Journal of Mathematical Research with Applications
Jan., 2013, Vol. 33, No. 1, pp. 79–89
DOI:10.3770/j.issn:2095-2651.2013.01.009
Http://jmre.dlut.edu.cn

A Categorification of the Vector Representation of $U(\mathfrak{so}(7,\mathbb{C}))$ via Projective Functors

Yongjun XU, Shilin YANG*
College of Applied Sciences, Beijing University of Technology, Beijing 100124*, P. R. China*

Abstract The aim of this paper is to categorify the n-th tensor power of the vector representation of $U(\mathfrak{so}(7,\mathbb{C}))$. The main tools are certain singular blocks and projective functors of the BGG category of the complex Lie algebra $\mathfrak{gl}_n$.

Keywords categorification; Lie algebra; vector representation; projective functor; BGG category.

MR(2010) Subject Classification 17B10; 17B20; 17B35

1. Introduction

The general idea of categorification was introduced by Crane and Frenkel [4, 5]. It refers to the process of finding category-theoretic analogues of ideas phrased in the language of set theory. One of the simplest examples of categorificaitons is that the set $\mathbb{N}$ of natural numbers can be categorified by the category of finite dimensional linear spaces, which lifts us from arithmetic in $\mathbb{N}$ to linear algebra. From this simplest example, we can see that the idea of categorification is very important. In fact, categorification has shown its power in low dimensional topology theory, knot theory and many other fields [1, 9]. Universal enveloping algebras of simple Lie algebras and their quantizations play an important role in various fields such as conformal field theory, low dimensional topology, etc. Therefore, categorifications of these algebras have been studied by many mathematicians in recent years [1, 10, 11].

Bernstein, Frenkel and Khovanov [1] investigated a categorification of the n-th tensor power of the fundamental representation of $U(\mathfrak{sl}_2)$ via certain singular blocks and projective functors of the BGG category of the complex Lie algebra $\mathfrak{gl}_n$. Meanwhile, they raised the more difficult problem: categorifications of the representation theory of $U(\mathfrak{g})$ for arbitrary simple Lie algebra $\mathfrak{g}$. In the present paper, we study a categorification of the n-th tensor power of the vector representation of $U(\mathfrak{so}(7,\mathbb{C}))$, which can be considered as a part of categorifications of the representation theory of $U(\mathfrak{g})$ for the simple Lie algebra $\mathfrak{g}$ of type B_3. In other words, we categorify the image of $U(\mathfrak{so}(7,\mathbb{C}))$ under the algebra homomorphism $\iota: U(\mathfrak{so}(7,\mathbb{C}))\to \mathrm{End}(\mathrm{V}^{\otimes \mathrm{n}})$ corresponding to the n-th tensor power of the vector representation V of $U(\mathfrak{so}(7,\mathbb{C}))$.

Received November 16, 2011; Accepted October 18, 2012
Supported by the Natural Science Foundation of Beijing (Grant No. 1122006), the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 201111103110011) & Science and Technology Foundation of BJUT (Grant No. ykj-4787).
* Corresponding author
E-mail address: yjxu2002@163.com (Yongjun XU); slyang@bjut.edu.cn (Shilin YANG)

The paper is organized in the following manner. In Section 2, we recall some basic concepts of $U(\mathfrak{so}(7,\mathbb{C}))$ and its vector representation, then give a brief introduction to the BGG category. In Section 3, we obtain a categorification of the n-th tensor power of the vector representation of $U(\mathfrak{so}(7,\mathbb{C}))$. The programme is arranged as follows. To begin with, we categorify the underlying space of the n-th tensor power $V^{\otimes n}$ of the vector representation V for $U(\mathfrak{so}(7,\mathbb{C}))$ by using certain singular blocks of the BGG category $\mathcal{O}(\mathfrak{gl}_n)$ (Theorem 8). Then we yield a categorification of the $U(\mathfrak{so}(7,\mathbb{C}))$ action on $V^{\otimes n}$ by projective functors of $\mathcal{O}(\mathfrak{gl}_n)$ (Theorem 10). Finally, we lift all the defining relations of $U(\mathfrak{so}(7,\mathbb{C}))$ to the natural isomorphisms between functors (Theorem 11).

Throughout the paper, we denote by $\mathbb{C}$, $\mathbb{N}$ and $\mathbb{Z}$ the complex number field, the natural number set and the integer number set, respectively.

2. Lie algebra $\mathfrak{so}(7,\mathbb{C})$ and the BGG category

We start by reviewing some basic results about the universal enveloping algebra of Lie algebra $\mathfrak{so}(7,\mathbb{C})$ and the BGG category of the complex reductive Lie algebra.

As an associative algebra, the universal enveloping algebra $U(\mathfrak{so}(7,\mathbb{C}))$ of the special orthogonal Lie algebra $\mathfrak{so}(7,\mathbb{C})$ is generated by h_i, e_i, f_i $(1 \le i \le 3)$ over $\mathbb{C}$ which are subject to the obvious relations (where $[\cdot,\cdot]$ denotes the usual commutator): $[h_i, h_j] = 0, [e_i, f_j] = \delta_{i,j} h_i, [h_i, e_j] = a_{i,j} e_j, [h_i, f_j] = -a_{i,j} f_j$, and the following Serre's relations:

$$\sum_{k=0}^{1-a_{i,j}} (-1)^k \binom{1-a_{i,j}}{k} e_i^{1-a_{i,j}-k} e_j e_i^k = 0 \quad \text{for} \quad i \ne j,$$
$$\sum_{k=0}^{1-a_{i,j}} (-1)^k \binom{1-a_{i,j}}{k} f_i^{1-a_{i,j}-k} f_j f_i^k = 0 \quad \text{for} \quad i \ne j,$$

where $a_{i,j}$ $(1 \le i, j \le 3)$ are the entries of the Cartan matrix $A = (a_{i,j})_{3\times 3}$ of $\mathfrak{so}(7,\mathbb{C})$ given by

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}.$$

As a coalgebra, the comultiplication Δ of $U(\mathfrak{so}(7,\mathbb{C}))$ is given by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in \{h_i, e_i, f_i | i = 1, 2, 3\}$.

Let $V = \oplus_{0 \le i \le 6} \mathbb{C} v_i$ be a 7 dimensional vector space over $\mathbb{C}$. Then V is the vector representation of $U(\mathfrak{so}(7,\mathbb{C}))$ under the algebra homomorphism $\iota : U(\mathfrak{so}(7,\mathbb{C})) \to \mathrm{End}_{\mathbb{C}}(\mathrm{V}) \cong \mathrm{M}_{7\times 7}(\mathbb{C})$ with

$$\begin{aligned}
&\iota(e_1) = E_{01} + E_{56},\ \iota(e_2) = E_{12} + E_{45},\ \iota(e_3) = 2E_{23} + E_{34},\\
&\iota(f_1) = E_{10} + E_{65},\ \iota(f_2) = E_{21} + E_{54},\ \iota(f_3) = E_{32} + 2E_{43},\\
&\iota(h_1) = E_{00} - E_{11} + E_{55} - E_{66},\ \iota(h_2) = E_{11} - E_{22} + E_{44} - E_{55},\ \iota(h_3) = 2(E_{22} - E_{44}),
\end{aligned}$$

where E_{ij} $(0 \le i, j \le 6)$ denote the 7×7 elementary matrices having 1 at the $(i+1, j+1)$-entry and 0 elsewhere.

