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Strong Law of Large Numbers of Partial Sums for Pairwise NQD Sequences

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Abstract By using the moment inequality, maximal inequality and the truncated method of random variables, we establish the strong law of large numbers of partial sums for pairwise NQD sequences, which extends the corresponding result of pairwise NQD random variables.

Keywords pairwise NQD sequences; strong law of large numbers.

MR(2010) Subject Classification 60F05; 60F15

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Lehmann [1] introduced the concept of negatively quadrant dependent (NQD) sequences.

Definition 1.1 *Two random variables X and Y are said to be NQD if for all real numbers x and y,*

$$P(X < x, Y < y) \leq P(X < x)P(Y < y).$$

A sequence of random variables $\{X_n, n \geq 1\}$ is said to be pairwise NQD if X_i and X_j are NQD for any i, $j \in \mathbf{N}^+$ and $i \neq j$.

Many known types of negative dependence such as negatively associated (NA) and negatively orthant dependence (NOD) etc. have developed on the notion of pairwise NQD. Joag-Dev and Proschan [2] pointed out that an NA sequence is NOD, and gave an example that is NOD but not NA. In particular, among them the negatively associated (NA) class is the most important and special case of pairwise NQD class and has wide applications in reliability theory and multivariate statistical analysis. Wang et al. [3] gave an example that is pairwise NQD but not NA. In addition, it is easily seen that an NOD sequence is pairwise NQD from the concept of NOD (see [3]), but the reverse is invalid (the counter-example refers to [3]). Thus, pairwise NQD sequences are sequences of wider scope which are weaker than NA and NOD sequences. So it is significant to study probabilistic properties of this wider pairwise NQD class. So far, many

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limiting properties on pairwise NQD sequences have been discussed, for instance, Matula [4] obtained the Kolmogorov strong law of large numbers for pairwise NQD random variable sequences with the same distribution. Wang et al. [5] obtained the Marcinkiewicz weak law of large numbers with the same distribution. Wang et al. [3] obtained the strong stability for Jamison type weighted product sums and the Marcinkiewicz strong law of large numbers for product sums of pairwise NQD sequences. Wu [6] gave Kolmogorov-type inequality and the three series theorem of pairwise NQD sequences and proved the Marcinkiewicz strong law of large numbers. Chen [7] generalized the results of Matula [4] to the case of nonidentical distributions under some mild condition. Wan [8] obtained law of large numbers and complete convergence for pairwise NQD sequences. Gan et al. [9] obtained the strong stability for pairwise NQD sequences. Zhao [10] obtained the almost surely convergence properties and growth rate for partial sums of a class of random variable sequences under moment condition. In addition, Wu [11] obtained the strong convergence rate of $\tilde{\rho}$ mixing sequence based on moment inequality and the truncation method of random variables, and so forth.

Inspired by the papers above, we study the strong law of large numbers for pairwise NQD by using the truncation method below, which extends the corresponding result of pairwise NQD random variables.

Put

$$X_i^{(k)}=-\frac{2^{\frac{k+1}{r}}}{(k+1)^{\frac{\mu}{r}}}I(X_i<-\frac{2^{\frac{k+1}{r}}}{(k+1)^{\frac{\mu}{r}}})+X_iI(|X_i|^r\le\frac{2^{k+1}}{(k+1)^{\mu}})+\frac{2^{\frac{k+1}{r}}}{(k+1)^{\frac{\mu}{r}}}I(X_i>\frac{2^{\frac{k+1}{r}}}{(k+1)^{\frac{\mu}{r}}}),$$

where $I(A)$ denotes the indicator function of the event A. Denote $S_n=\sum_{i=1}^n X_i$, $S_n^{(k)}=\sum_{i=1}^n X_i^{(k)}$.

The symbols C, C_1, C_2, $\ldots$ stand for generic positive constants not depending on n. α, μ and r are positive numbers not depending on n and $\log x$ represents $\log_2(\max(x,e))$.

