta\phi[v^{p_1} + v^{q_1}(0,t)],\ v_t \geq \eta\phi[u^{p_2} + u^{q_2}(0,t)], \quad (x,t) \in \bar{B} \times [0,T) \tag{2.1}$$

with $\eta \leq 1/2$.

Proof Introduce auxiliary functions

$$I(x,t) = u_t - \eta\phi[v^{p_1} + v^{q_1}(0,t)],\ J(x,t) = v_t - \eta\phi[u^{p_2} + u^{q_2}(0,t)]$$

with ϕ defined by (1.7). A simple computation shows

$$I_t - \Delta I - p_1 v^{p_1-1} J \geq 0,\ q_1 v^{q_1-1}(0,t) v_t(0,t)(1-\eta\phi) + 2\eta p_1 v^{p_1-1} \nabla v \cdot \nabla \phi.$$

Notice that $\nabla v \cdot \nabla \phi \geq 0$, since both v and ϕ are radially symmetric and monotonically decreasing with respect to $r = |x|$, and $v_t(0,t) \geq 0$. We have

$$I_t - \Delta I - p_1 v^{p_1-1} J \geq 0, \quad (x,t) \in B \times (0,T), \tag{2.2}$$

and similarly,

$$J_t - \Delta J - p_2 u^{p_2-1} I \geq 0, \quad (x,t) \in B \times (0,T). \tag{2.3}$$

On the other hand,

$$I = J = 0 \text{ on } \partial B \times [0,T) \tag{2.4}$$

due to $\phi = u = v = 0$ on $\partial B \times [0,T)$. The assumption (B) yields

$$I(x,0) = \Delta u_0 + v_0^{p_1}(x) + v_0^{q_1}(0) - \eta\varphi_0[v_0^{p_1}(x) + v_0^{q_1}(0)] \geq 0, \quad x \in \bar{B}, \tag{2.5}$$

$$J(x,0) = \Delta v_0 + u_0^{p_2}(x) + u_0^{q_2}(0) - \eta\varphi_0[u_0^{p_2}(x) + u_0^{q_2}(0)] \geq 0, \quad x \in \bar{B}. \tag{2.6}$$

The maximum principle with (2.2)–(2.6) concludes that $I, J \geq 0$ on $\bar{B} \times [0,T)$. $\Box$

Lemma 2.2 *Let (u,v) be a blow-up solution of (1.1). Then*

$$c \leq u^{-\frac{1}{2\alpha}}(0,t) v^{\frac{1}{2\beta}}(0,t) \leq C, \quad t \in (0,T) \tag{2.7}$$

where $(\alpha,\beta) = (\alpha_i,\beta_i)$, $i = 1,\ldots,4$, are defined by (1.6), and c and C denote positive constants independent of t, which may be different from line to line throughout the paper.

Proof Notice that $u(0,t), v(0,t)$ are nondecreasing in $(0,T)$ and any blow-up in (1.1) must be simultaneous. Thus, $\|u(\cdot,t)\|_\infty = u(0,t)$, $\|v(\cdot,t)\|_\infty = v(0,t)$ tend to infinity monotonously as $t \to T^-$.

We follow the technique in [4, 13]. If the lower bound estimate in (2.7) does not hold, then there exists a sequence $t_j \to T^-$ as $j \to +\infty$ such that

$$u^{-\frac{1}{2\alpha}}(0,t) v^{\frac{1}{2\beta}}(0,t) \to 0 \text{ as } j \to +\infty.$$

Let $\lambda_j = u^{-\frac{1}{2\alpha}}(0,t_j)$. Since $\alpha > 0$, $u(0,t_j)$ diverges as $j \to +\infty$, it follows that $\lambda_j = u^{-\frac{1}{2\alpha}}(0,t_j) \to 0$ as $j \to +\infty$. Scale (u,v) to $(\varphi^{\lambda_j}, \psi^{\lambda_j})$ as

$$\varphi^{\lambda_j}(y,s) = \lambda_j^{2\alpha} u(\lambda_j y, \lambda_j^2 s + t_j),\ \psi^{\lambda_j}(y,s) = \lambda_j^{2\beta} v(\lambda_j y, \lambda_j^2 s + t_j) \tag{2.8}$$

for $(y,s) \in \bar{B}_{\lambda_j} \times (-t_j/\lambda_j^2, (T-t_j)/\lambda_j^2)$ with $B_{\lambda_j} = \{y \in \mathbb{R}^N : \lambda_j y \in B\}$.