For convenience, we fix some notations we will use in the sequel.

All Lie algebras and their representations are defined over $\mathbb{C}$. The $U(\mathfrak{g})$ is the universal enveloping algebra of a finite dimensional reductive Lie algebra $\mathfrak{g}$. The notation Mod-$U(\mathfrak{g})$ denotes the category of all $U(\mathfrak{g})$-modules. Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be a triangular decomposition of $\mathfrak{g}$ and $(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ the Killing form of $\mathfrak{g}$. Denote by W the Weyl group of $\mathfrak{g}$. Let R be the set of roots of $\mathfrak{g}$ and R_+ the set of its positive roots. Let ρ be the half-sum of positive roots. For $\lambda \in \mathfrak{h}^*$, $M(\lambda)$ and $L(\lambda)$ denote the Verma module and irreducible module with the highest weight λ, respectively.

Now let us recall some concepts and properties of the BGG category for a reductive Lie algebra $\mathfrak{g}$. The definition of the BGG category is stated as follows [8, pp. 13–14].

Definition 1 *The BGG category $\mathcal{O}(\mathfrak{g})$ is defined to be the full subcategory of $U(\mathfrak{g})$-*Mod *whose objects are the modules satisfying the following three conditions.*

(1) M is a finitely generated $U(\mathfrak{g})$-module.

(2) M is $\mathfrak{h}$-diagonalizable, that is, M is a weight module : $M = \oplus_{\lambda\in\mathfrak{h}^*} M_\lambda$.

(3) M is locally $U(\mathfrak{n}_+)$-finite: for each $v \in M$, the subspace $U(\mathfrak{n}_+) \cdot v$ is finite dimensional.

Denote by

$$\Theta = \{\theta : Z(U(\mathfrak{g})) \to \mathbb{C} \mid \theta \text{ is an algebra homomorphism}\}$$

the central character set. We can naturally identify Θ with the quotient of the weight space $\mathfrak{h}^*$ by the map $\eta : \mathfrak{h}^* \to \Theta$ with $\eta(\lambda)(z) = \theta_\lambda(z) = \lambda(\xi(z)) = \lambda(\mathrm{pr}(z))$, where $\mathrm{pr} : U(\mathfrak{g}) \to U(\mathfrak{h})$ is the projection onto the subspace by setting all other monomials equal to 0 and ξ is the Harish-Chandra homomorphism, that is, ξ is the map from $Z(U(\mathfrak{g}))$ to $U(\mathfrak{h})$ with $\xi(z) = \mathrm{pr}(z)$. The above identification can be seen from the following proposition [8, p. 26].

Proposition 2 *(1) Every central character $\theta : Z(U(\mathfrak{g})) \to \mathbb{C}$ is of the form θ_λ for some $\lambda \in \mathfrak{h}^*$.*

(2) For all $\lambda, \mu \in \mathfrak{h}^*$, we have $\theta_\lambda = \theta_\mu$ if and only if $\mu = w \cdot \lambda$ for some $w \in W$, where $w \cdot \lambda = w(\lambda + \rho) - \rho$.

For any $\theta \in \Theta$, denote by $\mathcal{O}_\theta(\mathfrak{g})$ the full subcategory of $\mathcal{O}(\mathfrak{g})$ whose objects are the modules M where

$$M = \{m \in M | (z - \theta(z))^n \cdot m = 0 \text{ for some } n \in \mathbb{N} \text{ for each } z \in Z(U(\mathfrak{g}))\}.$$

If $\mathscr{A}$ is an additive category, denote by $K(\mathscr{A})$ the Grothendieck group of $\mathscr{A}$. Denote by $[M]$ the image of an object $M \in \mathrm{Ob}(\mathscr{A})$ in the Grothendieck group of $\mathscr{A}$.

The following proposition can be referred to [2, 3] and [8].

Proposition 3 *The BGG category $\mathcal{O}(\mathfrak{g})$ has the following properties.*

(1) The BGG category $\mathcal{O}(\mathfrak{g})$ is both artinian and noetherian, i.e., each $M \in Ob(\mathcal{O}(\mathfrak{g}))$ is both artinian and noetherian.

(2) The BGG category $\mathcal{O}(\mathfrak{g})$ is the direct sum of the subcategories $\mathcal{O}_\theta(\mathfrak{g})$ as θ ranges over the central characters, i.e., each $M \in Ob(\mathcal{O}(\mathfrak{g}))$ decomposes into a finite sum $M = \oplus_{\theta\in\Theta} M(\theta)$, where $M(\theta) \in Ob(\mathcal{O}_\theta(\mathfrak{g}))$, and for $\theta_1 \neq \theta_2 \in \Theta, \mathrm{Hom}(M_1, M_2) = 0$ for any $M_1 \in Ob(\mathcal{O}_{\theta_1}(\mathfrak{g})), M_2 \in Ob(\mathcal{O}_{\theta_2}(\mathfrak{g}))$.

(3) Fix a central charater θ and let $\{V^{(\lambda)}\}$ be a collection of modules in $\mathcal{O}_\theta(\mathfrak{g})$ indexed by the weights λ for which $\theta = \theta_\lambda$, satisfying: (i) $\dim V_\lambda^{(\lambda)} = 1$; (ii) $\mu \leq \lambda$ for all weights μ of $V^{(\lambda)}$. Then $\{[V^{(\lambda)}]|\theta = \theta_\lambda\}$ forms a $\mathbb{Z}$-basis of the Grothendieck group $K(\mathcal{O}_\theta(\mathfrak{g}))$. In particular, $\{[M(\lambda)]|\theta = \theta_\lambda\}$ and $\{[L(\lambda)]|\theta = \theta_\lambda\}$ form two $\mathbb{Z}$-basis of the Grothendieck group $K(\mathcal{O}_\theta(\mathfrak{g}))$.

Now we give a brief introduction to projective functors.

Denote by proj_θ the functor from $\mathcal{O}(\mathfrak{g})$ to $\mathcal{O}_\theta(\mathfrak{g})$ that, to a module $M = \oplus_{\theta\in\Theta} M(\theta)$, associates the θ-component summand $M(\theta)$ of M. Let F_V be the functor of tensoring with a finite-dimensional $\mathfrak{g}$-module V. The following definition can be found in [1] (see also [2] or [8], p. 214).

Definition 4 *$F : \mathcal{O}(\mathfrak{g}) \to \mathcal{O}(\mathfrak{g})$ is a projective functor if it is isomorphic to a direct summand of the functor F_V for some finite dimensional module V.*

Remarks (1) The functor proj_θ is an example of a projective functor, since it is a direct summand of the functor of tensoring with the one-dimensional representation. We have an isomorphism of functors

$$F_V \cong \bigoplus_{\theta_1,\theta_2\in\Theta} (\mathrm{proj}_{\theta_1} \circ F_V \circ \mathrm{proj}_{\theta_2}).$$

(2) Any projective functor takes projective objects in $\mathcal{O}(\mathfrak{g})$ to projective objects. The composition of projective functors is again a projective functor. Each projective functor splits as a direct sum of indecomposable projective functors.

(3) Projective functors are exact. Therefore, they induce endomorphisms of the Grothendieck group of the category $\mathcal{O}(\mathfrak{g})$.

