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Numerical Stability of Differential Equations with Piecewise Constant Arguments of Mixed Type

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Abstract This paper deals with the stability analysis of the Euler-Maclaurin method for differential equations with piecewise constant arguments of mixed type. The expression of analytical solution is derived and the stability regions of the analytical solution are given. The necessary and sufficient conditions under which the numerical solution is asymptotically stable are discussed. The conditions under which the analytical stability region is contained in the numerical stability region are obtained and some numerical examples are given.

Keywords piecewise constant arguments; Euler-Maclaurin method; stability.

MR(2010) Subject Classification 65L05; 65L07; 65L20

1. Introduction

In this paper we consider the differential equations with piecewise constant arguments (EPCA) of mixed type with the following form:

$$\begin{aligned} &u'(t) = au(t) + a_{-1}u([t-1]) + a_0u([t]) + a_1u([t+1]), \quad t \ge 0,\\ &u(-1) = u_{-1}, \quad u(0) = u_0, \end{aligned} \tag{1}$$

where $a, a_{-1}, a_0, a_1, u_{-1}$ and u_0 are real constants and $[\cdot]$ denotes the greatest integer function and $a_{-1} \neq 0, a_1 \neq 0$.

The study of EPCA was initiated in [1, 2]. From then on, stability, oscillation and existence of periodic solutions have been treated by several authors, see [3–5] and references therein. The extensive applications of EPCA were discussed in [6, 7]. Studies of such equations are motivated by the fact that they represent a hybrid of discrete and continuous dynamical systems and combine the properties of both differential and differential-difference equations. The general theory and basic results for EPCA have been thoroughly investigated in the book of Wiener [8].

In recent years, the numerical computation and analysis for this type of equations have made great progress. Song [9] derived several stability conditions for EPCA of advanced type by using the θ-methods. For the unbounded retarded EPCA, Liu [10] obtained a sufficient condition for the equation to be asymptotically stable and the stability of Runge-Kutta methods were discussed. In [11], the stability and oscillations of the θ-methods for EPCA of alternately advanced and retarded type were considered. However, until now very few results dealing with the

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numerical solution of (1) have been published except for [12] in which the Runge-Kutta methods were discussed. The main objective of this paper is to enrich the gap by considering stability of the Euler-Maclaurin method for (1). In the present paper, we investigate the asymptotical stability of the Euler-Maclaurin method. The necessary and sufficient conditions under which the numerical stability region contains the analytical stability region are obtained. In the end, the numerical examples further illustrate the theoretical results and effectiveness of the method.

2. Asymptotical stability of the analytical solution

In this section, we provide a formula for the solution of (1) and discuss the condition under which the analytical solution is asymptotically stable.

Definition 1 ([8]) *A function $u : [0, \infty) \to \mathbb{R}$ is a solution of (1) if the following conditions hold:*

(a) u is continuous on $[0, \infty)$,

(b) The derivative u' exists at each point $t \in [0, \infty)$, with the possible exception of the points $[t] \in [0, \infty)$ where one-sided derivatives exist,

(c) (1) is satisfied on each interval $[n, n+1) \subset [0, \infty)$ with integral end-points.

By using the similar methods in [8], we have

Theorem 1 *The unique solution of (1) on $[0, \infty)$ is given by*

$$u(t) = \alpha_{-1}(\{t\})c_{[t-1]} + \alpha_0(\{t\})c_{[t]} + \alpha_1(\{t\})c_{[t+1]},$$

where $\{t\}$ is the fractional part of t and

$$c_{[t]} = \frac{\lambda_1^{[t+1]}(u_0 - \lambda_2 u_{-1}) + (\lambda_1 u_{-1} - u_0)\lambda_2^{[t+1]}}{\lambda_1 - \lambda_2},$$

$$\alpha_{-1}(t) = (e^{at} - 1)a^{-1}a_{-1}, \alpha_0(t) = e^{at} + (e^{at} - 1)a^{-1}a_0, \alpha_1(t) = (e^{at} - 1)a^{-1}a_1,$$

$$b_{-1} = \alpha_{-1}(1), b_0 = \alpha_0(1), b_1 = \alpha_1(1),$$

λ_1 and λ_2 are the roots of equation

$$(1 - b_1)\lambda^2 - b_0\lambda - b_{-1} = 0. \tag{2}$$

Theorem 2 ([8]) *That the solution of (1) is asymptotically stable ($u(t) \to 0$ as $t \to \infty$) is equivalent to that the moduli of the roots of (2) satisfy the inequalities*

$$|\lambda_1| < 1, \quad |\lambda_2| < 1.$$

Lemma 1 ([13]) *The moduli of the roots of equation $\lambda^2 - \gamma\lambda - \beta = 0$ are smaller than 1 if and only if $|\beta| < 1$ and $|\gamma| < 1 - \beta$.*