For $s \in (-t_j/\lambda_j^2, 0]$, we have $0 \le \varphi^{\lambda_j} \le 1$, $\varphi^{\lambda_j}(0,0) = 1$,

$$0 \le \psi^{\lambda_j} \le (u(0,t_j))^{-\frac{\beta}{\alpha}} v(0,t_j) \to 0, \quad j \to +\infty. \tag{2.9}$$

Moreover, $(\varphi^{\lambda_j}, \psi^{\lambda_j})$ solves

$$\begin{cases} \varphi_s = \Delta\varphi + \lambda_j^{2+2\alpha-2p_1\beta}\psi^{p_1} + \lambda_j^{2+2\alpha-2q_1\beta}\psi^{q_1}(0,s), \\ \psi_s = \Delta\psi + \lambda_j^{2+2\beta-2p_2\alpha}\varphi^{p_2} + \lambda_j^{2+2\beta-2q_2\alpha}\varphi^{q_2}(0,s). \end{cases} \tag{2.10}$$

If $p_1 \ge q_1$, $p_2 \ge q_2$, then $\theta_1 = \theta_2 = 1$, i.e., $(\alpha,\beta) = (\alpha_1,\beta_1) = (\frac{p_1+1}{p_1p_2-1}, \frac{p_2+1}{p_1p_2-1})$, and thus for $j \to \infty$,

$$\begin{aligned} &\mu_1 = 2 + 2\alpha - 2p_1\beta = 0, && \varepsilon_1 = \lambda_j^{\mu_1} = 1; \\ &\mu_2 = 2 + 2\alpha - 2q_1\beta \ge 0, && \varepsilon_2 = \lambda_j^{\mu_2} \in \{0,1\}; \\ &\mu_3 = 2 + 2\beta - 2p_2\alpha = 0, && \varepsilon_3 = \lambda_j^{\mu_3} = 1; \\ &\mu_4 = 2 + 2\beta - 2q_2\alpha \ge 0, && \varepsilon_4 = \lambda_j^{\mu_4} \in \{0,1\}. \end{aligned}$$

If $p_1 \ge q_1$, $p_2 \le q_2$, then $\theta_1 = 1, \theta_2 = 0$, i.e., $(\alpha,\beta) = (\alpha_2,\beta_2)$, and

$$\mu_1 = \mu_4 = 0,\ \varepsilon_1 = \varepsilon_4 = 1; \quad \mu_2, \mu_3 \ge 0,\ \varepsilon_2, \varepsilon_3 \in \{0,1\}.$$

If $p_1 \le q_1$, $p_2 \ge q_2$, then $\theta_1 = 0, \theta_2 = 1$, i.e., $(\alpha,\beta) = (\alpha_3,\beta_3)$, and

$$\mu_2 = \mu_3 = 0,\ \varepsilon_2 = \varepsilon_3 = 1; \quad \mu_1, \mu_4 \ge 0,\ \varepsilon_1, \varepsilon_4 \in \{0,1\}.$$

If $p_1 \le q_1$, $p_2 \le q_2$, then $\theta_1 = 0, \theta_2 = 0$, i.e., $(\alpha,\beta) = (\alpha_4,\beta_4)$, and

$$\mu_2 = \mu_4 = 0,\ \varepsilon_2 = \varepsilon_4 = 1; \quad \mu_1, \mu_3 \ge 0,\ \varepsilon_1, \varepsilon_3 \in \{0,1\}.$$