Definition 5 *A weight $\lambda \in \mathfrak{h}^*$ is called integral if $\langle\lambda, \alpha^\vee\rangle \in \mathbb{Z}$ for all $\alpha \in R$; a weight $\lambda \in \mathfrak{h}^*$ is called dominant if $\langle\lambda + \rho, \alpha^\vee\rangle \notin \mathbb{Z}^{<0}$ for any coroot $\alpha^\vee$ of $\alpha \in R_+$, where $\langle\lambda, \alpha^\vee\rangle = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}$ for any $\lambda \in \mathfrak{h}^*$ and $\alpha \in R$.*

The following result can be found in [2].

Proposition 6 *Let λ be a dominant integral weight, $\theta = \eta(\lambda)$ and F, G projective functors from $\mathcal{O}_\theta(\mathfrak{g})$ to $\mathcal{O}(\mathfrak{g})$. Then*

(1) Functors F and G are isomorphic if and only if the endomorphisms of $K(\mathcal{O}(\mathfrak{g}))$ induced by F and G are equal.

(2) Functors F and G are isomorphic if and only if modules $FM(\lambda)$ and $GM(\lambda)$ are isomorphic.

We are going to compute the action of projective functors on Grothendieck groups of certain subcategories of the BGG category. By Proposition 3 (3), the simplest basis in the Grothendieck group of $\mathcal{O}(\mathfrak{g})$ is given by images of Verma modules. The following proposition shows that this basis is also handy for writing the action of projective functors on the Grothendieck group of $\mathcal{O}(\mathfrak{g})$ (see [1, 2]).

Proposition 7 *Let V be a finite-dimensional $\mathfrak{g}$-module, $\mu_1, \ldots, \mu_m$ a multiset of weights of V, i.e., there is a basis $v_1, v_2, \ldots, v_m$ of V such that the weight of the vector v_i equals μ_i, $M(\lambda - \rho)$ the Verma module with the highest weight $\lambda - \rho$. Then*

(1) The module $V \otimes M(\lambda-\rho)$ *admits a filtration with successive quotients isomorphic to Verma modules* $M(\lambda+\mu_1-\rho),\dots,M(\lambda+\mu_m-\rho)$ *(in some order).*

(2) In the Grothendieck group $K(\mathcal{O}(\mathfrak{g}))$ *we have* $[V\otimes M(\lambda-\rho)]=\sum_{i=1}^m [M(\lambda+\mu_i-\rho)]$.

3. Categorification of the vector representation of $U(\mathfrak{so}(7,\mathbb{C}))$

The purpose of this section is to obtain a categorification of the n-th tensor power $V^{\otimes n}$ of the vector representation V for $U(\mathfrak{so}(7,\mathbb{C}))$. We will go along the following three steps.

(1) Categorifying the underlying space of the n-th tensor power $V^{\otimes n}$ of the vector representation V of $U(\mathfrak{so}(7,\mathbb{C}))$ by using certain singular blocks of the BGG category $\mathcal{O}(\mathfrak{gl}_n)$.

(2) Yielding a categorification of the $U(\mathfrak{so}(7,\mathbb{C}))$ action on $V^{\otimes n}$ by projective functors of $\mathcal{O}(\mathfrak{gl}_n)$.

(3) Lifting all the defining relations of $U(\mathfrak{so}(7,\mathbb{C}))$ to the natural isomorphisms between functors.

The categorifications we proceed in the above three steps are really about the image Im ι of $U(\mathfrak{so}(7,\mathbb{C}))$ under the algebra homomorphism $\iota: U(\mathfrak{so}(7,\mathbb{C}))\to \mathrm{End}(\mathrm{V}^{\otimes n})$ corresponding to the n-th tensor power of the vector representation V of $U(\mathfrak{so}(7,\mathbb{C}))$.

We fix once and for all a triangular decomposition $\mathfrak{n}_+\oplus\mathfrak{h}\oplus\mathfrak{n}_-$ of the Lie algebra $\mathfrak{gl}_n$. The Weyl group of $\mathfrak{gl}_n$ is isomorphic to the symmetric group S_n. Choose a standard orthogonal basis $\varepsilon_1,\dots,\varepsilon_n$ in the Euclidean space $\mathbb{R}^n$ and identify the complexification $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}^n$ with the dual $\mathfrak{h}^*$ of Cartan subalgebra so that $R_+=\{\varepsilon_i-\varepsilon_j \mid 1\le i<j\le n\}$ is the set of positive roots and $\beta_i=\varepsilon_i-\varepsilon_{i+1}, 1\le i\le n-1$ are simple roots. The generator s_i of the Weyl group $W=S_n$ acts on $\mathfrak{h}^*$ by permuting ε_i and ε_{i+1}. Denote by ρ the half-sum of positive roots

$$\rho=\frac{n-1}{2}\varepsilon_1+\frac{n-3}{2}\varepsilon_2+\cdots+\frac{1-n}{2}\varepsilon_n.$$

We denote by $[0,6]$ the integers $0\le k\le 6$. For a sequence $(a_1,\dots,a_n)\in[0,6]^n$ we denote by $M(a_1,\dots,a_n)$ the Verma module with the highest weight $a_1\varepsilon_1+\cdots+a_n\varepsilon_n-\rho$. Accordingly, $L(a_1,\dots,a_n)$ denotes the simple quotient of $M(a_1,\dots,a_n)$.

Let $\mathbf{D}$ be the set of all 7-tuples of natural numbers $\mathbf{d}=(d_0,d_1,d_2,d_3,d_4,d_5,d_6)$ such that $\sum_{k=0}^6 d_k=n$. We define the following equivalent relation $\sim$ on $\mathbf{D}$:

$$\mathbf{d}\sim\mathbf{d}' \Leftrightarrow \begin{cases} d_0-d_1+d_5-d_6=d_0'-d_1'+d_5'-d_6', \\ d_1-d_2+d_4-d_5=d_1'-d_2'+d_4'-d_5', \\ d_2-d_4=d_2'-d_4', \end{cases} \Leftrightarrow \begin{cases} d_0-d_6=d_0'-d_6', \\ d_1-d_5=d_1'-d_5', \\ d_2-d_4=d_2'-d_4', \end{cases}$$

for any $\mathbf{d}=(d_0,d_1,d_2,d_3,d_4,d_5,d_6), \mathbf{d}'=(d_0',d_1',d_2',d_3',d_4',d_5',d_6')\in\mathbf{D}$. In the following $[\mathbf{d}]$ and $\widetilde{\mathbf{D}}$ represent the equivalent class of $\mathbf{d}$ and the set of all the the equivalent classes respectively. For each $\mathbf{d}\in\mathbf{D}$, set

$$\lambda_{\mathbf{d}}=\sum_{i=0}^{6}\sum_{j=1}^{d_i}(6-i)\varepsilon_{d_0+\cdots+d_{i-1}+j}.$$

Denote by $\theta_{\mathbf{d}}=\eta(\lambda_{\mathbf{d}}-\rho)$ the corresponding central character of $\mathfrak{gl}_n$ under the map $\eta:\mathfrak{h}^*\to\Theta$.

