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Pullback $\mathcal{D}$-Attractors for A Non-Autonomous Brinkman-Forchheimer System

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Abstract The asymptotic behavior of solutions of the three-dimensional nonautonomous Brinkman-Forchheimer equation is investigated. And the existence of pullback global attractors in $\mathbb{L}^2(\Omega)$ and $\mathbb{H}_0^1(\Omega)$ is proved, respectively.

Keywords pullback attractor; asymptotic compactness; Brinkman-Forchheimer equation.

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1. Introduction

In this paper, we consider the asymptotic behavior of the solutions of the following initial-boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu\Delta u + \alpha u + \beta|u|u + \gamma|u|^2u + \nabla p = f(x,t), & x \in \Omega, t > \tau; \\ \nabla \cdot u = 0, & x \in \Omega, t > \tau; \\ u|_{\partial\Omega} = 0, & t > \tau; \\ u(x,\tau) = u_\tau(x), & x \in \Omega, \end{cases} \tag{1}$$

where $\nu, \alpha, \beta, \gamma$ are positive constants and Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$.

The Brinkman-Forchheimer equation describes the motion of fluid flow in a saturated porous medium [1, 2], where u and p are the velocity and the pressure of the fluid, respectively, ν is the Brinkman coefficient, α is the Darcy coefficient, β and γ are the Forchheimer coefficients. The solutions of system (1) have been studied by many people [3–9]. We should note that most of these papers have been focused on the question of continuous dependence of solutions on the coefficients ν, β and γ (see [3–6]). The asymptotic behavior of solutions was examined in [7–9]. In [7] and [8], Uğurlu, Ouyang and Yang have proved the existence of global attractor in $\mathbb{H}_0^1$ for autonomous Brinkman-Forchheimer equation, respectively, with respect to initial data $u_0 \in \mathbb{H}_0^1$. In [9], Wang and Lin have showed that system (1) (in autonomous case) has a global attractor in $\mathbb{H}^2$ when the external forcing term belongs to $\mathbb{L}^2$. Here, $\mathbb{L}^2(\Omega)$ and $\mathbb{H}^s(\Omega)$ are defined by

$$\mathbb{L}^2(\Omega) = (L^2(\Omega))^3 \text{ and } \mathbb{H}^s(\Omega) = (H^s(\Omega))^3, \quad s > 0.$$

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For convenience, we reformulate system (1) as follows. Let

$$H = \{u \in \mathbb{L}^2(\Omega) : \nabla \cdot u = 0, u \cdot n|_{\partial\Omega} = 0\}$$

and

$$V = \{u \in \mathbb{H}^1(\Omega) : \nabla \cdot u = 0, u|_{\partial\Omega} = 0\},$$

where n is the unit outward normal vector at $\partial\Omega$. Let P be the Leray orthogonal projection from $\mathbb{L}^2(\Omega)$ onto H, and $A = P(-\Delta)$ the Stokes operator. Then system (1) is equivalent to the following functional equation:

$$u_t + \nu Au + \alpha u + B(u) = f \tag{2}$$

with the initial condition

$$u(\tau) = u_\tau, \tag{3}$$

where $B(u) = PF(u)$, and $F(u) = \beta|u|u + \gamma|u|^2u$.

In this paper, we investigate the asymptotic behavior of solutions of system (1). We prove the existence of pullback $\mathcal{D}$-attractors in H and V, respectively.

The following notations will be used throughout the paper. We denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and inner product in H and use $\|\cdot\|_p$ to denote the norm in $\mathbb{L}^p(\Omega)$. The letter C denotes generic positive constant which may change its values from line to line or even in the same line.

2. Preliminaries and abstract results

In this section, we recall some definitions and results concerning the pullback attractor. These definitions and results can be found in [10–12]. Let Θ be a nonempty set, X be a metric space with distance $d(\cdot,\cdot)$. And $\mathcal{D}$ be a collection of families of subsets of X:

$$\mathcal{D} = \{D = \{D(\omega)\}_{\omega\in\Theta} : D(\omega) \subseteq X \text{ for every } \omega \in \Theta\}.$$

Definition 2.1 *A family of mapping $\{\theta_t\}_{t\in\mathbb{R}}$ from Θ to itself is called a family of shift operators on Θ if $\{\theta_t\}_{t\in\mathbb{R}}$ satisfies the group properties:*

(i) $\theta_0\omega = \omega$, $\forall\omega \in \Theta$;

(ii) $\theta_t(\theta_\tau\omega) = \theta_{t+\tau}\omega$, $\forall\omega \in \Theta$ *and* $t, \tau \in \mathbb{R}$.