The general parabolic estimates yield a subsequence converging uniformly on compact subsets of $R^N \times (-\infty, 0]$ to $(\widetilde{\varphi}, \widetilde{\psi})$ such that

$$\begin{cases} \widetilde{\varphi}_s = \Delta\widetilde{\varphi} + \varepsilon_1\widetilde{\psi}^{p_1} + \varepsilon_2\widetilde{\psi}^{q_1}(0,s), & (y,s) \in R^N \times (-\infty,0], \\ \widetilde{\psi}_s = \Delta\widetilde{\psi} + \varepsilon_3\widetilde{\varphi}^{p_2} + \varepsilon_4\widetilde{\varphi}^{q_2}(0,s), & (y,s) \in R^N \times (-\infty,0] \end{cases}$$

with $\varepsilon_i = 0$ or 1 $(i = 1,2,3,4)$, and there always exist $i \in \{1,2\}, j \in \{3,4\}$ such that $\varepsilon_i = \varepsilon_j = 1$. On the other hand, $\widetilde{\psi} \equiv 0$ by (2.9). This contradicts the second equation with $\widetilde{\varphi}(0,0) = 1$.

If the upper bound estimate in (2.7) does not hold, then there exists a sequence $t_j \to T^-$ as $j \to +\infty$ such that

$$u^{-\frac{1}{2\alpha}}(0,t)v^{\frac{1}{2\beta}}(0,t) \to +\infty \text{ as } j \to +\infty.$$

Let $\lambda_j = v^{-\frac{1}{2\beta}}(0,t_j)$, and define $(\varphi^{\lambda_j}, \psi^{\lambda_j})$ as (2.8). Then $(\varphi^{\lambda_j}, \psi^{\lambda_j})$ is the solution of (2.10), such that

$$0 \le \psi^{\lambda_j} \le 1,\ \psi^{\lambda_j}(0,0) = 1,\ 0 \le \varphi^{\lambda_j} \le u(0,t_j)(v(0,t_j))^{-\frac{\alpha}{\beta}} \to 0,\ j \to +\infty.$$

Proceeding as before, we will get a contradiction. Thus (2.7) is established. □

Next, we study blow-up rates of maximum point for solutions to (1.1), which would be helpful for the further study on uniform blow-up profiles of solutions. The sources for u (v) in the model consist of v^{p_1} and $v^{q_1}(0,t)$ $(u^{p_2}$ and $u^{q_2}(0,t))$. There are four different dominations of the sources, corresponding to four possible simultaneous blow-up rates of solutions. All these are clearly described via the characteristic algebraic system (1.6). In the sequel, always denote by T the blow-up time for (1.1).

Theorem 2.1 *Let (u,v) be a blow-up solution of (1.1). Then there are positive constants c, C such that*

$$c \le u(0,t)(T-t)^{\alpha} \le C, \quad c \le v(0,t)(T-t)^{\beta} \le C, \quad t \in (0,T), \tag{2.11}$$

where $(\alpha,\beta) = (\alpha_i,\beta_i)$, $i = 1,\dots,4$, are defined by (1.6).

Proof Without loss of generality, we only consider the case with v^{p_1}, u^{p_2} dominating the system, i.e., $p_1 \ge q_1$, $p_2 \ge q_2$. Thus, $(\alpha,\beta) = (\alpha_1,\beta_1)$, defined by (1.6). For the component u, notice that $\max_{\bar{B}} u(\cdot,t) = u(0,t)$ implies $\Delta u(0,t) \le 0$ and $u(0,t)$ blows up as $t \to T$. We have from the first equation of (1.1) that

$$u_t(0,t) \le v^{p_1}(0,t) + v^{q_1}(0,t) \le 2v^{p_1}(0,t).$$

By Lemma 2.2 and the assumption of the theorem, we know $v(0,t) \le Cu^{\frac{\beta_1}{\alpha_1}}(0,t) = Cu^{\frac{p_2+1}{p_1+1}}$, and thus

$$u_t(0,t) \le Cu^{\frac{p_1(p_2+1)}{p_1+1}}(0,t) \text{ as } t \to T. \tag{2.12}$$