We denote the category $\mathcal{O}_{\theta_{\mathbf{d}}}(\mathfrak{gl}_n)$ by $\mathcal{O}_{\mathbf{d}}$ and set

$$\mathcal{O}_{[\mathbf{d}]}=\underset{\mathbf{d}'\in[\mathbf{d}]}{\oplus}\mathcal{O}_{\mathbf{d}'},\quad \mathcal{O}^n=\underset{[\mathbf{d}]\in\widetilde{\mathbf{D}}}{\oplus}\mathcal{O}_{[\mathbf{d}]}.$$

From now on, we always view V and $V^{\otimes n}$ as $U(\mathfrak{so}(7,\mathbb{C}))$-modules over $\mathbb{Z}$. As is known, V has the weight space decomposition: $V=\oplus_{k=0}^{6}V_k$, where $V_k=\mathbb{Z}v_k$ $(0\le k\le 6)$. It follows that $V^{\otimes n}$ has the weight space decomposition: $V^{\otimes n}=\oplus_{[\mathbf{d}]\in\widetilde{\mathbf{D}}}(V^{\otimes n})_{[\mathbf{d}]}$, where $(V^{\otimes n})_{[\mathbf{d}]}$ is the $\mathbb{Z}$-module spanned by $\{v_{a_1}\otimes v_{a_2}\otimes\cdots\otimes v_{a_n}|(a_1,\ldots,a_n)\in[0,6]^n$ satisfying $\exists\ \mathbf{d}'\in[\mathbf{d}]$ such that $\sharp\ \{a_m|a_m=k,1\le m\le n\}=d'_k$ for $0\le k\le 6\}$.

Now we are prepared to realize $V^{\otimes n}$ and its weight space $(V^{\otimes n})_{[\mathbf{d}]}$ for any $[\mathbf{d}]\in\widetilde{\mathbf{D}}$ as the Grothendieck groups of the categories $\mathcal{O}^n$ and $\mathcal{O}_{[\mathbf{d}]}$, respectively. Indeed, we have the following result.

Theorem 8 *There exists an isomorphism of abelian groups $\gamma_n: K(\mathcal{O}^n)\to V^{\otimes n}$ given by*

$$\gamma_n([M(a_1,\ldots,a_n)])=v_{a_1}\otimes v_{a_2}\otimes\cdots\otimes v_{a_n},$$

for any sequence $(a_1,\ldots,a_n)\in[0,6]^n$. Moreover, the restriction of γ_n on $K(\mathcal{O}_{[\mathbf{d}]})$ is an abelian group isomorphism between $K(\mathcal{O}_{[\mathbf{d}]})$ and $(V^{\otimes n})_{[\mathbf{d}]}$ for any $[\mathbf{d}]\in\widetilde{\mathbf{D}}$.

Proof To prove the theorem, it suffices to prove that $\gamma_n: K(\mathcal{O}_{[\mathbf{d}]})\to(V^{\otimes n})_{[\mathbf{d}]}$ is an abelian group isomorphism for any $[\mathbf{d}]\in\widetilde{\mathbf{D}}$. Indeed, the above abelian group isomorphism will be obvious if we note the following facts.

For any $[\mathbf{d}]\in\widetilde{\mathbf{D}}$ and $\mathbf{d}'\in[\mathbf{d}]$, on one hand, by Proposition 2 and 3 (3) we know that the set of all the symbols $[M(a_1,\ldots,a_n)]$ satisfying

$$\sharp\ \{\ a_m\mid a_m=k,1\le m\le n\}=d'_k\ \text{ for }\ 0\le k\le 6, \tag{1}$$

is a $\mathbb{Z}$-basis of the Grothendieck group $K(\mathcal{O}_{\mathbf{d}'})$. We denote this $\mathbb{Z}$-basis by $B_{\mathbf{d}'}$. It follows that $B_{[\mathbf{d}]}=\underset{\mathbf{d}'\in[\mathbf{d}]}{\cup}B_{\mathbf{d}'}$ is a $\mathbb{Z}$-basis of the Grothendieck group $K(\mathcal{O}_{[\mathbf{d}]})$. On the other hand, if we denote by $B'_{\mathbf{d}'}$ the set of $v_{a_1}\otimes v_{a_2}\otimes\cdots\otimes v_{a_n}$ such that the sequence $(a_1,\ldots,a_n)\in[0,6]^n$ satisfies (1), then $B'_{[\mathbf{d}]}=\cup_{\mathbf{d}'\in[\mathbf{d}]}B'_{\mathbf{d}'}$ is a $\mathbb{Z}$-basis of the weight space $(V^{\otimes n})_{[\mathbf{d}]}$. $\square$

To categorify the action of $U(\mathfrak{so}(7,\mathbb{C}))$ on $V^{\otimes n}$, we introduce a series of projective functors of $\mathcal{O}(\mathfrak{gl}_n)$.

Let L_n be the n-dimensional fundamental representation of $\mathfrak{gl}_n$ with weights $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n$ and the corresponding weight vectors $u_1,u_2,\ldots,u_n$. Then its dual representation L_n^* has weights $-\varepsilon_1,-\varepsilon_2,\ldots,-\varepsilon_n$. In the following, we define $\mathcal{O}_{\mathbf{d}}$ to be the trivial subcategory of $\mathcal{O}(\mathfrak{gl}_{\mathfrak{n}})$ for $\mathbf{d}\notin\mathbf{D}$. For $\mathbf{d}\in\mathbf{D}$ let $\mathbf{d}_i$ denote the fact that one subtracts 1 from the coefficient at place i, and $\mathbf{d}^i$ the fact that one adds 1 to the coefficient at place i. Then $\mathbf{d}_i^j$ means that one subtracts 1 from the coefficient at place i and adds 1 to the coefficient at place j.

For $\mathbf{d}=(d_0,d_1,d_2,d_3,d_4,d_5,d_6)\in\mathbf{D}$, define

$$\begin{aligned}c_1(\mathbf{d})&:=d_0-d_1+d_5-d_6,\\ c_2(\mathbf{d})&:=d_1-d_2+d_4-d_5,\\ c_3(\mathbf{d})&:=2(d_2-d_4),\end{aligned}$$

and for $1 \le i \le 3$, denote by $\mathrm{sgn}(c_i(\mathbf{d}))$ the sign function of $c_i(\mathbf{d})$, i.e.,

$$\mathrm{sgn}(c_i(\mathbf{d})) = \begin{cases} 1, & \text{if} \quad c_i(\mathbf{d}) > 0, \\ 0, & \text{if} \quad c_i(\mathbf{d}) = 0, \\ -1, & \text{if} \quad c_i(\mathbf{d}) < 0. \end{cases}$$

Then set

$$\mathcal{H}_i^{\mathrm{sgn}(c_i(\mathbf{d}))}([\mathbf{d}]) = (\mathrm{Id}_{\mathcal{O}_{[\mathbf{d}]}})^{\oplus \mathrm{sgn}(c_i(\mathbf{d}))c_i(\mathbf{d})} : \mathcal{O}_{[\mathbf{d}]} \to \mathcal{O}_{[\mathbf{d}]},$$

where $\mathrm{Id}_{\mathcal{O}_{[\mathbf{d}]}}$ is the identity functor of $\mathcal{O}_{[\mathbf{d}]}$. From the definition of the equivalent relation $\sim$ on $\mathbf{D}$ it is easy to see that the functors $\mathcal{H}_i^{\mathrm{sgn}(c_i(\mathbf{d}))}([\mathbf{d}])$ $(1 \le i \le 3)$ are independent of the choice of the representative $\mathbf{d}$ of $[\mathbf{d}]$.