Definition 2.2 *Let $\{\theta_t\}_{t\in\mathbb{R}}$ be a family of shift operators on Θ. Then a continuous θ-cocycle ϕ on X is a mapping*

$$\phi : \mathbb{R}^+ \times \Theta \times X \to X, (t, \omega, x) \mapsto \phi(t, \omega, x),$$

which satisfies, for all $\omega \in \Theta$ and $t, \tau \in \mathbb{R}^+$,

(i) $\phi(0, \omega, \cdot)$ *is the identity on* X;

(ii) $\phi(t + \tau, \omega, \cdot) = \phi(t, \theta_\tau\omega, \cdot) \circ \phi(\tau, \omega, \cdot)$;

(iii) $\phi(t, \omega, \cdot) : X \to X$ *is continuous.*

Definition 2.3 *Let $\mathcal{D}$ be a collection of families of subsets of X. Then $\mathcal{D}$ is called inclusion-closed if $D = \{D(\omega)\}_{\omega\in\Theta} \in \mathcal{D}$ and $\tilde{D} = \{\tilde{D}(\omega) \subseteq X : \omega \in \Theta\}$ with $\tilde{D}(\omega) \subseteq D(\omega)$ for all $\omega \in \Theta$ imply that $\tilde{D} \in \mathcal{D}$.*

Definition 2.4 *Let $\mathcal{D}$ be a collection of families of subsets of X and $\{K(\omega)\}_{\omega\in\Theta} \in \mathcal{D}$. Then $\{K(\omega)\}_{\omega\in\Theta}$ is called a pullback absorbing set for ϕ in $\mathcal{D}$ if, for every $B \in \mathcal{D}$ and $\omega \in \Theta$, there exists $T(\omega, B) > 0$ such that*

$$\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subseteq K(\omega) \text{ for all } t \geq T(\omega, B).$$

Definition 2.5 *Let $\mathcal{D}$ be a collection of families of subsets of X. Then ϕ is said to be $\mathcal{D}$-pullback asymptotically compact in X if, for every $\omega \in \Theta$, $\{\phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$, and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Theta} \in \mathcal{D}$.*

Definition 2.6 *Let $\mathcal{D}$ be a collection of families of subsets of X and $\{\mathcal{A}(\omega)\}_{\omega\in\Theta} \in \mathcal{D}$. Then $\{\mathcal{A}(\omega)\}_{\omega\in\Theta}$ is called a $\mathcal{D}$-pullback global attractor for ϕ if the following conditions are satisfied, for every $\omega \in \Theta$,*

(i) $\mathcal{A}(\omega)$ is compact;

(ii) $\{\mathcal{A}(\omega)\}_{\omega\in\Theta}$ is invariant, that is,

$$\phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega), \forall t \geq 0;$$

(iii) $\{\mathcal{A}(\omega)\}_{\omega\in\Theta}$ attracts every set in $\mathcal{D}$, that is, for every $B = \{B(\omega)\}_{\omega\in\Theta} \in \mathcal{D}$,

$$\lim_{t\to\infty} \operatorname{dist}(\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

where dist is the Hausdorff semi-metric given by $\operatorname{dist}(A, B) = \sup_{a\in A} \inf_{b\in B} \| a - b \|_X$ for any $A \subset X$ and $B \subset X$.

Proposition 2.1 *Let $\mathcal{D}$ be an inclusion-closed collection of families of subsets of X and ϕ a continuous θ-cocycle on X. Suppose that $\{K(\omega)\}_{\omega\in\Theta} \in \mathcal{D}$ is a closed absorbing set for ϕ in $\mathcal{D}$ and ϕ is $\mathcal{D}$-pullback asymptotically compact in X. Then ϕ has a unique $\mathcal{D}$-pullback global attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Theta} \in \mathcal{D}$ which is given by*

$$\mathcal{A}(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}.$$

Now we assume $f \in L^2_{\text{loc}}(\mathbb{R}; H)$ which satisfies the following condition:

$$\int_{-\infty}^{\tau} e^{\sigma\xi} \| f(\xi) \|^2 \, \mathrm{d}\xi < \infty, \ \forall \tau \in \mathbb{R}, \tag{4}$$

where $0 < \sigma < \alpha$ is a fixed constant.

The existence and uniqueness of solutions for system (1) can be proved by a standard method as in [9] for the autonomous case. So we give the following result.

Theorem 2.1 *If $f \in L^2_{\text{loc}}(\mathbb{R}; H)$ and $u_\tau \in H$, then system (1) has a unique solution u with*

$$u \in C([\tau, +\infty); H) \cap L^2(\tau, T; V) \cap L^4(\tau, T; \mathbf{L}^4(\Omega)).$$

In addition, the solution is continuous with respect to u_τ in H.

We now introduce a shift operator $\theta_t : \mathbb{R} \to \mathbb{R}$ for every $t \in \mathbb{R}$:

$$\theta_t(\tau) = t + \tau, \forall \tau \in \mathbb{R}.$$

Let $\phi : \mathbb{R}^+ \times \mathbb{R} \times H \mapsto H$ be defined as

$$\phi(t, \tau, u_\tau) = u(t + \tau, \tau, u_\tau),$$

where $t \geq 0$, $\tau \in \mathbb{R}$, $u_\tau \in H$, and u is the solution of system (1). By the uniqueness of solutions, it is easy to verify that ϕ is a continuous θ-cocycle defined on H. Let $D = \{D(t)\}_{t\in\mathbb{R}}$ be a family of subsets of H, i.e., $D(t) \subset H$ for every $t \in \mathbb{R}$. In this paper, we are interested in a family $D = \{D(t)\}_{t\in\mathbb{R}}$ satisfying

$$\lim_{t\to-\infty} e^{\sigma t} \parallel D(t) \parallel^2 = 0, \tag{5}$$

where σ is a positive constant corresponding to (4) and $\parallel D(t) \parallel = \sup_{u\in D(t)} \parallel u \parallel$. We write the collection of all families satisfying (5) as $\mathcal{D}_\sigma$, that is,

$$\mathcal{D}_\sigma = \{D = \{D(t)\}_{t\in\mathbb{R}} : D \text{ satisfies } (5)\}. \tag{6}$$

The existence of $\mathcal{D}_\sigma$-pullback attractors for system (1) in H and V will be proved in the last section of this paper.