Applying Theorem 2 and Lemma 1 to (2) leads to the following theorem.

Theorem 3 *The solution of (1) is asymptotically stable as $t \to \infty$ if and only if*

$$\begin{cases} (a_1 + a_{-1} - \frac{a}{e^a-1})(a_1 - a_{-1} - \frac{a}{e^a-1}) > 0, \\ (a + a_0 + a_1 + a_{-1})(a_1 - a_0 + a_{-1} - \frac{a(e^a+1)}{e^a-1}) > 0; \end{cases} \quad \text{if} \quad a \neq 0,$$
$$\begin{cases} (a_1 + a_{-1} - 1)(a_1 - a_{-1} - 1) > 0, \\ (a_0 + a_1 + a_{-1})(a_1 - a_0 + a_{-1} - 2) > 0, \end{cases} \quad \text{if} \quad a = 0.$$

Definition 2 *The set of all points (a, a_{-1}, a_0, a_1) at which (1) is asymptotically stable is called the asymptotic stability region, denoted by H.*

Therefore, we have

$$H_* = \left\{ (a, a_{-1}, a_0, a_1) : \begin{array}{l} (a_1 + a_{-1} - \frac{a}{e^a-1})(a_1 - a_{-1} - \frac{a}{e^a-1}) > 0 \\ (a + a_0 + a_1 + a_{-1})(a_1 - a_0 + a_{-1} - \frac{a(e^a+1)}{e^a-1}) > 0 \end{array} \right\},$$

$$H_0 = \left\{ (a, a_{-1}, a_0, a_1) : \begin{array}{l} (a_1 + a_{-1} - 1)(a_1 - a_{-1} - 1) > 0 \\ (a_0 + a_1 + a_{-1})(a_1 - a_0 + a_{-1} - 2) > 0 \end{array} \right\},$$

for $a \neq 0$ and $a = 0$, respectively. For the first case, we also introduce the following two regions

$$H_1 = \left\{ (a, a_{-1}, a_0, a_1) : \begin{array}{l} a + a_{-1} + a_0 + a_1 > 0 \\ a_1 - \frac{a}{e^a-1} > a_{-1} > -(a_1 - \frac{a}{e^a-1}) \\ a_0 < a_1 + a_{-1} - \frac{a(e^a+1)}{e^a-1} \end{array} \right\}, \tag{3}$$

$$H_2 = \left\{ (a, a_{-1}, a_0, a_1) : \begin{array}{l} a + a_{-1} + a_0 + a_1 < 0 \\ a_1 - \frac{a}{e^a-1} < a_{-1} < -(a_1 - \frac{a}{e^a-1}) \\ a_0 > a_1 + a_{-1} - \frac{a(e^a+1)}{e^a-1} \end{array} \right\}. \tag{4}$$

3. Euler-Maclaurin method

3.1. Bernoulli's numbers and Bernoulli's polynomial

It is known to us that

$$\frac{z}{e^z - 1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} z^j, \quad |z| < 2\pi,$$

$$\frac{ze^{xz}}{e^z - 1} = \sum_{j=0}^{\infty} \frac{B_j(x)}{j!} z^j, \quad |z| < 2\pi,$$

where B_j and $B_j(x)$ are called Bernoulli's number and Bernoulli's polynomial, respectively.

Lemma 2 ([14]) *B_j satisfies the following properties:*

(a) $B_0 = 1, B_1 = -\frac{1}{2}$,

(b) $B_{2j} = 2(-1)^{j+1}(2j)! \sum_{k=1}^{\infty} (2k\pi)^{-2j}$, $B_{2j+1} = 0, j \geq 1$,

(c) $\frac{B_{2(2i-1)}}{(2(2i-1))!} + \frac{B_{2(2i)}}{(2(2i))!} > 0$, $i = 1, 2, \ldots$,

(d) $\frac{B_{2(2i+1)}}{(2(2i+1))!} + \frac{B_{2(2i)}}{(2(2i))!} < 0$, $i = 1, 2, \ldots$.