For $\mathbf{d} \in \mathbf{D}$ and the finite dimensional representation $L = L_n$ or L_n^* of $\mathfrak{gl}_n$, we define the following projective functor

$$\varphi(\mathbf{d}_i^j, L) = (\mathrm{proj}_{\theta_{\mathbf{d}_i^j}}) \circ F_L : \quad \mathcal{O}_{\mathbf{d}} \to \mathcal{O}_{\mathbf{d}_i^j}$$

given by tensoring with L and then taking the largest submodule of this tensor product that lies in $\mathcal{O}_{\mathbf{d}_i^j}$. For any $[M(a_1, \ldots, a_n)] \in B_{\mathbf{d}}$, by Proposition 7 we deduce

$$[\varphi(\mathbf{d}_i^j, L_n)]([M(a_1, \ldots, a_n)]) = \sum_{\substack{m=1, \\ a_m = i}}^{n} [M(a_1, \ldots, a_{m-1}, a_m + 1, a_{m+1}, \ldots, a_n)], \tag{2}$$

$$[\varphi(\mathbf{d}_i^j, L_n^*)]([M(a_1, \ldots, a_n)]) = \sum_{\substack{m=1, \\ a_m = i}}^{n} [M(a_1, \ldots, a_{m-1}, a_m - 1, a_{m+1}, \ldots, a_n)]. \tag{3}$$

For any $[\mathbf{d}] \in \widetilde{\mathbf{D}}$, we set

$$\begin{aligned}
\mathcal{E}_1([\mathbf{d}]) &= \bigoplus_{\mathbf{d}' \in [\mathbf{d}]} (\varphi(\mathbf{d'}_1^0, L_n^*) \oplus \varphi(\mathbf{d'}_6^5, L_n^*)) : \quad \mathcal{O}_{[\mathbf{d}]} \to \mathcal{O}_{[\overleftarrow{\mathbf{d_1}}]}, \\
\mathcal{E}_2([\mathbf{d}]) &= \bigoplus_{\mathbf{d}' \in [\mathbf{d}]} (\varphi(\mathbf{d'}_2^1, L_n^*) \oplus \varphi(\mathbf{d'}_5^4, L_n^*)) : \quad \mathcal{O}_{[\mathbf{d}]} \to \mathcal{O}_{[\overleftarrow{\mathbf{d_2}}]}, \\
\mathcal{E}_3([\mathbf{d}]) &= \bigoplus_{\mathbf{d}' \in [\mathbf{d}]} (\varphi(\mathbf{d'}_3^2, L_n^*)^{\oplus 2} \oplus \varphi(\mathbf{d'}_4^3, L_n^*)) : \quad \mathcal{O}_{[\mathbf{d}]} \to \mathcal{O}_{[\overleftarrow{\mathbf{d_3}}]}, \\
\mathcal{F}_1([\mathbf{d}]) &= \bigoplus_{\mathbf{d}' \in [\mathbf{d}]} (\varphi(\mathbf{d'}_0^1, L_n) \oplus \varphi(\mathbf{d'}_5^6, L_n)) : \quad \mathcal{O}_{[\mathbf{d}]} \to \mathcal{O}_{[\overrightarrow{\mathbf{d_1}}]}, \\
\mathcal{F}_2([\mathbf{d}]) &= \bigoplus_{\mathbf{d}' \in [\mathbf{d}]} (\varphi(\mathbf{d'}_1^2, L_n) \oplus \varphi(\mathbf{d'}_4^5, L_n)) : \quad \mathcal{O}_{[\mathbf{d}]} \to \mathcal{O}_{[\overrightarrow{\mathbf{d_2}}]}, \\
\mathcal{F}_3([\mathbf{d}]) &= \bigoplus_{\mathbf{d}' \in [\mathbf{d}]} (\varphi(\mathbf{d'}_2^3, L_n) \oplus \varphi(\mathbf{d'}_3^4, L_n)^{\oplus 2}) : \quad \mathcal{O}_{[\mathbf{d}]} \to \mathcal{O}_{[\overrightarrow{\mathbf{d_3}}]},
\end{aligned}$$

where $[\overleftarrow{\mathbf{d_1}}] = [\mathbf{d}_1^0] = [\mathbf{d}_6^5]$, $[\overleftarrow{\mathbf{d_2}}] = [\mathbf{d}_2^1] = [\mathbf{d}_5^4]$, $[\overleftarrow{\mathbf{d_3}}] = [\mathbf{d}_3^2] = [\mathbf{d}_4^3]$, $[\overrightarrow{\mathbf{d_1}}] = [\mathbf{d}_0^1] = [\mathbf{d}_5^6]$, $[\overrightarrow{\mathbf{d_2}}] = [\mathbf{d}_1^2] = [\mathbf{d}_4^5]$, and $[\overrightarrow{\mathbf{d_1}}] = [\mathbf{d}_2^3] = [\mathbf{d}_3^4]$.

Now a categorification of the action of $U(\mathfrak{so}(7,\mathbb{C}))$ on the n-th tensor power of its vector representation can be obtained as follows.

Proposition 9 *(1) For any $1 \le i \le 3$ and $[\mathbf{d}] \in \widetilde{\mathbf{D}}$, the action of h_i on $(V^{\otimes n})_{[\mathbf{d}]}$ can be categorified by the exact functor $\mathcal{H}_i^{\mathrm{sgn}(c_i(\mathbf{d}))}([\mathbf{d}])$, which means that the following diagram is* *commutative*:

$$
\begin{array}{ccc}
K(\mathcal{O}_{[\mathbf{d}]}) & \xrightarrow{\quad\gamma_n\quad} & (V^{\otimes n})_{[\mathbf{d}]} \\
\Big\downarrow [\mathcal{H}_i^{\mathrm{sgn}(c_i(\mathbf{d}))}([\mathbf{d}])] & & \Big\downarrow \mathrm{sgn}(c_i(\mathbf{d}))h_i \\
K(\mathcal{O}_{[\mathbf{d}]}) & \xrightarrow{\quad\gamma_n\quad} & (V^{\otimes n})_{[\mathbf{d}]}.
\end{array}
$$

(2) For any $1 \le i \le 3$ and $[\mathbf{d}] \in \widetilde{\mathbf{D}}$, the restriction of e_i from $(V^{\otimes n})_{[\mathbf{d}]}$ to $(V^{\otimes n})_{[\overleftarrow{\mathbf{d_i}}]}$ can be categorified by the exact functor $\mathcal{E}_i([\mathbf{d}])$, which means that the following diagram is commutative:

$$
\begin{array}{ccc}
K(\mathcal{O}_{[\mathbf{d}]}) & \xrightarrow{\gamma_n} & (V^{\otimes n})_{[\mathbf{d}]} \\
\Big\downarrow [\mathcal{E}_i([\mathbf{d}])] & & \Big\downarrow e_i \\
K(\mathcal{O}_{[\overleftarrow{\mathbf{d_i}}]}) & \xrightarrow{\gamma_n} & (V^{\otimes n})_{[\overleftarrow{\mathbf{d_i}}]}.
\end{array}
$$

(3) For any $1 \le i \le 3$ and $[\mathbf{d}] \in \widetilde{\mathbf{D}}$, the restriction of f_i from $(V^{\otimes n})_{[\mathbf{d}]}$ to $(V^{\otimes n})_{[\overrightarrow{\mathbf{d_i}}]}$ can be categorified by the exact functor $\mathcal{F}_i([\mathbf{d}])$, which means that the following diagram is commutative:

$$
\begin{array}{ccc}
K(\mathcal{O}_{[\mathbf{d}]}) & \xrightarrow{\gamma_n} & (V^{\otimes n})_{[\mathbf{d}]} \\
\Big\downarrow [\mathcal{F}_i([\mathbf{d}])] & & \Big\downarrow f_i \\
K(\mathcal{O}_{[\overrightarrow{\mathbf{d_i}}]}) & \xrightarrow{\gamma_n} & (V^{\otimes n})_{[\overrightarrow{\mathbf{d_i}}]}.
\end{array}
$$

Proof Here we only check some cases in (1) and (2). Other cases can be verified similarly.