3. Uniform estimates of solutions

In this section, we derive uniform estimates of solutions of system (1). These estimates are necessary for proving the existence of bounded pullback absorbing sets and the pullback asymptotic compactness of the θ-cocycle ϕ associated with the system.

Lemma 3.1 *For every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\sigma$, there exists $T = T(\tau, D) > 0$ such that, for all $t \geq T$,*

$$\parallel u(\tau, \tau - t, u_0(\tau - t)) \parallel^2 \leq Ce^{-\sigma\tau} \int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi;$$
$$\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel u(\xi, \tau - t, u_0(\tau - t)) \parallel_3^3 \mathrm{d}\xi \leq C \int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi;$$
$$\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel u(\xi, \tau - t, u_0(\tau - t)) \parallel_4^4 \mathrm{d}\xi \leq C \int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi;$$
$$\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel \nabla u(\xi, \tau - t, u_0(\tau - t)) \parallel^2 \mathrm{d}\xi \leq C \int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi,$$

where $u_0(\tau - t) \in D(\tau - t)$, and C is a positive constant that does not depend on τ or D.

Proof Taking the inner product of (2) with u yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \parallel u \parallel^2 + \nu \parallel \nabla u \parallel^2 + \alpha \parallel u \parallel^2 + \beta \parallel u \parallel_3^3 + \gamma \parallel u \parallel_4^4 = \int_\Omega f(t)u\mathrm{d}x. \tag{7}$$

By Young inequality, it follows from (7) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \parallel u \parallel^2 + 2\nu \parallel \nabla u \parallel^2 + \alpha \parallel u \parallel^2 + 2\beta \parallel u \parallel_3^3 + 2\gamma \parallel u \parallel_4^4 \leq \frac{\parallel f(t) \parallel^2}{\alpha}. \tag{8}$$

Multiplying (8) by $e^{\sigma t}$ and then integrating over $(\tau - t, \tau)$ with $t \geq 0$, we obtain

$$\parallel u(\tau, \tau - t, u_0(\tau - t)) \parallel^2 + 2\nu e^{-\sigma\tau} \int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel \nabla u(\xi, \tau - t, u_0(\tau - t)) \parallel^2 \mathrm{d}\xi +$$

$$2\beta e^{-\sigma\tau}\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel u(\xi,\tau-t,u_0(\tau-t)) \parallel_3^3 \mathrm{d}\xi + 2\gamma e^{-\sigma\tau}\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel u(\xi,\tau-t,u_0(\tau-t)) \parallel_4^4 \mathrm{d}\xi$$
$$\le (\sigma-\alpha)e^{-\sigma\tau}\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel u(\xi,\tau-t,u_0(\tau-t)) \parallel^2 \mathrm{d}\xi + \frac{1}{\alpha}e^{-\sigma\tau}\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi +$$
$$e^{-\sigma\tau}e^{\sigma(\tau-t)} \parallel u_0(\tau-t) \parallel^2 . \tag{9}$$

And since $0<\sigma<\alpha$, we have

$$\parallel u(\tau,\tau-t,u_0(\tau-t)) \parallel^2 + 2\nu e^{-\sigma\tau}\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel \nabla u(\xi,\tau-t,u_0(\tau-t)) \parallel^2 \mathrm{d}\xi +$$
$$2\beta e^{-\sigma\tau}\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel u(\xi,\tau-t,u_0(\tau-t)) \parallel_3^3 \mathrm{d}\xi + 2\gamma e^{-\sigma\tau}\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel u(\xi,\tau-t,u_0(\tau-t)) \parallel_4^4 \mathrm{d}\xi$$
$$\le \frac{1}{\alpha}e^{-\sigma\tau}\int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi + e^{-\sigma\tau}e^{\sigma(\tau-t)} \parallel u_0(\tau-t) \parallel^2 . \tag{10}$$

Since $u_0(\tau-t)\in D(\tau-t)$ and $D=\{D(t)\}_{t\in\mathbb{R}}\in\mathcal{D}_\sigma$, for every $\tau\in\mathbb{R}$, there exists $T=T(\tau,D)>0$ such that, for all $t\ge T$,

$$e^{\sigma(\tau-t)} \parallel u_0(\tau-t) \parallel^2 \le \frac{1}{\alpha}\int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi. \tag{11}$$

By (10) and (11), we find that

$$\parallel u(\tau,\tau-t,u_0(\tau-t)) \parallel^2 + 2\nu e^{-\sigma\tau}\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel \nabla u(\xi,\tau-t,u_0(\tau-t)) \parallel^2 \mathrm{d}\xi +$$
$$2\beta e^{-\sigma\tau}\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel u(\xi,\tau-t,u_0(\tau-t)) \parallel_3^3 \mathrm{d}\xi + 2\gamma e^{-\sigma\tau}\int_{\tau-t}^{\tau} e^{\sigma\xi} \parallel u(\xi,\tau-t,u_0(\tau-t)) \parallel_4^4 \mathrm{d}\xi$$
$$\le \frac{2}{\alpha}e^{-\sigma\tau}\int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi, \tag{12}$$

from which, Lemma 3.1 follows.