Lemma 3 ([14]) *$B_j(x)$ satisfies the following properties:*

(a) $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_k(x) = \sum_{j=0}^{k} C_k^j B_j x^{k-j}$,

(b) $B_{2k+1}(1) = B_{2k+1}(0) = B_{2k+1} = 0$,

(c) $B_{2k}(1) = B_{2k}(0) = B_{2k}$,

(d) $B_k(x) = \frac{1}{k+1} B'_{k+1}(x)$, $k = 1, 2, \ldots$.

3.2. The numerical scheme

Let step size $h = 1/m$ with integer $m \geq 1$ and the grid points t_i be defined by $t_i = ih$ ($i = 0, 1, 2, \ldots$), and let $i = km + l$ ($l = 0, 1, \ldots, m-1$). Applying the Euler-Maclaurin formula to (1), we have

$$\begin{aligned} u_{i+1} =& u_i + \frac{ha}{2}(u_{i+1} + u_i) - \sum_{j=1}^{n} \frac{B_{2j}(ha)^{2j}}{(2j)!}(u_{i+1} - u_i) + ha_{-1}u_{(k-1)m} + \\ & ha_0 u_{km} + ha_1 u_{(k+1)m}. \end{aligned} \tag{5}$$

Lemma 4 ([15]) *Assume that $f(x)$ has $2n+3$rd continuous derivative on the interval $[t_i, t_{i+1}]$, then we have*

$$\begin{aligned} &\Big| \int_{t_i}^{t_{i+1}} f(t)\mathrm{d}t - \frac{h}{2}(f(t_{i+1}) + f(t_i)) + \sum_{j=1}^{n} \frac{B_{2j}h^{2j}}{(2j)!}(f^{(2j-1)}(t_{i+1}) - f^{(2j-1)}(t_i)) \Big| \\ &= O(h^{2n+3}). \end{aligned} \tag{6}$$

Then from Lemma 4 and (5), we know that the following convergence theorem is true.

Theorem 4 *For any given $n \in \mathbb{N}$, the Euler-Maclaurin method is of order $2n+2$.*

4. Numerical stability

(5) reduces to the following recurrence relation

$$\begin{aligned} u_{km+l+1} =& R(x)u_{km+l} + \frac{a_{-1}}{a}(R(x) - 1)u_{(k-1)m} + \\ & \frac{a_0}{a}(R(x) - 1)u_{km} + \frac{a_1}{a}(R(x) - 1)u_{(k+1)m} \quad \text{if} \quad a \neq 0, \\ u_{km+l+1} =& u_{km+l} + ha_{-1}u_{(k-1)m} + ha_0 u_{km} + ha_1 u_{(k+1)m} \quad \text{if} \quad a = 0, \end{aligned} \tag{7}$$

where

$$x = ha, R(x) = 1 + \frac{x}{\phi(x)}, \quad \phi(x) = 1 - \frac{x}{2} + \sum_{j=1}^{n} \frac{B_{2j}x^{2j}}{(2j)!}.$$

It is easy to see that (7) is equivalent to

$$\begin{aligned} u_{(k+1)m} =& \frac{R(x)^m + \frac{a_0}{a}(R(x)^m - 1)}{1 - \frac{a_1}{a}(R(x)^m - 1)} u_{km} + \frac{\frac{a_{-1}}{a}(R(x)^m - 1)}{1 - \frac{a_1}{a}(R(x)^m - 1)} u_{(k-1)m} \quad \text{if} \quad a \neq 0, \\ u_{km+l+1} =& [R(x)^{l+1} + \frac{a_0}{a}(R(x)^{l+1} - 1)]u_{km} + \frac{a_{-1}}{a}(R(x)^{l+1} - 1)u_{(k-1)m} + \\ & \frac{a_1}{a}(R(x)^{l+1} - 1)u_{(k+1)m} \quad \text{if} \quad a \neq 0, \\ u_{(k+1)m} =& \frac{1 + a_0}{1 - a_1}u_{km} + \frac{a_{-1}}{1 - a_1}u_{(k-1)m} \quad \text{if} \quad a = 0, \\ u_{km+l+1} =& [1 + (l+1)ha_0]u_{km} + (l+1)ha_{-1}u_{(k-1)m} + (l+1)ha_1 u_{(k+1)m} \quad \text{if} \quad a = 0, \end{aligned} \tag{8}$$

where $0 \le l \le m-2$. To guarantee the process (8) can be going on, we require that $1-(R(x)^m - 1)a_1/a \neq 0$.