(1) To check $\gamma_n \circ [\mathcal{H}_i^{\mathrm{sgn}(c_i(\mathbf{d}))}([\mathbf{d}])] = \mathrm{sgn}(c_i(\mathbf{d}))h_i \circ \gamma_n$ is equivalent to checking

$$\gamma_n \circ [\mathcal{H}_i^{\mathrm{sgn}(c_i(\mathbf{d}))}([\mathbf{d}])]([M(a_1,\ldots,a_n)]) = \mathrm{sgn}(c_i(\mathbf{d}))h_i \circ \gamma_n([M(a_1,\ldots,a_n)])$$

for any $[M(a_1,\ldots,a_n)] \in B_{[\mathbf{d}]} = \cup_{\mathbf{d}'\in[\mathbf{d}]}B_{\mathbf{d}'}$. In fact, we have

$$
\begin{aligned}
\gamma_n \circ [\mathcal{H}_i^{\mathrm{sgn}(c_i(\mathbf{d}))}([\mathbf{d}])]([M(a_1,\ldots,a_n)]) &= |c_i(\mathbf{d})|\gamma_n([M(a_1,\ldots,a_n)]) \\
&= |c_i(\mathbf{d})|(v_{a_1} \otimes v_{a_2} \otimes \cdots \otimes v_{a_n}) \\
&= \mathrm{sgn}(c_i(\mathbf{d}))c_i(\mathbf{d})(v_{a_1} \otimes v_{a_2} \otimes \cdots \otimes v_{a_n}) \\
&= \mathrm{sgn}(c_i(\mathbf{d}))h_i \circ \gamma_n([M(a_1,\ldots,a_n)]).
\end{aligned}
$$

(2) We give the proof of the case $i = 3$.

To verify the commutativity of the diagram in this case, it suffices to check $\gamma_n \circ [\mathcal{E}_3([\mathbf{d}])] = e_3 \circ \gamma_n$. Indeed, for any $[\mathbf{d}] \in \widetilde{\mathbf{D}}$ and $[M(a_1,\ldots,a_n)] \in B_{[\mathbf{d}]} = \cup_{\mathbf{d}'\in[\mathbf{d}]}B_{\mathbf{d}'}$, we assume $[M(a_1,\ldots,a_n)] \in B_{\mathbf{d}_0}$ for some $\mathbf{d}_0 \in [\mathbf{d}]$, by (17)–(20), then one has

$$
\begin{aligned}
[\mathcal{E}_3([\mathbf{d}])]([M(a_1,\ldots,a_n)]) =&[\bigoplus_{\mathbf{d}'\in[\mathbf{d}]}(\varphi(\mathbf{d}'^2_3, L_n^*)^{\oplus 2} \oplus \varphi(\mathbf{d}'^3_4, L_n^*))]([M(a_1,\ldots,a_n)]) \\
=&[(\varphi(\mathbf{d_0}^2_3, L_n^*)^{\oplus 2} \oplus \varphi(\mathbf{d_0}^3_4, L_n^*))M(a_1,\ldots,a_n)] \\
=&[\varphi(\mathbf{d_0}^2_3, L_n^*)^{\oplus 2}M(a_1,\ldots,a_n)] + [\varphi(\mathbf{d_0}^3_4, L_n^*)M(a_1,\ldots,a_n)] \\
=&\sum_{\substack{m=1,\\ a_m=3}}^{n} 2[M(a_1,\ldots,a_{m-1},a_m-1,a_{m+1},\ldots,a_n)]+ \\
&\sum_{\substack{m=1,\\ a_m=4}}^{n} [M(a_1,\cdots,a_{m-1},a_m-1,a_{m+1},\ldots,a_n)].
\end{aligned}
$$

It follows that

$$\begin{aligned}
\gamma_n\circ[\mathcal{E}_3([\mathbf{d}])]([M(a_1,\dots,a_n)]) =&\gamma_n(\sum_{\substack{m=1,\\ a_m=3}}^{n} 2[M(a_1,\dots,a_{m-1},a_m-1,a_{m+1},\dots,a_n)])+\\
&\gamma_n(\sum_{\substack{m=1,\\ a_m=4}}^{n} [M(a_1,\dots,a_{m-1},a_m-1,a_{m+1},\dots,a_n)])\\
=&2\sum_{\substack{m=1,\\ a_m=3}}^{n}(v_{a_1}\otimes\cdots\otimes v_{a_{m-1}}\otimes v_{a_m-1}\otimes v_{a_{m+1}}\otimes\cdots\otimes v_{a_n})+\\
&\sum_{\substack{m=1,\\ a_m=4}}^{n}(v_{a_1}\otimes\cdots\otimes v_{a_{m-1}}\otimes v_{a_m-1}\otimes v_{a_{m+1}}\otimes\cdots\otimes v_{a_n}).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
(e_3\circ\gamma_n)([M(a_1,\dots,a_n)]) =&e_3(v_{a_1}\otimes\cdots\otimes v_{a_{m-1}}\otimes v_{a_m}\otimes v_{a_{m+1}}\otimes\cdots\otimes v_{a_n})\\
=&\sum_{m=1}^{n}(v_{a_1}\otimes\cdots\otimes v_{a_{m-1}}\otimes e_3v_{a_m}\otimes v_{a_{m+1}}\otimes\cdots\otimes v_{a_n})\\
=&2\sum_{\substack{m=1,\\ a_m=3}}^{n}(v_{a_1}\otimes\cdots\otimes v_{a_{m-1}}\otimes v_{a_m-1}\otimes v_{a_{m+1}}\otimes\cdots\otimes v_{a_n})+\\
&\sum_{\substack{m=1,\\ a_m=4}}^{n}(v_{a_1}\otimes\cdots\otimes v_{a_{m-1}}\otimes v_{a_m-1}\otimes v_{a_{m+1}}\otimes\cdots\otimes v_{a_n}).
\end{aligned}$$

Hence, the diagram is commutative. □

Theorem 10 *For any* $1\le i\le 3$ *and* $1\le j\le 3$*, let*

$$\mathcal{H}_i^+=\bigoplus_{\substack{[\mathbf{d}]\in\widetilde{\mathbf{D}},\\ \mathrm{sgn}(c_i(\mathbf{d}))=1\ \text{or}\ 0}}\mathcal{H}_i^{\mathrm{sgn}(c_i(\mathbf{d}))}([\mathbf{d}]):\mathcal{O}^n\to\mathcal{O}^n,\quad \mathcal{H}_i^-=\bigoplus_{\substack{[\mathbf{d}]\in\widetilde{\mathbf{D}},\\ \mathrm{sgn}(c_i(\mathbf{d}))=-1}}\mathcal{H}_i^{\mathrm{sgn}(c_i(\mathbf{d}))}([\mathbf{d}]):\mathcal{O}^n\to\mathcal{O}^n,$$

$$\mathcal{E}_j=\bigoplus_{[\mathbf{d}]\in\widetilde{\mathbf{D}}}\mathcal{E}_j([\mathbf{d}]):\mathcal{O}^n\to\mathcal{O}^n,\quad \mathcal{F}_j=\bigoplus_{[\mathbf{d}]\in\widetilde{\mathbf{D}}}\mathcal{F}_j([\mathbf{d}]):\mathcal{O}^n\to\mathcal{O}^n.$$

Then we have the following results.