Lemma 3.2 *For all $\tau\in\mathbb{R}$ and $D=\{D(t)\}_{t\in\mathbb{R}}\in\mathcal{D}_\sigma$, there exists $T=T(\tau,D)>2$ such that, for all $t\ge T$,*

$$\int_{\tau-2}^{\tau} e^{\sigma\xi} \parallel u(\xi,\tau-t,u_0(\tau-t)) \parallel^2 \mathrm{d}\xi \le C\int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi;$$
$$\int_{\tau-2}^{\tau} e^{\sigma\xi} \parallel \nabla u(\xi,\tau-t,u_0(\tau-t)) \parallel^2 \mathrm{d}\xi \le C\int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi;$$
$$\int_{\tau-2}^{\tau} e^{\sigma\xi} \parallel u(\xi,\tau-t,u_0(\tau-t)) \parallel_3^3 \mathrm{d}\xi \le C\int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi;$$
$$\int_{\tau-2}^{\tau} e^{\sigma\xi} \parallel u(\xi,\tau-t,u_0(\tau-t)) \parallel_4^4 \mathrm{d}\xi \le C\int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi,$$

where $u_0(\tau-t)\in D(\tau-t)$, and C is a positive constant that does not depend on τ or D.

Proof It follows from (8) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \parallel u \parallel^2 + \alpha \parallel u \parallel^2 \le \frac{\parallel f(t) \parallel^2}{\alpha}. \tag{13}$$

Let $s\in[\tau-2,\tau]$ and $t\ge 2$. Multiplying (13) by $e^{\sigma t}$, then relabeling t as ξ and integrating with respect to ξ over $(\tau - t, s)$, we get

$$\begin{aligned}
&e^{\sigma s} \parallel u(s, \tau - t, u_0(\tau - t)) \parallel^2 \\
&\leq e^{\sigma(\tau - t)} \parallel u_0(\tau - t) \parallel^2 + (\sigma - \alpha) \int_{\tau - t}^{s} e^{\sigma\xi} \parallel u(\xi, \tau - t, u_0(\tau - t)) \parallel^2 \mathrm{d}\xi + \\
&\frac{1}{\alpha} \int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi \\
&\leq e^{\sigma(\tau - t)} \parallel u_0(\tau - t) \parallel^2 + \frac{1}{\alpha} \int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi. \qquad (14)
\end{aligned}$$

In light of (14), there exists $T = T(\tau, D) > 2$, such that, for all $t \geq T$ and $s \in [\tau - 2, \tau]$,

$$e^{\sigma s} \parallel u(s, \tau - t, u_0(\tau - t)) \parallel^2 \leq \frac{2}{\alpha} \int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi. \qquad (15)$$

Integrating (15) with respect to s over the interval $(\tau - 2, \tau)$ produces

$$\int_{\tau-2}^{\tau} e^{\sigma s} \parallel u(s, \tau - t, u_0(\tau - t)) \parallel^2 \mathrm{d}s \leq \frac{4}{\alpha} \int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi. \qquad (16)$$

Multiplying (8) by $e^{\sigma t}$ and then integrating over $(\tau - 2, \tau)$, by (15) we obtain that, for all $t \geq T$,

$$\begin{aligned}
&e^{\sigma\tau} \parallel u(\tau, \tau - t, u_0(\tau - t)) \parallel^2 + 2\nu \int_{\tau-2}^{\tau} e^{\sigma\xi} \parallel \nabla u(\xi, \tau - t, u_0(\tau - t)) \parallel^2 \mathrm{d}\xi \\
&\quad 2\beta \int_{\tau-2}^{\tau} e^{\sigma\xi} \parallel u(\xi, \tau - t, u_0(\tau - t)) \parallel_3^3 \mathrm{d}\xi + 2\gamma \int_{\tau-2}^{\tau} e^{\sigma\xi} \parallel u(\xi, \tau - t, u_0(\tau - t)) \parallel_4^4 \mathrm{d}\xi \\
&\leq (\sigma - \alpha) \int_{\tau-2}^{\tau} e^{\sigma\xi} \parallel u(\xi, \tau - t, u_0(\tau - t)) \parallel^2 \mathrm{d}\xi + \frac{1}{\alpha} \int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi \\
&\quad + e^{\sigma(\tau-2)} \parallel u(\tau - 2, \tau - t, u_0(\tau - t)) \parallel^2 \\
&\leq \frac{3}{\alpha} \int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi,
\end{aligned}$$

which along with (16) completes the proof. □

Corollary 3.1 *For all $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\sigma$, there exists $T = T(\tau, D) > 2$ such that, for all $t \geq T$,*

$$\begin{aligned}
&\int_{\tau-2}^{\tau} \parallel u(\xi, \tau - t, u_0(\tau - t)) \parallel^2 \mathrm{d}\xi \leq Ce^{-\sigma\tau} \int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi; \\
&\int_{\tau-2}^{\tau} \parallel \nabla u(\xi, \tau - t, u_0(\tau - t)) \parallel^2 \mathrm{d}\xi \leq Ce^{-\sigma\tau} \int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi; \\
&\int_{\tau-2}^{\tau} \parallel u(\xi, \tau - t, u_0(\tau - t)) \parallel_3^3 \mathrm{d}\xi \leq Ce^{-\sigma\tau} \int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi; \\
&\int_{\tau-2}^{\tau} \parallel u(\xi, \tau - t, u_0(\tau - t)) \parallel_4^4 \mathrm{d}\xi \leq Ce^{-\sigma\tau} \int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi,
\end{aligned}$$

where $u_0(\tau - t) \in D(\tau - t)$, and C is a positive constant that does not depend on τ or D.