Definition 3 *Process (5) for (1) is called asymptotically stable at* (a, a_{-1}, a_0, a_1) *if* $u_n \to 0$ *as* $n \to \infty$ *for all* $m \ge M$ *and* $h = 1/m$.

Definition 4 *The set of all points* (a, a_{-1}, a_0, a_1) *at which the Euler-Maclaurin method is asymptotically stable is called the asymptotical stability region, denoted by* S.

By (8), the following lemma is obtained.

Lemma 5 *Assume that* $1-(R(x)^m - 1)a_1/a \neq 0$, *then for all* k *and* $0 \le l \le m-2$, *there exists a constant* $C > 0$ *independent of* k *and* l *such that*

$$|u_{km+l+1}| \le C(|u_{km}| + |u_{(k-1)m}| + |u_{(k+1)m}|) \tag{9}$$

where

$$C = \max_{0\le l\le m-2}\{|R(x)^{l+1} + \frac{a_0}{a}(R(x)^{l+1} - 1)|, |\frac{a_{-1}}{a}(R(x)^{l+1} - 1)|, |\frac{a_1}{a}(R(x)^{l+1} - 1)|\}$$

and

$$C = 1 + |a_{-1}| + |a_0| + |a_1|$$

for $a \neq 0$ *and* $a = 0$, *respectively.*

It follows from Lemma 5 that $u_n \to 0$ as $n \to \infty$ if and only if $u_{km} \to 0$ as $k \to \infty$, and it is well known by (8) that $u_{km} \to 0$ as $k \to \infty$ if and only if the roots of the equations

$$\begin{aligned}
&\lambda^2 - \frac{R(x)^m + \frac{a_0}{a}(R(x)^m - 1)}{1 - \frac{a_1}{a}(R(x)^m - 1)}\lambda - \frac{\frac{a_{-1}}{a}(R(x)^m - 1)}{1 - \frac{a_1}{a}(R(x)^m - 1)} = 0 \quad \text{if} \quad a \neq 0,\\
&\lambda^2 - \frac{1 + a_0}{1 - a_1}\lambda - \frac{a_{-1}}{1 - a_1} = 0 \quad \text{if} \quad a = 0
\end{aligned} \tag{10}$$

are inside the unit disk of the complex plane. To simplify the notation, define

$$f_1(a, a_{-1}, a_0, a_1) = \begin{cases} \frac{a+a_{-1}+a_0+a_1}{a}(R(x)^m - 1) & \text{if} \quad a \neq 0,\\ a_0 + a_1 + a_{-1} & \text{if} \quad a = 0,\end{cases}$$

$$f_2(a, a_{-1}, a_0, a_1) = \begin{cases} \frac{a_{-1}+a_1}{a}(R(x)^m - 1) - 1 & \text{if} \quad a \neq 0,\\ a_1 + a_{-1} - 1 & \text{if} \quad a = 0,\end{cases}$$

$$g_1(a, a_{-1}, a_0, a_1) = \begin{cases} \frac{a_{-1}-a_0+a_1-a}{a}(R(x)^m - 1) - 2 & \text{if} \quad a \neq 0,\\ a_1 + a_{-1} - a_0 - 2 & \text{if} \quad a = 0,\end{cases}$$

$$g_2(a, a_{-1}, a_0, a_1) = \begin{cases} \frac{a_1-a_{-1}}{a}(R(x)^m - 1) - 1 & \text{if} \quad a \neq 0,\\ a_1 - a_{-1} - 1 & \text{if} \quad a = 0,\end{cases}$$

and

$$F_i = \text{sign}(1 - \frac{a_1}{a}(R(x)^m - 1))f_i,$$
$$G_i = \text{sign}(1 - \frac{a_1}{a}(R(x)^m - 1))g_i.$$

Then by Lemma 1, we have

Theorem 5 *$(a, a_{-1}, a_0, a_1) \in S$ if and only if $F_i(a, a_{-1}, a_0, a_1) < 0$ and $G_i(a, a_{-1}, a_0, a_1) < 0$ $(i = 1, 2)$.*

In the rest of this section, we assume $M > |a|$, which implies that $|x| < 1$ for $h = 1/m$ with $m \geq M$. Now we give two useful lemmas in the proofs of our main results.