Lemma 1.1 ([1]) *Let random variables X and Y be NQD. Then*

(i) $EXY\le EXEY$;

(ii) $P(X>x,Y>y)\le P(X>x)P(Y>y)$, $\forall x,y\in R$;

(iii) If f and g are both nondecreasing (or nonincreasing) functions, then $f(X)$ and $g(Y)$ are NQD.

Lemma 1.2 ([6]) *Let $\{X_n,n\ge1\}$ be a pairwise NQD sequence with $EX_n=0$ and $EX_n^2<\infty$ for all $n\ge1$. Denote $T_j(k)=\sum_{i=j+1}^{j+k}X_i$, $j\ge0$. Then*

$$E(T_j(k))^2\le\sum_{i=j+1}^{j+k}EX_i^2$$

and

$$E\Big(\max_{1\le k\le n}(T_j(k))^2\Big)\le C\log^2 n\sum_{i=j+1}^{j+n}EX_i^2. \tag{1}$$

Lemma 1.3 ([12]) *Let $\{X_n,n\ge1\}$ be an arbitrary random variable sequence. If there exists some random variable X such that $P(|X_n|\ge x)\le CP(|X|\ge x)$ for any $x>0$ and $n\ge1$, then* *for any $\beta > 0$ and $t > 0$,*

$$E|X_n|^\beta I(|X_n| \le t) \le C(E|X|^\beta I(|X| \le t) + t^\beta P(|X| > t)) \tag{2}$$

and

$$E|X_n|^\beta I(|X_n| > t) \le CE|X|^\beta I(|X| > t). \tag{3}$$

2. Main result and its proof

Theorem 2.1 *Let $\{X_n, n \ge 1\}$ be a pairwise NQD sequence with $EX_n = 0$ for all $n \ge 1$. Suppose that there exists a random variable X such that for any $x > 0$ and $n \ge 1$,*

$$P(|X_n| \ge x) \le CP(|X| \ge x). \tag{4}$$

If there exist constants $1 \le r < 2$ and $\alpha > r + 1$ such that

$$E(|X|^r \log^\alpha |X|) < \infty, \tag{5}$$

then

$$\lim_{n\to\infty} n^{-1/r} S_n = 0, \quad \text{a.s.} \tag{6}$$

Proof For any integer n, there exists some integer $k = k(n)$ such that $2^k \le n < 2^{k+1}$, hence $n^{-1/r}|S_n| \le \max_{2^k \le n < 2^{k+1}} (2^{-k/r}|S_n|)$. It suffices to show that

$$\max_{2^k \le n < 2^{k+1}} 2^{-k/r}|S_n| \to 0, \quad \text{a.s.}, \quad k \to \infty. \tag{7}$$

Take $r < \mu < r + 1$ and for any $\varepsilon > 0$, denote

$$A_k = \bigcap_{i=1}^{2^{k+1}} \Big(|X_i|^r \le 2^{k+1}/(k+1)^\mu\Big), \quad A_k^c = \bigcup_{i=1}^{2^{k+1}} \Big(|X_i|^r > 2^{k+1}/(k+1)^\mu\Big),$$

$$E_k = \Big(\max_{2^k \le n < 2^{k+1}} |S_n| > 2^{k/r}\varepsilon\Big).$$

It is clear to check that

$$E_k = E_k A_k + E_k A_k^c \subset \Big(\max_{2^k \le n < 2^{k+1}} |S_n^{(k)}| > 2^{k/r}\varepsilon\Big) \bigcup \Big(\bigcup_{i=1}^{2^{k+1}} (|X_i|^r > 2^{k+1}/(k+1)^\mu)\Big).$$

Hence

$$\begin{aligned}
&\sum_{k=1}^\infty P\Big(\max_{2^k \le n < 2^{k+1}} |S_n| > 2^{k/r}\varepsilon\Big) \\
&\le \sum_{k=1}^\infty P\Big(\bigcup_{i=1}^{2^{k+1}} (|X_i|^r > \frac{2^{k+1}}{(k+1)^\mu})\Big) + \sum_{k=1}^\infty P\Big(\max_{2^k \le n < 2^{k+1}} |S_n^{(k)}| > 2^{k/r}\varepsilon\Big) \doteq I_1 + I_2.
\end{aligned}$$

If we can obtain that $I_1 < \infty$ and $I_2 < \infty$, by Borel-Cantelli Lemma, (7) holds.