Proof The above estimates are straightforward from Lemma 3.2.

Lemma 3.3 *For all $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\sigma$, there exists $T = T(\tau, D) > 2$ such that,* *for all $t \geq T$,*

$$\| \nabla u(\tau,\tau-t,u_0(\tau-t)) \|^2 \leq Ce^{-\sigma\tau}\int_{-\infty}^{\tau} e^{\sigma\xi} \| f(\xi) \|^2 \,\mathrm{d}\xi;$$

$$\| u(\tau,\tau-t,u_0(\tau-t)) \|_3^3 \leq Ce^{-\sigma\tau}\int_{-\infty}^{\tau} e^{\sigma\xi} \| f(\xi) \|^2 \,\mathrm{d}\xi;$$

$$\| u(\tau,\tau-t,u_0(\tau-t)) \|_4^4 \leq Ce^{-\sigma\tau}\int_{-\infty}^{\tau} e^{\sigma\xi} \| f(\xi) \|^2 \,\mathrm{d}\xi;$$

$$\int_{\tau-1}^{\tau} \| u_\xi(\xi,\tau-t,u_0(\tau-t)) \|^2 \,\mathrm{d}\xi \leq Ce^{-\sigma\tau}\int_{-\infty}^{\tau} e^{\sigma\xi} \| f(\xi) \|^2 \,\mathrm{d}\xi,$$

where $u_0(\tau-t) \in D(\tau-t)$, and C is a positive constant that does not depend on τ or D.

Proof For convenience, we will write $u_0(\tau-t)$ as u_0 in what follows. Taking the inner product of (2) with u_t in H, we get

$$\begin{aligned}&\| u_\xi(\xi,\tau-t,u_0) \|^2 + \frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}\xi} \| \nabla u(\xi,\tau-t,u_0) \|^2 + \frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}\xi} \| u(\xi,\tau-t,u_0) \|^2 + \\ &\quad \frac{\beta}{3}\frac{\mathrm{d}}{\mathrm{d}\xi} \| u(\xi,\tau-t,u_0) \|_3^3 + \frac{\gamma}{4}\frac{\mathrm{d}}{\mathrm{d}\xi} \| u(\xi,\tau-t,u_0) \|_4^4 \\ &= (f(\xi),u_\xi(\xi,\tau-t,u_0)) \leq \frac{1}{2} \| u_\xi(\xi,\tau-t,u_0) \|^2 + \frac{1}{2} \| f(\xi) \|^2 .\end{aligned}$$

Then

$$\begin{aligned}&\| u_\xi(\xi,\tau-t,u_0) \|^2 + \frac{\mathrm{d}}{\mathrm{d}\xi}\Big(\nu \| \nabla u(\xi,\tau-t,u_0) \|^2 + \alpha \| u(\xi,\tau-t,u_0) \|^2 + \\ &\quad \frac{2}{3}\beta \| u(\xi,\tau-t,u_0) \|_3^3 + \frac{\gamma}{2} \| u(\xi,\tau-t,u_0) \|_4^4 \Big) \leq \| f(\xi) \|^2 .\end{aligned} \tag{17}$$

Let $s \leq \tau$ and $t \geq 2$. Integrating (17) over (s,τ), it follows that

$$\begin{aligned}&\nu \| \nabla u(\tau,\tau-t,u_0) \|^2 + \alpha \| u(\tau,\tau-t,u_0) \|^2 + \frac{2}{3}\beta \| u(\tau,\tau-t,u_0) \|_3^3 + \frac{\gamma}{2} \| u(\tau,\tau-t,u_0) \|_4^4 \\ &\leq \nu \| \nabla u(s,\tau-t,u_0) \|^2 + \alpha \| u(s,\tau-t,u_0) \|^2 + \frac{2}{3}\beta \| u(s,\tau-t,u_0) \|_3^3 + \\ &\quad \frac{\gamma}{2} \| u(s,\tau-t,u_0) \|_4^4 + \int_s^{\tau} \| f(\xi) \|^2 \,\mathrm{d}\xi.\end{aligned} \tag{18}$$

Integrating (18) with respect to s on $(\tau-1,\tau)$ produces

$$\begin{aligned}&\nu \| \nabla u(\tau,\tau-t,u_0) \|^2 + \alpha \| u(\tau,\tau-t,u_0) \|^2 + \frac{2}{3}\beta \| u(\tau,\tau-t,u_0) \|_3^3 + \frac{\gamma}{2} \| u(\tau,\tau-t,u_0) \|_4^4 \\ &\leq \nu \int_{\tau-1}^{\tau} \| \nabla u(s,\tau-t,u_0) \|^2 \,\mathrm{d}s + \alpha \int_{\tau-1}^{\tau} \| u(s,\tau-t,u_0) \|^2 \,\mathrm{d}s + \\ &\quad \frac{2}{3}\beta \int_{\tau-1}^{\tau} \| u(s,\tau-t,u_0) \|_3^3 \,\mathrm{d}s + \frac{\gamma}{2}\int_{\tau-1}^{\tau} \| u(s,\tau-t,u_0) \|_4^4 \,\mathrm{d}s + \int_{\tau-1}^{\tau} \| f(\xi) \|^2 \,\mathrm{d}\xi.\end{aligned}$$

