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A Note on Linearly Isometric Extension for 1-Lipschitz and Anti-1-Lipschitz Mappings between Unit Spheres of $AL_P(\mu, H)$ Spaces

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Abstract In this paper, we show that if V_0 is a 1-Lipschitz mapping between unit spheres of $L_P(\mu, H)$ and $L_P(\nu, H)(p > 2,\ H$ is a Hilbert space), and $-V_0(S(L_p(\mu, H))) \subset V_0(S(L_p(\mu, H)))$, then V_0 can be extended to a linear isometry defined on the whole space. If $1 < p < 2$ and V_0 is an "anti-1-Lipschitz" mapping, then V_0 can also be linearly and isometrically extended.

Keywords isometric extension; strictly convex; Bochner integral.

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1. Introduction

Tingley posed the problem of extending an isometry between unit spheres in [1] as follows: Let E and F be two real Banach spaces. If V_0 is a surjective isometry between the two unit spheres $S(E)$ and $S(F)$, does V_0 have an isometric affine extension? It will be very difficult to answer this question, even in a two-dimensional case. In [1], Tingley showed that isometries between the unit spheres of finite dimensional Banach spaces necessarily map antipodal points to antipodal points. Professor Guanggui Ding and his students kept on working on this topic and have obtained many important results [2]. For example, Ding and some people have obtained an affirmative answer to Tingley's problem for the classical Banach spaces, e. g., $l^p(\Gamma)(1 \leq p \leq \infty)$, $L^p(\mu)$ and generally, for the AL^p-space $(1 \leq p < \infty)$ (see [3–7]). In [8–10], the authors discussed the Tingley's problem on spaces of different types and obtained an affirmative answer.

Subsequently, Ding and some people also considered "into" mappings between two Banach spaces of different types (for example, from $S(l(\Gamma))$ or $S(L^\infty(\Gamma))$ into $S(E)$ for a normed space E) in the context of Tingley's isometric extension problem, and they obtained some useful results in [11, 12]. In [13], Ding first obtained an affirmative answer for 1-Lipshchtz mapping between two unit spheres of Hilbert space. Yang [14] also obtained the affirmative answer to the above problem for some vector-valued space $L^p(H)$ (where H is a Hilbert space, $1 < p \neq 2$). Recently, Ding [2] studied the linearly isometric extension problem for 1-Lipschitz (respectively, "anti-1-Lipschtz")

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mappings between unit spheres of AL^p-spaces with $p > 2$ (respectively, $1 < p < 2$). (We recall that T is called "1-Lipschitz" (respectively, "anti-1-Lipschitz"), which satisfies $\|T(x) - T(y)\| \leq \|x-y\|$ (respectively, $\|T(x)-T(y)\| \geq \|x-y\|$)). He obtained that if V_0 is a 1-Lipschitz mapping between unit spheres of two AL^p-spaces with $p > 2$ and $-V_0(S(L^p)) \subset V_0(S(L^p))$, then V_0 can be extended to a linear isometry defined on the whole space. If $1 < p < 2$ and V_0 is an "anti-1-Lipschitz" mapping, then V_0 can also be linearly and isometrically extended.

In this paper, we shall combine the above problem with the linearity problem for Lipschitz mappings to study the linearly isometric extension problem for (respectively, "anti-Lipschitz") mapping between unit spheres of vector-valued space $AL^p(H)$ (where H is a Hilbert space), we generalize the corresponding main results of [2] and [14].

2. Some lemmas

Throughout this paper, we always assume that $1 < p \neq 2$, $S(E) = \{x \in E : \|x\| = 1\}$, $\mathrm{supp}(f) = \{t : f(t) \neq 0\}$. We mainly concern function spaces $AL_p(\Omega, \Sigma, \mu, H)$, where H is a Hilbert space. For convenience we denote by $L_p(\mu, H)$ the space of all (equivalent class of) H-valued Bochner integrable function f defined on Ω with $\int_\Omega \|f(t)\|^p \mathrm{d}\mu < \infty$. The norm $\|\cdot\|_p$ is defined by

$$\|f\|_p = \Big(\int_\Omega \|f(t)\|^p \mathrm{d}\mu\Big)^{\frac{1}{p}}, \quad f \in L_p(\mu, H).$$

We first introduce a famous conclusion as follows.