It follows from (2.12) that

$$u(0,t) \ge c(T-t)^{-\frac{p_1+1}{p_1p_2-1}} \text{ as } t \to T.$$

On the other hand, Lemma 2.1 says

$$\begin{aligned} u_t(0,t) &\ge \eta\phi(0,t)[v^{p_1} + v^{q_1}(0,t)] \\ &\ge \eta\phi(0,t)v^{p_1}(0,t) \ge \eta\phi(0,t)cu^{\frac{p_1(p_2+1)}{p_1+1}}(0,t), \end{aligned}$$

and so $u(0,t) \le C(T-t)^{-\frac{p_1+1}{p_1p_2-1}} = C(T-t)^{-\alpha_1}$ is true. For the component v, similarly to above, we also have

$$c \le v(0,t)(T-t)^{\beta_1} \le C. \ \square$$

3. Uniform blow-up profiles

This section considers uniform blow-up profiles of solutions to (1.1). We will use the technique in [9–11] with Theorem 2.1 to establish the uniform blow-up profiles of solutions. There are three cases to be considered: (a) $p_1 \ge q_1$, $p_2 < q_2$; (b) $p_1 < q_1$, $p_2 \ge q_2$; (c) $p_1 < q_1$, $p_2 < q_2$.

Let us first treat the case (a) with $p_1 \ge q_1$, $p_2 < q_2$, where v^{p_1} and $u^{q_2}(0,t)$ play a dominance role:

Theorem 3.1 *(i) If $p_1 > q_1$, $p_2 < q_2$, then*

$$\lim_{t\to T}(T-t)^{\frac{p_1+1}{p_1q_2-1}}u(x,t)=\Big(\frac{q_2+1}{p_1+1}\Big)^{\frac{p_1}{p_1q_2-1}}\Big(\frac{p_1+1}{p_1q_2-1}\Big)^{\frac{p_1+1}{p_1q_2-1}}, \tag{3.1}$$

$$\lim_{t\to T}(T-t)^{\frac{q_2+1}{p_1q_2-1}}v(x,t)=\Big(\frac{p_1+1}{q_2+1}\Big)^{\frac{q_2}{p_1q_2-1}}\Big(\frac{q_2+1}{p_1q_2-1}\Big)^{\frac{q_2+1}{p_1q_2-1}} \tag{3.2}$$

uniformly on compact subsets of Ω.

(ii) If $p_1 = q_1$, $p_2 < q_2$, then

$$\lim_{t\to T}(T-t)^{\frac{p_1+1}{p_1q_2-1}}u(x,t)=2^{-\frac{1}{p_1q_2-1}}\Big(\frac{q_2+1}{p_1+1}\Big)^{\frac{p_1}{p_1q_2-1}}\Big(\frac{p_1+1}{p_1q_2-1}\Big)^{\frac{p_1+1}{p_1q_2-1}}, \tag{3.3}$$

$$\lim_{t\to T}(T-t)^{\frac{q_2+1}{p_1q_2-1}}v(x,t)=2^{-\frac{q_2}{p_1q_2-1}}\Big(\frac{p_1+1}{q_2+1}\Big)^{\frac{q_2}{p_1q_2-1}}\Big(\frac{q_2+1}{p_1q_2-1}\Big)^{\frac{q_2+1}{p_1q_2-1}} \tag{3.4}$$

uniformly on compact subsets of Ω.