By Corollary 3.1, there exists $T = T(\tau,D) > 2$ such that, for all $t \geq T$,

$$\begin{aligned}&\nu \| \nabla u(\tau,\tau-t,u_0) \|^2 + \alpha \| u(\tau,\tau-t,u_0) \|^2 + \frac{2}{3}\beta \| u(\tau,\tau-t,u_0) \|_3^3 + \frac{\gamma}{2} \| u(\tau,\tau-t,u_0) \|_4^4 \\ &\leq Ce^{-\sigma\tau}\int_{-\infty}^{\tau} e^{\sigma\xi} \| f(\xi) \|^2 \,\mathrm{d}\xi.\end{aligned} \tag{19}$$

Similarly, we can also prove that, for all $t \geq T$,

$$\nu \parallel \nabla u(\tau-1,\tau-t,u_0) \parallel^2 + \alpha \parallel u(\tau-1,\tau-t,u_0) \parallel^2 + \frac{2}{3}\beta \parallel u(\tau-1,\tau-t,u_0) \parallel_3^3 + \frac{\gamma}{2} \parallel u(\tau-1,\tau-t,u_0) \parallel_4^4 \leq Ce^{-\sigma\tau}\int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi. \tag{20}$$

We now integrate (17) with respect to ξ on $(\tau-1,\tau)$ to obtain

$$\begin{aligned}
&\int_{\tau-1}^{\tau} \parallel u_\xi(\xi,\tau-t,u_0) \parallel^2 \mathrm{d}\xi + \nu \parallel \nabla u(\tau,\tau-t,u_0) \parallel^2 + \alpha \parallel u(\tau,\tau-t,u_0) \parallel^2 + \\
&\quad \frac{2}{3}\beta \parallel u(\tau,\tau-t,u_0) \parallel_3^3 + \frac{\gamma}{2} \parallel u(\tau,\tau-t,u_0) \parallel_4^4 \\
&\leq \int_{\tau-1}^{\tau} \parallel f(\xi) \parallel^2 \mathrm{d}\xi + \nu \parallel \nabla u(\tau-1,\tau-t,u_0) \parallel^2 + \alpha \parallel u(\tau-1,\tau-t,u_0) \parallel^2 + \\
&\quad \frac{2}{3}\beta \parallel u(\tau-1,\tau-t,u_0) \parallel_3^3 + \frac{\gamma}{2} \parallel u(\tau-1,\tau-t,u_0) \parallel_4^4 \\
&\leq Ce^{-\sigma\tau}\int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi.
\end{aligned}$$

Lemma 3.4 *Let $\frac{\mathrm{d}f}{\mathrm{d}t} \in L^2_{\mathrm{loc}}(\mathbb{R};H)$. Then for all $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\sigma$, there exists $T = T(\tau, D) > 2$ such that, for all $t \geq T$,*

$$\parallel u_\tau(\tau,\tau-t,u_0(\tau-t)) \parallel^2 \leq Ce^{-\sigma\tau}\int_{-\infty}^{\tau} e^{\sigma\xi} \parallel f(\xi) \parallel^2 \mathrm{d}\xi + C\int_{\tau-1}^{\tau} \parallel f_\xi(\xi) \parallel^2 \mathrm{d}\xi,$$

where $u_0(\tau-t) \in D(\tau-t)$ and C is a positive constant independent of τ or D.

Proof Let $u_t = v$ and differentiate (2) with respect to t to get

$$\frac{\partial v}{\partial t} + \nu A v + \alpha v + B'(u)v = f_t(x,t). \tag{21}$$

Taking the inner product of (21) with v yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \parallel v \parallel^2 + \nu \parallel \nabla v \parallel^2 + \alpha \parallel v \parallel^2 + \int_\Omega (F'(u)v)\cdot v \mathrm{d}x = (f_t(t), v). \tag{22}$$

From Lemma 2.1 in [9], $\int_\Omega (F'(u)v)\cdot v\mathrm{d}x$ is positive definite, and according to Young inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \parallel v \parallel^2 \leq \frac{\parallel f_t(t) \parallel^2}{2\alpha}. \tag{23}$$

Let $s \in [\tau-1,\tau]$ and $t \geq 1$. Relabeling t as ξ and then integrating (23) with respect to ξ on (s,τ), by $v = u_t$ we get

$$\begin{aligned}
\parallel u_\tau(\tau,\tau-t,u_0(\tau-t)) \parallel^2 &\leq \parallel u_s(s,\tau-t,u_0(\tau-t)) \parallel^2 + \frac{1}{2\alpha}\int_s^\tau \parallel f_\xi(\xi) \parallel^2 \mathrm{d}\xi \\
&\leq \parallel u_s(s,\tau-t,u_0(\tau-t)) \parallel^2 + \frac{1}{2\alpha}\int_{\tau-1}^\tau \parallel f_\xi(\xi) \parallel^2 \mathrm{d}\xi.
\end{aligned} \tag{24}$$