Lemma 1 ([2]) *Let E and F be two normed spaces and E be strictly convex, and V_0 be a 1-Lipshitz mapping from the unit sphere $S(E)$ into $S(F)$. If $-V_0(S(E)) \subset V_0(S(E))$, then V_0 is one-to-one and $V_0(-x) = -V_0(x),\ \forall\, x \in S(E)$.*

Lemma 2 ([14]) *Let $f, g \in S(L_p(\mu, H))$. Then $\|f+g\|_p^p + \|f-g\|_p^p = 2(\|f\|_p^p + \|g\|_p^p)$ if and only if $\mu[\mathrm{supp}\, f \cap \mathrm{supp}\, g] = 0$.*

Lemma 3 *Let V_0 be a 1-Lipshitz mapping from the unit sphere $S(L_p(\mu, H))$ into $S(L_p(\nu, H))$ with $-V_0(S(L_p(\mu, H))) \subset V_0(S(L_p(\mu, H)))(p > 2)$. Then $\mu[\mathrm{supp}\, f \cap \mathrm{supp}\, g] = 0$ implies $\nu[\mathrm{supp}\, V_0(f) \cap \mathrm{supp}\, V_0(g)] = 0$.*

Proof From the hypotheses, by Lemma 1, we first obtain

$$V_0(-f) = -V_0(f), \quad \forall\, f \in S(L_p(\mu, H)).$$

Thus

$$\|V_0(f) \pm V_0(g)\|_p = \|V_0(f) - V_0(\mp g)\|_p \leq \|f \pm g\|_p, \quad \forall f, g \in S(L_p(\mu, H)).$$

Suppose that $\mu[\mathrm{supp} f \cap \mathrm{supp} g] = 0$. By Lemma 2 we have

$$\|f+g\|_p^p + \|f-g\|_p^p = 2(\|f\|_p^p + \|g\|_p^p) = 4,$$

hence

$$\|V_0(f) + V_0(g)\|_p^p + \|V_0(f) - V_0(g)\|_p^p \leq \|f+g\|_p^p + \|f-g\|_p^p = 2(\|V_0(f)\|_p^p + \|V_0(g)\|_p^p).$$

That is,

$$\int_S [(\|V_0(f)(s)+V_0(g)(s)\|^p+\|V_0(f)(s)-V_0(g)(s)\|^p)-2(\|V_0(f)(s)\|^p+\|V_0(g)(s)\|^p)]\mathrm{d}\nu \le 0. \quad (1)$$

By the convexity of $u^{p/2}(p>2)$, that is

$$(\frac{|a|^2+|b|^2}{2})^{p/2} \le \frac{|a|^p+|b|^p}{2}$$

and the characterization of inner product space, we have

$$\begin{aligned}
&\|V_0(f)+V_0(g)\|_p^p+\|V_0(f)-V_0(g)\|_p^p \\
&= \int_S (\|V_0(f)(s)+V_0(g)(s)\|^p+\|V_0(f)(s)-V_0(g)(s)\|^p)\mathrm{d}\nu \\
&\ge 2^{1-p/2}\int_S (\|V_0(f)(s)+V_0(g)(s)\|^2+\|V_0(f)(s)-V_0(g)(s)\|^2)^{p/2}\mathrm{d}\nu \\
&= 2\int_S (\|V_0(f)(s)\|^2+\|V_0(g)(s)\|^2)^{p/2}\mathrm{d}\nu.
\end{aligned}$$