Firstly, we will check $I_1 < \infty$. By (4), (5) and $1 \le r < \mu < \alpha$, it follows that

$$I_1 \le \sum_{k=1}^\infty \sum_{i=1}^{2^{k+1}} P\Big(|X_i|^r > \frac{2^{k+1}}{(k+1)^\mu}\Big) \le C \sum_{k=1}^\infty 2^{k+1} P\Big(|X|^r > \frac{2^{k+1}}{(k+1)^\mu}\Big)$$

$$
\begin{aligned}
&\le C_1\sum_{k=1}^{\infty}2^{k+1}\sum_{j=k}^{\infty}P\Big(\frac{2^j}{j^\mu}\le|X|^r<\frac{2^{j+1}}{(j+1)^\mu}\Big)=C_1\sum_{j=1}^{\infty}\sum_{k=1}^{j}2^{k+1}P\Big(\frac{2^j}{j^\mu}\le|X|^r<\frac{2^{j+1}}{(j+1)^\mu}\Big)\\
&=4C_1\sum_{j=1}^{\infty}2^jP\Big(\frac{2^j}{j^\mu}\le|X|^r<\frac{2^{j+1}}{(j+1)^\mu}\Big)\\
&\le C_2+4C_1\sum_{j=j_0}^{\infty}\frac{2^j}{j^\mu}(j-\mu\log j)^\alpha E\Big(I(\frac{2^j}{j^\mu}\le|X|^r<\frac{2^{j+1}}{(j+1)^\mu})\Big)\\
&\quad\text{(where } j_0 \text{ satisfies that for } j\ge j_0,\ (j-\mu\log j)^\alpha>0 \text{ and } 1<\frac{(j-\mu\log j)^\alpha}{j^\mu})\\
&\le C_2+C_3\sum_{j=j_0}^{\infty}E\Big\{|X|^r\log^\alpha|X|I(\frac{2^j}{j^\mu}\le|X|^r<\frac{2^{j+1}}{(j+1)^\mu})\Big\}\\
&\le C_2+C_3E(|X|^r\log^\alpha|X|)<\infty. \qquad (8)
\end{aligned}
$$

Next, we will check $I_2<\infty$. By $EX_i=0$, (3)–(5), Lemma 3 and taking k sufficiently large such that $(k+1-\mu\log(k+1))^\alpha>0$, one has

$$
\begin{aligned}
&\frac{\max\limits_{2^k\le n<2^{k+1}}|ES_n^{(k)}|}{2^{k/r}}\le\sum_{i=1}^{2^{k+1}}\frac{|EX_i^{(k)}|}{2^{k/r}}\\
&\le 2^{-k/r}\sum_{i=1}^{2^{k+1}}\Big\{E|X_i|I(|X_i|^r>\frac{2^{k+1}}{(k+1)^\mu})+\frac{2^{(k+1)/r}}{(k+1)^{\mu/r}}P\Big(|X_i|>\frac{2^{(k+1)/r}}{(k+1)^{\mu/r}}\Big)\Big\}\\
&\le C2^{-k/r}\sum_{i=1}^{2^{k+1}}\Big\{E|X|I(|X|^r>\frac{2^{k+1}}{(k+1)^\mu})+\frac{2^{(k+1)/r}}{(k+1)^{\mu/r}}P\Big(|X|>\frac{2^{(k+1)/r}}{(k+1)^{\mu/r}}\Big)\Big\}\\
&\le 2C2^{-k/r}\sum_{i=1}^{2^{k+1}}E|X|I(|X|^r>\frac{2^{k+1}}{(k+1)^\mu})\\
&\le C_4\frac{2^{k+1}(k+1)^{\mu(r-1)/r}E(|X|^r\log^\alpha|X|)}{2^{k/r}2^{(k+1)(r-1)/r}(k+1-\mu\log(k+1))^\alpha}\le C_5\frac{1}{k^{\alpha-\mu+\mu/r}}\to0\ \text{ as } k\to\infty.
\end{aligned}
$$