Lemma 6 ([14]) *If $|x| \leq 1$, then $\phi(x) \geq 1/2$ for $x > 0$ and $\phi(x) \geq 1$ for $x \leq 0$.*

Lemma 7 ([14]) *If $|x| \leq 1$, then*

$$\phi(x) \leq \frac{x}{e^x - 1}$$

and

$$\phi(x) \geq \frac{x}{e^x - 1}$$

for even and odd n, respectively.

In order to analyze the stability, we simplify the conditions (3) and (4). In H_1, the condition

$$\begin{cases} -(a_1 - \frac{a}{e^a-1}) < a_{-1} < a_1 - \frac{a}{e^a-1}, \\ a_0 < a_{-1} + a_1 - \frac{a(e^a+1)}{e^a-1}, \end{cases}$$

can be reduced to

$$\begin{cases} a_1 - a_{-1} > \frac{a}{e^a-1}, \\ a_{-1} - a_0 + a_1 - a > \frac{2a}{e^a-1}. \end{cases} \tag{11}$$

Similarly, the condition

$$\begin{cases} -(a_1 - \frac{a}{e^a-1}) > a_{-1} > a_1 - \frac{a}{e^a-1}, \\ a_0 > a_{-1} + a_1 - \frac{a(e^a+1)}{e^a-1}, \end{cases}$$

in H_2 becomes

$$\begin{cases} a_1 - a_{-1} < \frac{a}{e^a-1}, \\ a_{-1} - a_0 + a_1 - a < \frac{2a}{e^a-1}. \end{cases} \tag{12}$$

Case I $1 - \frac{a_1}{a}(R(x)^m - 1) > 0.$

Lemma 8 *If $(a, a_{-1}, a_0, a_1) \in H_2$, then $f_1(a, a_{-1}, a_0, a_1) < 0$ is equivalent to*

$$\begin{cases} R(x)^m > 1 \quad \text{if} \quad a > 0, \\ R(x)^m < 1 \quad \text{if} \quad a < 0. \end{cases} \tag{13}$$

It is not difficult to derive the following formula by means of the notations of f_i, g_i $(i = 1, 2)$

$$2f_2(a, a_{-1}, a_0, a_1) = f_1(a, a_{-1}, a_0, a_1) + g_1(a, a_{-1}, a_0, a_1),$$

so we only consider the following inequalities hold for all $m \geq M$

$$f_1(a, a_{-1}, a_0, a_1) < 0, \;\; g_1(a, a_{-1}, a_0, a_1) < 0, \;\; g_2(a, a_{-1}, a_0, a_1) < 0.$$

Then the first main theorem of this paper is obtained.

Theorem 6 *For the Euler-Maclaurin method, $H_2 \subseteq S$ if and only if n is odd.*

Proof Let $(a,a_{-1},a_0,a_1)\in H_2$. Then $a+a_{-1}+a_0+a_1<0$. If $a>0$, it follows from Theorem 5 and Lemma 8 that $R(x)^m>1$, thus,

$$\frac{R(x)^m-1}{a}>0.$$

In view of (12) and Theorem 5, we know that for any given $a>0$

$$\sup_{(a,a_{-1},a_0,a_1)\in H_2} g_1(a,a_{-1},a_0,a_1)=2(\frac{R(x)^m-1}{e^a-1}-1)<0,$$
$$\sup_{(a,a_{-1},a_0,a_1)\in H_2} g_2(a,a_{-1},a_0,a_1)=\frac{R(x)^m-1}{e^a-1}-1<0,$$

which is equivalent to $R(x)^m<e^a$. So we have

$$R(x)<e^x,\quad 0<x\le 1,$$

that is

$$\phi(x)>\frac{x}{e^x-1},\quad 0<x\le 1.$$

As the consequence of Lemma 7, we have that n is odd. If $a<0$, we get the result by using the same proof as mentioned above. This completes the proof. $\Box$

Case II $1-\frac{a_1}{a}(R(x)^m-1)<0$.