Moreover, by the convexity of $u^{p/2}$ that

$$(|a|^2+|b|^2)^{p/2} \ge |a|^p+|b|^p,$$

we device

$$\|V_0(f)+V_0(g)\|_p^p+\|V_0(f)-V_0(g)\|_p^p \ge 2\int_S (\|V_0(f)(s)\|^p+\|V_0(g)(s)\|^p)\mathrm{d}\nu.$$

Combining this with (2.1), we get

$$\int_S [(\|V_0(f)(s)+V_0(g)(s)\|^p+\|V_0(f)(s)-V_0(g)(s)\|^p)-2(\|V_0(f)(s)\|^p+\|V_0(g)(s)\|^p)]\mathrm{d}\nu = 0.$$

In view of Lemma 2, we obtain $\nu[\operatorname{supp} V_0(f)\cap \operatorname{supp} V_0(g)]=0$, and complete this proof. $\square$

Lemma 2.4 in [2] is a key lemma. We find this lemma also holds for $AL^p(\mu,H)$. Hence, we have the following lemma.

Lemma 4 *Let V_0 be a 1-Lipshitz mapping from the unit sphere $S(L_p(\mu,H))$ into $S(L_p(\nu,H))$ with $-V_0(S(L_p(\mu,H)))\subset V_0(S(L_p(\mu,H)))(p>2)$. For every disjoint f_1 and f_2 in $S(L_p(\mu,H))$ and $\xi_1,\xi_2\in\mathbb{R}$ with $|\xi_1|^p+|\xi_2|^p=1$, if we have $V_0(f)=\xi_1V_0(f_1)+\xi_2V_0(f_2)$, then $f=\xi_1f_1+\xi_2f_2$.*

Proof By Lemma 2 and Lemma 3, the proof is similar to that of Lemma 2.4 in [2].

Lemma 5 *In the assumptions of Lemma 4, if for every mutual disjoint element $f_1,f_2,\ldots,f_n$ in $S(L_p(\mu,H))$ and $\xi_1,\xi_2,\ldots,\xi_n\in\mathbb{R}$ with $\sum_{k=1}^n|\xi_k|^p=1$, satisfying $V_0(f)=\xi_1V_0(f_1)+\xi_2V_0(f_2)+\cdots+\xi_nV_0(f_n)$, then $f=\xi_1f_1+\xi_2f_2+\cdots+\xi_nf_n$.*

Proof We prove the above conclusion by induction. Indeed, for $n=2$ the above conclusion holds because of Lemma 4. Suppose that the result holds for $n\le m-1$. Since

$$\sum_{k=1}^m \xi_kV_0(f_k)=\xi_1V_0(f_1)+(1+|\xi_1|^p)^{1/p}\sum_{k=2}^m\frac{\xi_k}{(1+|\xi_1|^p)^{1/p}}V_0(f_k),$$

we have

$$V_0\Big(\sum_{k=1}^{m}\xi_k f_k\Big) = \xi_1 V_0(f_1) + (1+|\xi_1|^p)^{1/p} V_0\Big(\sum_{k=2}^{m}\frac{\xi_k}{(1+|\xi_1|^p)^{1/p}} f_k\Big)$$
$$= \xi_1 V_0(f_1) + (1+|\xi_1|^p)^{1/p} \sum_{k=1}^{m}\frac{\xi_k}{(1+|\xi_1|^p)^{1/p}} V_0(f_k)$$
$$= \sum_{k=1}^{m}\xi_k V_0(f_k). \quad \Box$$

3. Main results

Theorem 1 *Let V_0 be a 1-Lipshitz mapping from the unit sphere $S(L_p(\mu,H))$ into $S(L_p(\nu,H))$ with $-V_0(S(L_p(\mu,H)))\subset V_0(S(L_p(\mu,H)))(p>2)$. Then V_0 can be extended to a (real) linear isometry on the whole space $L_p(\mu,H)$.*