Hence $2^{-k/r}\max_{2^k\le n<2^{k+1}}|ES_n^{(k)}|<\varepsilon/2$ for k sufficiently large. Thus,

$$I_2\le C_6+\sum_{k=1}^{\infty}P\Big(\max_{2^k\le n<2^{k+1}}|S_n^{(k)}-ES_n^{(k)}|>2^{k/r}\varepsilon/2\Big). \qquad (9)$$

Since $X_i^{(k)}-EX_i^{(k)}$ is a nondecreasing function, we have by applying Lemma 1.1(iii) that $\{X_i^{(k)}-EX_i^{(k)},1\le i\le n\}$ is still a pairwise NQD sequence with mean zero. Hence by (9), Markov's inequality, (1), (2) and C_r inequality, it follows that

$$
\begin{aligned}
I_2&\le C_6+\sum_{k=1}^{\infty}2^{-2k/r}E\Big(\max_{2^k\le n<2^{k+1}}|S_n^{(k)}-ES_n^{(k)}|^2\Big)\\
&\le C_6+\sum_{k=1}^{\infty}2^{-2k/r}E\Big(\max_{1\le n\le2^{k+1}}|\sum_{i=1}^{n}(X_i^{(k)}-EX_i^{(k)})|^2\Big)\\
&\le C_6+\sum_{k=1}^{\infty}\frac{(\log2^{k+1})^2}{2^{2k/r}}\Big\{\sum_{i=1}^{2^{k+1}}E|X_i^{(k)}-EX_i^{(k)}|^2\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\le C_6 + C_7 \sum_{k=1}^{\infty} \frac{k^2}{2^{2k/r}} \sum_{i=1}^{2^{k+1}} \Big\{ E\Big(X_i^2 I(|X_i|^r \le \frac{2^{k+1}}{(k+1)^{\mu}}) \Big) + \frac{2^{2(k+1)/r}}{(k+1)^{2\mu/r}} P(|X_i|^r > \frac{2^{k+1}}{(k+1)^{\mu}}) \Big\} \\
&\le C_6 + C_7 \sum_{k=1}^{\infty} \frac{k^2}{2^{2k/r}} 2^{k+1} \Big\{ E\Big(X^2 I(|X|^r \le \frac{2^{k+1}}{(k+1)^{\mu}}) \Big) + \\
&\quad C_7 \sum_{k=1}^{\infty} \frac{k^2}{2^{2k/r}} 2^{k+1} \frac{2^{2(k+1)/r}}{(k+1)^{2\mu/r}} P(|X|^r > \frac{2^{k+1}}{(k+1)^{\mu}}) \Big\} \\
&=: C_6 + C_7 I_{21} + C_7 I_{22}.
\end{aligned} \tag{10}
$$

For I_{21}, it is clear to check the fact that $\sum_{n=m}^{\infty} \frac{n^2}{2^{\delta n}} \le C \frac{m^2}{2^{\delta m}}$ for any $m \ge 1$ and $\delta > 0$. Without loss of generality, we assume $\frac{2^m}{m^{\mu}} < \frac{2^{m+1}}{(m+1)^{\mu}}$, $m \ge 1$ and $A_m := \{ \frac{2^m}{m^{\mu}} < |X|^r \le \frac{2^{m+1}}{(m+1)^{\mu}} \}$. Noting that $1 \le r < 2$, $r < \mu < r+1$, $\alpha > r+1$ and $E(|X|^r \log^{\alpha} |X|) < \infty$, one has