Lemma 9 *If $(a,a_{-1},a_0,a_1)\in H_1$, then $f_1(a,a_{-1},a_0,a_1)>0$ is equivalent to*

$$\begin{cases} R(x)^m>1 & \text{if} \quad a>0,\\ R(x)^m<1 & \text{if} \quad a<0.\end{cases} \tag{14}$$

Theorem 7 *For the Euler-Maclaurin method, $H_1\subseteq S$ if and only if n is even.*

Theorem 8 *For all Euler-Maclaurin methods, we have $H_0\subseteq S$.*

5. Numerical examples

We consider the following two problems:

$$\begin{aligned} &u_1'(t)=u_1(t)+0.5u_1([t-1])-0.2u_1([t])+3u_1([t+1]),\quad t\ge 0,\\ &u_1(-1)=1,\quad u_1(0)=1,\end{aligned} \tag{15}$$

$$\begin{aligned} &u_2'(t)=-2u_2(t)-u_2([t-1])+0.9u_2([t])+u_2([t+1]),\quad t\ge 0,\\ &u_2(-1)=1,\quad u_2(0)=1.\end{aligned} \tag{16}$$

According to (3) and (4), it is not difficult to see that $(1,0.5,-0.2,3)\in H_1$ and $(-2,-1,0.9,1)\in H_2$ by direct calculations. We shall use the Euler-Maclaurin method with step size $h=1/m$ and $n=2$ to get the numerical solution at $t=10$, where the exact solutions are $u_1(10)\approx -7.1587\times 10^{-5}$, $u_2(10)\approx -0.1278$. In Table 1 we have listed the absolute errors (AE) and the relative errors (RE) at $t=10$ and the ratio of the errors of the case $m=20$ over that of $m=40$. We can see from this table that the Euler-Maclaurin method with $n=2$ is of order 6, that is, the method preserves its order of convergence.

	(15)		(16)	
	AE	*RE*	*AE*	*RE*
$m = 2$	2.4192E-9	3.3794E-5	7.3798E-6	5.7724E-5
$m = 3$	2.1067E-10	2.9429E-6	6.2728E-7	4.9065E-6
$m = 5$	9.7885E-12	1.3674E-7	2.8786E-8	2.2516E-7
$m = 10$	1.5268E-13	2.1327E-9	4.4665E-10	3.4936E-9
$m = 20$	2.3850E-15	3.3316E-11	6.9664E-12	5.4491E-11
$m = 40$	4.1715E-17	5.8271E-13	1.0911E-13	8.5343E-13
Ratio	57.1737	57.1742	63.8475	63.8494

Table 1 Errors of the Euler-Maclaurin method with $n = 2$

In Figures 1 and 2, we draw the numerical solutions of the Euler-Maclaurin method with $m = 2$ and $m = 15$ for (15) and (16), respectively. It is easy to see that the numerical solutions are asymptotically stable.

All above numerical examples are in agreement with the main results in the paper.

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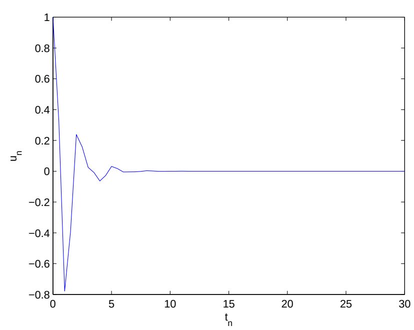

Figure 1 The numerical solution of (15) with $n = 4, m = 2$

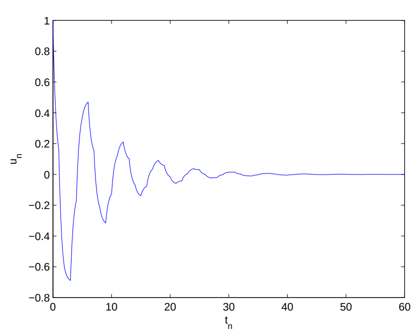

Figure 2 The numerical solution of (16) with $n = 3, m = 15$

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