Proof By Lemma 5, we find that for every finite family of mutually disjoint measurable subsets $\{A_1,A_2,\dots,A_n\}$ of T with $0<\mu A_k<\infty(1\le k\le n)$, for every family of numbers $\{\xi_1,\xi_2,\dots,\xi_n\}$ in $\mathbb{R}$ with $\sum_{k=1}^{n}|\xi_k|^p=1$, and for every family of elements $\{x_1,x_2,\dots,x_n\}$ in $S(H)$,

$$V_0\Big(\sum_{k=1}^{n}\xi_k\frac{x_k\chi_{A_k}}{[\mu(A_k)]^{1/p}}\Big) = \sum_{k=1}^{n}\xi_k V_0(\frac{x_k\chi_{A_k}}{[\mu(A_k)]^{1/p}}).$$

That is, V_0 is linear on the subset of all simple functions of unit sphere $S(L_p(\mu,H))$.

We define in a similar fashion a mapping on the subspace X which consists of all simple functions of $L_p(\mu,H)$ as follows:

$$V_1(f) = V_1\Big(\sum_{k=1}^{n}\lambda_k\frac{x_k\chi_{A_k}}{[\mu(A_k)]^{1/p}}\Big) = \sum_{k=1}^{n}\lambda_k V_0(\frac{x_k\chi_{A_k}}{[\mu(A_k)]^{1/p}}),$$

where $x_1,x_2,\dots,x_n\in S(H)$, $f=\sum_{k=1}^{n}\lambda_k\frac{x_k\chi_{A_k}}{[\mu(A_k)]^{1/p}}\in X(\subset L_p(\mu,H))$, and $\{A_1,A_2,\dots,A_n\}$ are mutually disjoint non-zero measurable subsets of T having a finite measure for each $1\le k\le n(n\in\mathbb{N})$.

By Lemma 3, we have

$$\|V_1(f)\| = \sum_{k=1}^{n}|\lambda_k|^p = \|f\|, \quad \forall f=\sum_{k=1}^{n}\lambda_k\frac{x_k\chi_{A_k}}{[\mu(A_k)]^{1/p}}\in X.$$

That is, we have obtained a linear isometry on the subspace X of $L_p(\mu,H)$. Recall that f lies in $L_p(\mu,H)$, then $\mu\{t\in T:\|f(t)\|>\lambda\}<\infty$ for every $\lambda>0$. Hence X is dense in the space $L_p(\mu,H)$. Moreover, since V_1 is isometric on X, and since $L_p(\mu,H)$ and $L_p(\nu,H)$ are complete, we can conclude that V_1 has a unique linear isometric extension V on $L_p(\mu,H)$. Thus, V is the desired extension of V_0, and this completes the proof. $\Box$

By a similar argument, we can also obtain the following theorem.

Theorem 2 *Let V_0 be an "anti-1-Lipschitz" mapping from the unit sphere $S(L_p(\mu,H))$ into the unit sphere $S(L_p(\nu,H))$, $1<p<2$. Then V_0 can be extended to a (real) linear isometry on* *the whole space* $L_p(\mu, H)$.

Remark (i) In Theorem 2, we do not need the assumption

$$-V_0(S(L_p(\mu, H))) \subset V_0(S(L_p(\mu, H))).$$

In fact, from the following inequalities

$$2 \geq \|V_0(f) - V_0(-f)\| \geq \|f - (-f)\| = 2\|f\| = 2,$$

we have

$$\|V_0(f) - V_0(-f)\| = \|V_0(f)\| + \| - V_0(-f)\|.$$

Since H is a strictly convex space, and by [15], $L_p(\mu, H)$ is also a strictly convex space, we know that $V_0(-f) = -V_0(f)$.

(ii) Since $L_p(\nu, H)$ is strictly convex, if V_0 is isometric, we may obtain that $-V_0(S(L_p(\mu, H))) \subset V_0(S(L_p(\mu, H)))$. Thus we generalize the main result Theorem 2.4 in [14].

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