Integrating (24) with respect to s on $(\tau-1,\tau)$ produces

$$\parallel u_\tau(\tau,\tau-t,u_0(\tau-t)) \parallel^2 \leq \int_{\tau-1}^{\tau} \parallel u_s(s,\tau-t,u_0(\tau-t)) \parallel^2 \mathrm{d}s + \frac{1}{2\alpha}\int_{\tau-1}^{\tau} \parallel f_\xi(\xi) \parallel^2 \mathrm{d}\xi, \tag{25}$$

which together with Lemma 3.3 shows that there exists $T=T(\tau,D)>2$ such that, for all $t\geq T$,

$$\| u_\tau(\tau,\tau-t,u_0)\|^2\leq Ce^{-\sigma\tau}\int_{-\infty}^{\tau} e^{\sigma\xi}\| f(\xi)\|^2\,\mathrm{d}\xi+\frac{1}{2\alpha}\int_{\tau-1}^{\tau}\| f_\xi(\xi)\|^2\,\mathrm{d}\xi.$$

4. Existence of pullback attractors

In this section, we first prove the existence of a $\mathcal{D}_\sigma$-pullback attractor $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ for system (1) in H, and then prove that $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ is actually a $\mathcal{D}_\sigma$-pullback attractor in V.

Lemma 4.1 *ϕ is $\mathcal{D}_\sigma$-pullback asymptotically compact in H, meaning that, for every $\tau\in\mathbb{R}, D=\{D(t)\}_{t\in\mathbb{R}}\in\mathcal{D}_\sigma$, and $t_n\to\infty$, $u_{0n}\in D(\tau-t_n)$, the sequence $\phi(t_n,\tau-t_n,u_{0n})$ has a convergent subsequence in H.*

Proof By Lemma 3.3, there exist $C=C(\tau)>0$ and $N=N(\tau,D)>0$ such that, for all $n\geq N$,

$$\|\nabla\phi(t_n,\tau-t_n,u_{0n})\|\leq C. \tag{26}$$

By (26) and the compactness of embedding $V\hookrightarrow H$, the sequence $\phi(t_n,\tau-t_n,u_{0n})$ is precompact in H. So for every $\tau\in\mathbb{R}$, $D=\{D(t)\}_{t\in\mathbb{R}}\in\mathcal{D}_\sigma$, and $t_n\to\infty$, $u_{0n}\in D(\tau-t_n)$, the sequence $\phi(t_n,\tau-t_n,u_{0n})$ has a convergent subsequence in H.

Theorem 4.1 *System (1) has a unique $\mathcal{D}_\sigma$-pullback global attractor $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}\in\mathcal{D}_\sigma$ in H.*

Proof For $\tau\in\mathbb{R}$, denote

$$B(\tau)=\{u:\| u\|^2\leq Ce^{-\sigma\tau}\int_{-\infty}^{\tau}e^{\sigma\xi}\| f(\xi)\|^2\,\mathrm{d}\xi\},$$

where C corresponds to the positive constant in Lemma 3.1. Then $B=\{B(\tau)\}_{\tau\in\mathbb{R}}\in\mathcal{D}_\sigma$ is a $\mathcal{D}_\sigma$-pullback absorbing set for ϕ in H by Lemma 3.1. And from Lemma 4.1 we know ϕ is $\mathcal{D}_\sigma$-pullback asymptotically compact. Then the existence of a $\mathcal{D}_\sigma$-pullback global attractor for ϕ in H follows from Proposition 2.1 immediately.

Lemma 4.2 *Let $\frac{\mathrm{d}f}{\mathrm{d}t}\in L^2_{\mathrm{loc}}(\mathbb{R};H)$. Then ϕ is $\mathcal{D}_\sigma$-pullback asymptotically compact in V, meaning that, for all $\tau\in\mathbb{R}$, $D=\{D(t)\}_{t\in\mathbb{R}}\in\mathcal{D}_\sigma$, and $t_n\to+\infty$, $u_{0n}\in D(\tau-t_n)$, the sequence $\phi(t_n,\tau-t_n,u_{0n})$ has a convergent subsequence in V.*

Proof We will show that the sequence $\phi(t_n,\tau-t_n,u_{0n})$ has a Cauchy subsequence in V. By Lemma 4.1, up to a subsequence,

$$\phi(t_n,\tau-t_n,u_{0n})=u(\tau,\tau-t_n,u_{0n})\text{ is a Cauchy sequence in }H. \tag{27}$$

From (2), we find that, for any $n,m\geq 1$,

$$\begin{aligned}&\nu(Au(\tau,\tau-t_n,u_{0n})-Au(\tau,\tau-t_m,u_{0m}))\\&\quad=-u_\tau(\tau,\tau-t_n,u_{0n})+u_\tau(\tau,\tau-t_m,u_{0m})-\alpha u(\tau,\tau-t_n,u_{0n})+\\&\qquad\alpha u(\tau,\tau-t_m,u_{0m})-B(u(\tau,\tau-t_n,u_{0n}))+B(u(\tau,\tau-t_m,u_{0m})).\end{aligned} \tag{28}$$

Taking the inner product of (28) with $u(\tau,\tau-t_n,u_{0n})-u(\tau,\tau-t_m,u_{0m})$, and according to the monotony of $F(u)$, we get

$$\begin{aligned}&\nu \parallel \nabla u(\tau,\tau-t_n,u_{0n})-\nabla u(\tau,\tau-t_m,u_{0m})\parallel^2\\&\quad\le\parallel u_\tau(\tau,\tau-t_n,u_{0n})-u_\tau(\tau,\tau-t_m,u_{0m})\parallel\cdot\parallel u(\tau,\tau-t_n,u_{0n})-u(\tau,\tau-t_m.u_{0m})\parallel+\\&\qquad a\parallel u(\tau,\tau-t_n,u_{0n})-u(\tau,\tau-t_m.u_{0m})\parallel^2 .\end{aligned}\tag{29}$$