(1) For any $1\le i\le 3$*, the action of* h_i *on* $V^{\otimes n}$ *can be categorified by a pair of exact functors* $(\mathcal{H}_i^+,\mathcal{H}_i^-)$*, which means that the following diagram is commutative*:

$$\begin{array}{ccc}
K(\mathcal{O}^n) & \xrightarrow{\ \gamma_n\ } & V^{\otimes n}\\
\big\downarrow{\scriptstyle [\mathcal{H}_i^+]-[\mathcal{H}_i^-]} & & \big\downarrow{\scriptstyle h_i}\\
K(\mathcal{O}^n) & \xrightarrow{\ \gamma_n\ } & V^{\otimes n}.
\end{array}$$

(2) For any $1\le i\le 3$*, the action of* e_i *on* $V^{\otimes n}$ *can be categorified by the exact functor* $\mathcal{E}_i$*, which means that the following diagram is commutative:*

$$\begin{array}{ccc}
K(\mathcal{O}^n) & \xrightarrow{\gamma_n} & V^{\otimes n}\\
\big\downarrow{\scriptstyle [\mathcal{E}_i]} & & \big\downarrow{\scriptstyle e_i}\\
K(\mathcal{O}^n) & \xrightarrow{\gamma_n} & V^{\otimes n}.
\end{array}$$

(3) For any $1 \le i \le 3$, the action of f_i on $V^{\otimes n}$ can be categorified by the exact functor $\mathcal{F}_i$, which means that the following diagram is commutative:

$$\begin{array}{ccc} K(\mathcal{O}^n) & \xrightarrow{\gamma_n} & V^{\otimes n} \\ \Big\downarrow [\mathcal{F}_i] & & \Big\downarrow f_i \\ K(\mathcal{O}^n) & \xrightarrow{\gamma_n} & V^{\otimes n}. \end{array}$$

Proof It is not difficult to check the diagrams are commutative by Proposition 9. □

Now we categorify all the defining relations of $U(\mathfrak{so}(7,\mathbb{C}))$ as the natural isomorphisms between some projective functors of $\mathcal{O}(\mathfrak{gl}_n)$. In fact, we have the following result.

Theorem 11 *(1)* $\mathcal{H}_i^+ \circ \mathcal{H}_j^+ \oplus \mathcal{H}_i^- \circ \mathcal{H}_j^- \oplus \mathcal{H}_j^+ \circ \mathcal{H}_i^- \oplus \mathcal{H}_j^- \circ \mathcal{H}_i^+ \cong \mathcal{H}_i^+ \circ \mathcal{H}_j^- \oplus \mathcal{H}_i^- \circ \mathcal{H}_j^+ \oplus \mathcal{H}_j^+ \circ \mathcal{H}_i^+ \oplus \mathcal{H}_j^- \circ \mathcal{H}_i^-$ *for* $1 \le i, j \le 3$.

(2) $\mathcal{E}_i \circ \mathcal{F}_j \oplus \delta_{i,j}\mathcal{H}_i^- \cong \mathcal{F}_j \circ \mathcal{E}_i \oplus \delta_{i,j}\mathcal{H}_i^+$ for $1 \le i, j \le 3$.

(3) (a) $\mathcal{H}_i^+ \circ \mathcal{E}_i \oplus \mathcal{E}_i \circ \mathcal{H}_i^- \cong \mathcal{H}_i^- \circ \mathcal{E}_i \oplus \mathcal{E}_i \circ \mathcal{H}_i^+ \oplus \mathcal{E}_i^{\oplus 2}$ *for* $i = 1, 2, 3$;

(b) $\mathcal{H}_i^+ \circ \mathcal{E}_j \oplus \mathcal{E}_j \circ \mathcal{H}_i^- \oplus (\mathcal{E}_j)^{\oplus(-a_{i,j})} \cong \mathcal{H}_i^- \circ \mathcal{E}_j \oplus \mathcal{E}_j \circ \mathcal{H}_i^+$ *for* $1 \le i \ne j \le 3$.

(4) (a) $\mathcal{H}_i^+ \circ \mathcal{F}_i \oplus \mathcal{F}_i \circ \mathcal{H}_i^- \oplus \mathcal{F}_i^{\oplus 2} \cong \mathcal{H}_i^- \circ \mathcal{F}_i \oplus \mathcal{F}_i \circ \mathcal{H}_i^+$ *for* $i = 1, 2, 3$;

(b) $\mathcal{H}_i^+ \circ \mathcal{F}_j \oplus \mathcal{F}_j \circ \mathcal{H}_i^- \cong \mathcal{H}_i^- \circ \mathcal{F}_j \oplus \mathcal{F}_j \circ \mathcal{H}_i^+ \oplus (\mathcal{F}_j)^{\oplus(-a_{i,j})}$ *for* $1 \le i \ne j \le 3$.

(5) $$\sum_{\substack{k=0,\\ even}}^{1-a_{i,j}} (E_i^{(1-a_{i,j}-k)} \circ E_j \circ E_i^k)^{\oplus \binom{1-a_{i,j}}{k}} \cong \sum_{\substack{k=0,\\ odd}}^{1-a_{i,j}} (E_i^{(1-a_{i,j}-k)} \circ E_j \circ E_i^k)^{\oplus \binom{1-a_{i,j}}{k}}$$
for $1 \le i \ne j \le 3$.

(6) $$\sum_{\substack{k=0,\\ even}}^{1-a_{i,j}} (F_i^{(1-a_{i,j}-k)} \circ F_j \circ F_i^k)^{\oplus \binom{1-a_{i,j}}{k}} \cong \sum_{\substack{k=0,\\ odd}}^{1-a_{i,j}} (F_i^{(1-a_{i,j}-k)} \circ F_j \circ F_i^k)^{\oplus \binom{1-a_{i,j}}{k}}$$
for $1 \le i \ne j \le 3$.

Proof Note that $\mathcal{O}^n = \oplus_{[\mathbf{d}]\in\widetilde{\mathbf{D}}}\mathcal{O}_{[\mathbf{d}]}$, $\mathcal{O}_{[\mathbf{d}]} = \oplus_{\mathbf{d}'\in[\mathbf{d}]}\mathcal{O}_{\mathbf{d}'}$, $\mathcal{O}_{\mathbf{d}} = \mathcal{O}_{\theta_{\mathbf{d}}}(\mathfrak{gl}_n)$, $\theta_{\mathbf{d}} = \eta(\lambda_{\mathbf{d}} - \rho)$, and for any $\alpha \in R_+$, $\langle \lambda_{\mathbf{d}} - \rho + \rho, \alpha^\vee \rangle = \langle \lambda_{\mathbf{d}}, \alpha^\vee \rangle \in \mathbb{Z}^{\ge 0}$, this means that the weight $\lambda_{\mathbf{d}} - \rho$ is integral and dominant.