Proof (i) By Theorem 2.1 with $p_1 \ge q_1$ and $p_2 \le q_2$,

$$c\le u(0,t)(T-t)^{\alpha_2}\le C,\ \ c\le v(0,t)(T-t)^{\beta_2}\le C,\quad t\in(0,T).$$

Set

$$F(t)=\int_0^t v^{p_1}(0,\tau)\mathrm{d}\tau,\quad G(t)=\int_0^t u^{q_2}(0,\tau)\mathrm{d}\tau, \tag{3.5}$$

and hence $F(t), G(t)\to\infty$, as $t\to T^-$. Since $\Delta v(0,t)\le 0$ by $v(0,t)=\max_{\bar\Omega}u(\cdot,t)$, it follows that

$$v_t(0,t)\le u^{p_2}(0,t)+u^{q_2}(0,t),\quad 0<t<T. \tag{3.6}$$

Integrate (3.6) over $(0,t)$ to get

$$v(0,t)-v_0(0)\le\int_0^t u^{p_2}(0,s)\mathrm{d}s+\int_0^t u^{q_2}(0,s)\mathrm{d}s,\quad 0<t<T,$$

which implies

$$\limsup_{t\to T}\frac{v(0,t)}{\int_0^t u^{p_2}(0,s)\mathrm{d}s+G(t)}\le 1.$$

Since $p_2<q_2$, we have

$$\lim_{t\to T}\frac{\int_0^t u^{p_2}(0,s)\mathrm{d}s}{G(t)}=0.$$

So, there holds

$$\limsup_{t\to T}\frac{v(0,t)}{G(t)}\le 1. \tag{3.7}$$

Let λ_0 and ψ_0 be the first eigenvalue and eigenfunction of $-\Delta$ with the null Dirichlet boundary condition, normalized by $\int_\Omega\psi_0(x)\mathrm{d}x=1$. Multiplying the second equation of (1.1) by ψ_0, and then integrating over $Q_t=\Omega\times(0,t)$ for $0<t<T$, we obtain

$$\int_\Omega v\psi_0\mathrm{d}x-\int_\Omega v_0\psi_0\mathrm{d}x=-\lambda_0\iint_{Q_t}v\psi_0\mathrm{d}x\mathrm{d}s+\iint_{Q_t}u^{p_2}\psi_0\mathrm{d}x\mathrm{d}s+G(t). \tag{3.8}$$

By (i), we know $v(0,t) \geq cu^{\frac{q_2+1}{p_1+1}}(0,t)$, and thus

$$0 \leq \lim_{t\to T} \frac{\iint_{Q_t} v\psi_0 \mathrm{d}x\mathrm{d}s}{G(t)} \leq \lim_{t\to T} \frac{\int_0^t v(0,s)\mathrm{d}s}{G(t)} = 0,$$

$$0 \leq \lim_{t\to T} \frac{\iint_{Q_t} u^{p_2}\psi_0 \mathrm{d}x\mathrm{d}s}{G(t)} \leq 0.$$

Combining (3.8) gives

$$\liminf_{t\to T} \frac{v(0,t)}{G(t)} \geq \lim_{t\to T} \frac{\int_\Omega v\psi_0 \mathrm{d}x}{G(t)} = 1. \tag{3.9}$$

Due to (3.7) and (3.9), we conclude

$$\lim_{t\to T} \frac{v(0,t)}{G(t)} = 1, \tag{3.10}$$

namely,

$$v(0,t) \sim G(t), \quad t \to T. \tag{3.11}$$

On the other hand, by (3.9) and (3.10),

$$\lim_{t\to T} \frac{\int_\Omega v\psi_0 \mathrm{d}x}{v(0,t)} = 1,$$

and hence

$$\lim_{t\to T} \frac{v(x,t)}{v(0,t)} = 1 \quad \text{for a.e. } x \in \Omega$$

due to $\int_\Omega \psi_0 \mathrm{d}x = 1$. Since $u_r \leq 0$, we have furthermore

$$v(x,t) \sim v(0,t) \sim G(t), \ x \in \Omega, \quad t \to T. \tag{3.12}$$

Similarly to (3.7), we have

$$\limsup_{t\to T} \frac{u(0,t)}{F(t)} \leq 1. \tag{3.13}$$

Multiplying the first equation of (1.1) by ψ_0, and then integrating over $Q_t = \Omega \times (0,t)$ for $t \in (0,T)$, we obtain