By Lemma 3.4, there is $N=N(\tau,D)>0$ such that, for all $n\ge N$,

$$\parallel u_\tau(\tau,\tau-t_n,u_{0n})\parallel\le C,$$

which along with (27) and (29) implies that

$$\nabla u(\tau,\tau-t_n,u_{0n}) \text{ is a Cauchy sequence in } H,$$

i.e.,

$$u(\tau,\tau-t_n,u_{0n}) \text{ is a Cauchy sequence in } V.$$

We are now in a position to present the existence of pullback attractor in V.

Theorem 4.2 *Let $\frac{\mathrm{d}f}{\mathrm{d}t}\in L^2_{\mathrm{loc}}(\mathbb{R};H)$. Then system (1) has a $\mathcal{D}_\sigma$-pullback global attractor $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ in V, meaning that, for all $\tau\in\mathbb{R}$,*

(a) $\mathcal{A}(\tau)$ is compact in V;

(b) $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ is invariant, that is,

$$\phi(t,\tau,\mathcal{A}(\tau))=\mathcal{A}(t+\tau),\quad \forall t\ge 0;$$

(c) $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ attracts every set in $\mathcal{D}_\sigma$ with respect to the V norm, that is, for all $B=\{B(\tau)\}_{\tau\in\mathbb{R}}\in\mathcal{D}_\sigma$,

$$\lim_{t\to\infty}\mathrm{dist}_V(\phi(t,\tau-t,B(\tau-t)),\mathcal{A}(\tau))=0.$$

Proof We will prove that the attractor $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ in H obtained in Theorem 4.1 is actually a $\mathcal{D}_\sigma$-pullback global attractor for ϕ in V. The invariant of $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ is already given in Theorem 4.1. So we need only to prove (a) and (c).

Given a sequence $\{v_n\}\subset\mathcal{A}(\tau)$, by the compactness of $\mathcal{A}(\tau)$ in H, there exists $v\in\mathcal{A}(\tau)$ such that, up to a subsequence,

$$v_n\to v \text{ in } H.\tag{30}$$

We now prove that the above convergence is also valid in V. Let $\{t_n\}$ be a sequence with $t_n\to\infty$. By the invariance of $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$, for every $n\ge 1$, there exists $w_n\in\mathcal{A}(\tau-t_n)$ such that

$$v_n=\phi(t_n,\tau-t_n,w_n).\tag{31}$$

By (31) and Lemma 4.2, we find that there exist $v\in V$ such that, up to a subsequence,

$$v_n=\phi(t_n,\tau-t_n,w_n)\to\tilde v \text{ in } V.\tag{32}$$

It follows from (30) and (32) that $\tilde v=v\in\mathcal{A}(\tau)$, which along with (32) implies (a).

Suppose (c) is not true. Then there are $\tau\in\mathbb{R}$, $B=\{B(\tau)\}_{\tau\in\mathbb{R}}\in\mathcal{D}_\sigma,\varepsilon>0$ and $t_n\to\infty$ such that, for all $n\ge 1$,

$$\mathrm{dist}_V(\phi(t_n,\tau-t_n,B(\tau-t_n)),\mathcal{A}(\tau))\ge 2\varepsilon,$$

which implies that, for every $n \geq 1$, there exists $v_n \in B(\tau - t_n)$ such that

$$\mathrm{dist}_V(\phi(t_n, \tau - t_n, v_n), \mathcal{A}(\tau)) \geq \varepsilon. \tag{33}$$

By Lemma 4.2, there are $w \in V$ and a subsequence of $\phi(t_n, \tau - t_n, v_n)$ (not relabeled) such that

$$\phi(t_n, \tau - t_n, v_n) \to w \text{ in } V. \tag{34}$$

On the other hand, by Lemma 4.1, there are $v \in H$ and a subsequence of $\phi(t_n, \tau - t_n, v_n)$ (not relabeled) such that

$$\phi(t_n, \tau - t_n, v_n) \to v \text{ in } H. \tag{35}$$

By (34) and (35) we find that $v = w$, and hence by (34) we have

$$\phi(t_n, \tau - t_n, v_n) \to v \text{ in } V. \tag{36}$$

Since $\{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ attracts $B = \{B(\tau)\}_{\tau\in\mathbb{R}}$ in H, by Theorem 4.1, we get

$$\lim_{n\to\infty} \mathrm{dist}_H(\phi(t_n, \tau - t_n, v_n), \mathcal{A}(\tau)) = 0. \tag{37}$$

By (35), (37) and the compactness of $\mathcal{A}(\tau)$ in H, we must have $v \in \mathcal{A}(\tau)$, which along with (36) shows that

$$\lim_{n\to\infty} \mathrm{dist}_V(\phi(t_n, \tau - t_n, v_n), \mathcal{A}(\tau)) \leq \lim_{n\to\infty} \mathrm{dist}_V(\phi(t_n, \tau - t_n, v_n), v) = 0, \tag{38}$$

a contradiction with (33).

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