$$
\begin{aligned}
I_{21} =& \sum_{k=1}^{\infty} k^2 2^{k+1-\frac{2k}{r}} \big(EX^2 I(|X|^r \le 2) + \sum_{m=1}^{k} EX^2 I(A_m) \big) \\
=& C_8 + \sum_{m=1}^{\infty} (\sum_{k=m}^{\infty} k^2 2^{k+1-\frac{2k}{r}}) EX^2 I(A_m) \\
\le& C_8 + C_9 \sum_{m=1}^{\infty} m^2 2^{m-\frac{2m}{r}} E\big(|X|^r \log^{\alpha} |X| \cdot \frac{|X|^{2-r}}{\log^{\alpha} |X|} I(A_m) \big) \\
\le& C_8 + C_{10} \sum_{m=1}^{\infty} m^2 2^{m-\frac{2m}{r}} \Big(\frac{2^{m+1}}{(m+1)^{\mu}} \Big)^{\frac{2-r}{r}} \frac{1}{(\log \frac{2^m}{m^{\mu}})^{\alpha}} E|X|^r \log^{\alpha} |X| I(A_m) \\
\le& C_8 + C_{11} \sum_{m=1}^{\infty} m^{2+\mu-\frac{2\mu}{r}-\alpha} E|X|^r \log^{\alpha} |X| I(A_m).
\end{aligned}
$$

Since $\alpha > r$, we can take μ such that $\alpha > 2 + \mu - \frac{2\mu}{r}$. Thereby,

$$
I_{21} \le C_8 + C_{11} E(|X|^r \log^{\alpha} |X|) < \infty. \tag{11}
$$

For I_{22}, by $\mu < r+1 < \frac{3r}{2-r}$, it follows that $2 + \mu - 2\mu/r > -1$. Consequently, by $\alpha > r+1$, we have

$$
\begin{aligned}
I_{22} \le& C \sum_{k=1}^{\infty} k^{2-\frac{2\mu}{r}} 2^{k+1} EI(|X|^r > \frac{2^{k+1}}{(k+1)^{\mu}}) \le C_{12} \sum_{k=1}^{\infty} k^{2+\mu-\frac{2\mu}{r}} E\Big(|X|^r I(|X|^r > \frac{2^{k+1}}{(k+1)^{\mu}}) \Big) \\
=& C_{12} \sum_{k=1}^{\infty} k^{2-\mu(\frac{2}{r}-1)} \sum_{m=k}^{\infty} E\big(|X|^r I(A_{m+1}) \big) = C_{12} \sum_{m=1}^{\infty} E\big(|X|^r I(A_{m+1}) \big) \sum_{k=1}^{m} k^{2+\mu-\frac{2\mu}{r}} \\
\le& C_{13} \sum_{m=1}^{\infty} m^{3-\mu(\frac{2}{r}-1)} E\big(|X|^r I(A_{m+1}) \big) \le C_{13} \sum_{m=1}^{\infty} m^{r+1} E\big(|X|^r I(A_{m+1}) \big) \\
\le& C_{14} + C_{15} \sum_{m=m_0}^{\infty} \frac{m^{r+1}}{(m+1-\mu \log(m+1))^{\alpha}} E\big(|X|^r \log^{\alpha} |X| I(A_{m+1}) \big) \\
\le& C_{14} + C_{15} \sum_{m=1}^{\infty} E(|X|^r \log^{\alpha} |X| I(A_{m+1})) \\
\le& C_{14} + C_{15} E(|X|^r \log^{\alpha} |X|) < \infty.
\end{aligned} \tag{12}
$$

By (10)–(12), we have that $I_2 < \infty$. Combining (8) with (7), we get (6). The proof of the desired result is completed. $\Box$

Remark In the process of proving $I_2 < \infty$, we refer to the method on the proof of Theorem 5.4.2 in [12], but the choices of truncation random variables $\{X_i^{(k)}, 1 \le i \le n\}$ and the specific parameter μ are different.

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