$$\int_\Omega u\psi_0 \mathrm{d}x - \int_\Omega u_0\psi_0 \mathrm{d}x = -\lambda_0 \iint_{Q_t} u\psi_0 \mathrm{d}x\mathrm{d}s + \iint_{Q_t} v^{p_1}\psi_0 \mathrm{d}x\mathrm{d}s + \iint_{Q_t} v^{q_1}(0,t)\psi_0 \mathrm{d}x\mathrm{d}s. \tag{3.14}$$

Due to $u(0,t) \geq cv^{\frac{p_1+1}{q_2+1}}(0,t)$ by (i), we have

$$\lim_{t\to T} \frac{\iint_{Q_t} u\psi_0 \mathrm{d}x\mathrm{d}s}{F(t)} = 0. \tag{3.15}$$

We know from (3.12) that $\int_0^t v^{p_1}(x,s)\mathrm{d}s \sim \int_0^t u^{p_1}(0,s)\mathrm{d}s$ uniformly on compact subsets of $\Omega = B_1$. Denoting $\Omega_n = B_{1-1/n}$, we have

$$\lim_{t\to T} \frac{\iint_{Q_t} v^{p_1}\psi_0 \mathrm{d}x\mathrm{d}s}{F(t)} = \lim_{n\to+\infty} \int_{\Omega_n} \lim_{t\to T} \frac{\int_0^t v^{p_1}(x,s)\mathrm{d}s}{F(t)} \psi_0 \mathrm{d}x = 1. \tag{3.16}$$

It follows from (3.14)–(3.16) that

$$\liminf_{t\to T} \frac{u(0,t)}{F(t)} \geq \lim_{t\to T} \frac{\int_\Omega u\psi_0 \mathrm{d}x}{F(t)} = 1. \tag{3.17}$$

Combining (3.13) with (3.17), we conclude

$$u(0,t) \sim F(t), \quad t \to T. \tag{3.18}$$

Similarly to above, we have

$$\lim_{t\to T} \frac{u(x,t)}{u(0,t)} = 1 \text{ for a.e. } x \in \Omega,$$

and thus

$$u(x,t) \sim u(0,t) \sim F(t), \quad x \in \Omega,\ t \to T \tag{3.19}$$

due to $u_r(r,t) \le 0$. In summary of (3.11), (3.18) and (3.5),

$$F'(t) \sim G^{p_1}(t),\ G'(t) \sim F^{q_2}(t), \quad t \to T. \tag{3.20}$$

It follows from (3.20) that $G(t) \sim (\frac{p_1+1}{q_2+1})^{\frac{1}{p_1+1}} F^{\frac{q_2+1}{p_1+1}}(t)$ $(t \to T)$, and consequently,

$$\lim_{t\to T}(T-t)^{\frac{p_1+1}{p_1q_2-1}}F(t) = \Big(\frac{q_2+1}{p_1+1}\Big)^{\frac{p_1}{p_1q_2-1}}\Big(\frac{p_1+1}{p_1q_2-1}\Big)^{\frac{p_1+1}{p_1q_2-1}},$$
$$\lim_{t\to T}(T-t)^{\frac{q_2+1}{p_1q_2-1}}G(t) = \Big(\frac{p_1+1}{q_2+1}\Big)^{\frac{q_2}{p_1q_2-1}}\Big(\frac{q_2+1}{p_1q_2-1}\Big)^{\frac{q_2+1}{p_1q_2-1}}.$$

Combined with (3.12) and (3.19), the required uniform blow-up profiles are proved.