By Proposition 6 (1), to check the natural isomorphisms in Theorem 11 is equivalent to checking the equalities among the abelian group homomorphisms $[\mathcal{H}_i^+]$ and $[\mathcal{H}_i^-]$ $(1 \le i \le 3)$, $[\mathcal{E}_j]$ and $[\mathcal{F}_j]$ $(1 \le j \le 3)$ on the Grothendieck group $K(\mathcal{O}^n)$. In the following, we will check some cases, while the other cases can be similarly verified.

(1) By Proposition 6 (1), we only need to check that

$$\begin{aligned} &[\mathcal{H}_i^+] \circ [\mathcal{H}_j^+] + [\mathcal{H}_i^-] \circ [\mathcal{H}_j^-] + [\mathcal{H}_j^+] \circ [\mathcal{H}_i^-] + [\mathcal{H}_j^-] \circ [\mathcal{H}_i^+] \\ &\quad = [\mathcal{H}_i^+] \circ [\mathcal{H}_j^-] + [\mathcal{H}_i^-] \circ [\mathcal{H}_j^+] + [\mathcal{H}_j^+] \circ [\mathcal{H}_i^+] + [\mathcal{H}_j^-] \circ [\mathcal{H}_i^-]. \end{aligned} \tag{4}$$

Indeed, it is easy to see that $h_i h_j \gamma_n = h_j h_i \gamma_n$. By Theorem 10 (1), we have

$$h_i h_j \gamma_n = \gamma_n \circ ([\mathcal{H}_i^+] - [\mathcal{H}_i^-]) \circ ([\mathcal{H}_j^+] - [\mathcal{H}_j^-]),$$

$$h_j h_i \gamma_n = \gamma_n \circ ([\mathcal{H}_j^+] - [\mathcal{H}_j^-]) \circ ([\mathcal{H}_i^+] - [\mathcal{H}_i^-]).$$

Therefore,

$$([\mathcal{H}_i^+]-[\mathcal{H}_i^-])\circ([\mathcal{H}_j^+]-[\mathcal{H}_j^-])=([\mathcal{H}_j^+]-[\mathcal{H}_j^-])\circ([\mathcal{H}_i^+]-[\mathcal{H}_i^-]). \tag{5}$$

Immediately we get (4) by expanding (5).

(6) We only check the case $i=3, j=2$ and the remaining cases in (6) can be verified similarly. By Proposition 6 (1), it is equivalent to checking

$$[\mathcal{F}_3]\circ[\mathcal{F}_3]\circ[\mathcal{F}_3]\circ[\mathcal{F}_2]+3[\mathcal{F}_3]\circ[\mathcal{F}_2]\circ[\mathcal{F}_3]\circ[\mathcal{F}_3]=3[\mathcal{F}_3]\circ[\mathcal{F}_3]\circ[\mathcal{F}_2]\circ[\mathcal{F}_3]+[\mathcal{F}_2]\circ[\mathcal{F}_3]\circ[\mathcal{F}_3]\circ[\mathcal{F}_3].$$

By Theorem 10 (3), we have

$$\begin{aligned}
&\gamma_n\circ([\mathcal{F}_3]\circ[\mathcal{F}_3]\circ[\mathcal{F}_3]\circ[\mathcal{F}_2]+3[\mathcal{F}_3]\circ[\mathcal{F}_2]\circ[\mathcal{F}_3]\circ[\mathcal{F}_3])\\
=&\quad (f_3^3f_2+3f_3f_2f_3^2)\circ\gamma_n=(3f_3^2f_2f_3+f_2f_3^3)\circ\gamma_n\\
&=\gamma_n\circ(3[\mathcal{F}_3]\circ[\mathcal{F}_3]\circ[\mathcal{F}_2]\circ[\mathcal{F}_3]+[\mathcal{F}_2]\circ[\mathcal{F}_3]\circ[\mathcal{F}_3]\circ[\mathcal{F}_3]).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\gamma_n\circ([\mathcal{F}_3]\circ[\mathcal{F}_3]\circ[\mathcal{F}_3]\circ[\mathcal{F}_2]+3[\mathcal{F}_3]\circ[\mathcal{F}_2]\circ[\mathcal{F}_3]\circ[\mathcal{F}_3])\\
&=\gamma_n\circ(3[\mathcal{F}_3]\circ[\mathcal{F}_3]\circ[\mathcal{F}_2]\circ[\mathcal{F}_3]+[\mathcal{F}_2]\circ[\mathcal{F}_3]\circ[\mathcal{F}_3]\circ[\mathcal{F}_3]).
\end{aligned}$$

Therefore, $\mathcal{F}_3\circ\mathcal{F}_3\circ\mathcal{F}_3\circ\mathcal{F}_2\oplus(\mathcal{F}_3\circ\mathcal{F}_2\circ\mathcal{F}_3\circ\mathcal{F}_3)^{\oplus 3}\cong(\mathcal{F}_3\circ\mathcal{F}_3\circ\mathcal{F}_2\circ\mathcal{F}_3)^{\oplus 3}\oplus\mathcal{F}_2\circ\mathcal{F}_3\circ\mathcal{F}_3\circ\mathcal{F}_3$.

The proof of the theorem is finished. $\square$

Acknowledgement We are grateful to the referee and the editor for many useful comments, suggestions and information about relevant references.

References

[1] J. BERNSTEIN, I. FRENKEL, M. KHOVANOV. *A categorification of the Temperley-Lieb algebra and Schur quotients of $U(\mathfrak{sl}_2)$ via projective and Zuckerman functors*. Selecta Math. (N.S.), 1999, **5**(2): 199–241.

[2] J. N. BERNSTEIN, S. I. GELFAND. *Tensor products of finite and infinite dimensional representations of semisimple Lie algebras*. Compositio Math., 1980, **41**(2): 245–285.

[3] J. N. BERNSTEIN, I. M. GELFAND, S. I. GELFAND. *Category of $\mathfrak{g}$-modules*. Functional Anal. and Appl., 1976, **2**: 87–92.

[4] L. CRANE. *Clock and category: is quantum gravity algebraic?*. J. Math. Phys., 1995, **36**(11): 6180–6193.

[5] L. CRANE, I. FRENKEL. *Four-dimensional topological quantum field theory, Hopf categories, and the canonical bases, in: Topology and Physics*. J. Math. Phys., 1994, **35**(10): 5136–5154.

[6] W. FULTON, J. HARRIS. *Representation Theory: A First Course*. Springer-Verlag, New York, 1991.

[7] Jin HONG, Seok-Jin KANG. *Introduction to Quantum Groups and Crystal Bases*. Amer. Math. Soc., Rhode Island, 2002.

[8] J. E. HUMPHREYS. *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$*. Amer. Math. Soc., Rhode Island, 2008.

[9] M. KHOVANOV. *A categorification of the Jones polynomial*. Duke Math. J., 2000, **101**(3): 359–426.

[10] M. KHOVANOV, A. LAUDA. *A diagrammatic approach to categorification of quantum groups (I)*. Representation Theory, 2009, **13**: 309–347.

[11] V. MAZORCHUK. *Lectures on algebraic categorification*. European Mathematical Society, Zürich, Switzerland, 2012.

[12] Yucai SU, Caihui LU, Yimin CUI. *A Comprehensive Textbook on Finite-dimensional Semisimple Lie Algebras*. Science Press, Beijing, 2008.