(ii) Similarly to (3.12),

$$v(x,t) \sim v(0,t) \sim G(t), \quad x \in \Omega,\ t \to T. \tag{3.21}$$

By (3.6), we get

$$\limsup_{t\to T} \frac{u(0,t)}{F(t)} \le 2. \tag{3.22}$$

On the other hand, multiplying the first equation of (1.1) by ψ_0, and then integrating over $Q_t = \Omega \times (0,t)$ for $t \in (0,T)$, we have

$$\int_\Omega u\psi_0 \mathrm{d}x - \int_\Omega u_0\psi_0 \mathrm{d}x = -\lambda_0 \iint_{Q_t} u\psi_0 \mathrm{d}x\mathrm{d}s + \iint_{Q_t} v^{p_1}\psi_0 \mathrm{d}x\mathrm{d}s + \iint_{Q_t} v^{q_1}(0,t)\psi_0 \mathrm{d}x\mathrm{d}s. \tag{3.23}$$

Repeating the argument for (i), we can get

$$\liminf_{t\to T} \frac{u(0,t)}{F(t)} \ge \lim_{t\to T} \frac{\int_\Omega u\psi_0 \mathrm{d}x}{F(t)} = 2. \tag{3.24}$$

Combining (3.22) with (3.24) gives

$$u(0,t) \sim 2F(t), \quad t \to T. \tag{3.25}$$

Similarly to (3.10),

$$u(x,t) \sim u(0,t) \sim 2F(t), \quad x \in \Omega,\ t \to T \tag{3.26}$$

due to $u_r(r,t) \le 0$. In summary of (3.21), (3.25) and (3.5),

$$F'(t) \sim G^{p_1}(t),\ G'(t) \sim (2F)^{q_2}(t), \quad t \to T. \tag{3.27}$$

Clearly, (3.27) implies that $G(t) \sim 2^{\frac{q_2}{p_1+1}}(\frac{p_1+1}{q_2+1})^{\frac{1}{p_1+1}} F^{\frac{q_2+1}{p_1+1}}(t)$ $(t \to T)$. Combining with (3.21) and (3.26), we obtain

$$\lim_{t\to T}(T-t)^{\frac{p_1+1}{p_1q_2-1}}u(x,t) = 2^{-\frac{1}{p_1q_2-1}}\Big(\frac{q_2+1}{p_1+1}\Big)^{\frac{p_1}{p_1q_2-1}}\Big(\frac{p_1+1}{p_1q_2-1}\Big)^{\frac{p_1+1}{p_1q_2-1}},$$

$$\lim_{t\to T}(T-t)^{\frac{q_2+1}{p_1q_2-1}}v(x,t) = 2^{-\frac{q_2}{p_1q_2-1}}\Big(\frac{p_1+1}{q_2+1}\Big)^{\frac{q_2}{p_1q_2-1}}\Big(\frac{q_2+1}{p_1q_2-1}\Big)^{\frac{q_2+1}{p_1q_2-1}}.$$

This completes the proof. □

The case (b) with $p_1 < q_1$, $p_2 \geq q_2$ can be treated by exchanging the roles of u and v in Theorem 3.1.

Finally we consider the third situation with $v^{q_1}(0,t)$ and $u^{q_2}(0,t)$ dominating the system. That is the following theorem. The proof is similar to (i) of Theorem 3.1, and omitted here.

Theorem 3.2 *Assume $p_1 < q_1$, $p_2 < q_2$. Then there holds*

$$\lim_{t\to T}(T-t)^{\frac{q_1+1}{q_1q_2-1}}u(x,t) = \Big(\frac{q_2+1}{q_1+1}\Big)^{\frac{q_1}{q_1q_2-1}}\Big(\frac{q_1+1}{q_1q_2-1}\Big)^{\frac{q_1+1}{q_1q_2-1}},$$

$$\lim_{t\to T}(T-t)^{\frac{q_2+1}{q_1q_2-1}}v(x,t) = \Big(\frac{q_1+1}{q_2+1}\Big)^{\frac{q_2}{q_1q_2-1}}\Big(\frac{q_2+1}{q_1q_2-1}\Big)^{\frac{q_2+1}{q_1q_2-1}}$$

uniformly on all compact subsets of Ω. □

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