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Product and Commutativity of Slant Toeplitz Operators

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Abstract In this paper, the product and commutativity of slant Toeplitz operators are discussed. We show that the product of k_1^{th}-order slant Toeplitz operators and k_2^{th}-order slant Toeplitz operators must be a $(k_1k_2)^{th}$-order slant Toeplitz operator except for zero operators, and the commutativity and essential commutativity of two slant Toeplitz operators with different orders are the same.

Keywords Toeplitz operator; slant Toeplitz operator; product; commutativity.

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1. Preliminaries

In the year 1995, Ho [1] introduced one class of operators, which have the property that the matrices of such operators with respect to the standard orthonormal basis could be obtained from those of Toeplitz operators just by eliminating every other row. Such operators were termed as slant Toeplitz operators [1].

In the past few years, slant Toeplitz operators have appeared in connection with many applications where they go under other names. Villemoes associated the Besov regularity of solutions of the refinement equation with the spectral radii of an associated slant Toeplitz operator [11] and Goodman, Micchelli and Ward [12] showed the connection between the spectral radii and conditions for the solutions of certain differential equations being in Lipschitz classes.

Ever since the introduction of the class of slant Toeplitz operators, Ho and many other researchers began a systematic study of such operators and their various generalizations [1–10].

Throughout this paper, let k, k_1 and k_2 be integers and $\min\{k,k_1,k_2\}\geq 2$. Let $\varphi(z)=\sum_{i=-\infty}^{\infty}a_iz^i$ be a bounded measurable function on the unit circle $\mathbb{T}$, where $a_i=\langle\varphi,z^i\rangle$ is the i^{th} Fourier coefficient of φ and $\{z^i:i\in\mathbb{Z}\}$ is the standard orthonormal basis of $L^2(\mathbb{T})$, $\mathbb{Z}$ being the set of integers. The k^{th}-order slant Toeplitz operator U_φ^k with symbol φ in $L^\infty(\mathbb{T})$ is defined on $L^2(\mathbb{T})$ as follows

$$U_\varphi^k(z^l)=\sum_{i=-\infty}^{\infty}a_{ki-l}z^i.$$

It is proved in [1] and [5] that $U_\varphi^k=W_kM_\varphi$, where M_φ is the multiplication operator on $L^2(\mathbb{T})$

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induced by φ and W_k is a bounded operator on $L^2(\mathbb{T})$ defined as

$$W_k(z^i) = \begin{cases} z^{i/k}, & \text{if } i \text{ is divisible by } k, \\ 0, & \text{otherwise.} \end{cases}$$

Some properties for the product of two k^{th}-order slant Toeplitz operators were investigated in [5] and [10]. Motivated by these, we have proved some properties for the product of slant Toeplitz operators.

2. Product and commutativity of slant Toepltiz operators

In this section, the following problems are examined:

(1) What is the product of slant Toeplitz operators?

(2) When do slant Toeplitz operators with different orders commute?

Now we begin with the following Proposition.

Proposition 2.1 *Let $\varphi, \psi \in L^\infty(\mathbb{T})$. Then the following statements hold:*

(1) $W_{k_1}W_{k_2} = W_{k_1k_2}$;

(2) $U_\psi^{k_1}U_\varphi^{k_2} = U^{k_1k_2}_{\psi(z^{k_2})\varphi}$.

Proof (1) By the properties of W_k and W_k^*, we get that for any integer n,

$$W^*_{k_1k_2}z^n = z^{k_1k_2n},$$
$$(W_{k_1}W_{k_2})^*z^n = W^*_{k_2}(W^*_{k_1}z^n) = W^*_{k_2}(z^{k_1n}) = z^{k_1k_2n}.$$

This implies that for any integer n,

$$W^*_{k_1k_2}z^n = (W_{k_1}W_{k_2})^*z^n.$$

Thus we get that $W^*_{k_1k_2} = (W_{k_1}W_{k_2})^*$, since $\{z^i : i \in \mathbb{Z}\}$ is the standard orthonormal basis of $L^2(\mathbb{T})$. So the required result holds.

(2) By the properties of U^k_φ and W_k, we get that

$$U_\psi^{k_1}U_\varphi^{k_2} = W_{k_1}M_\psi W_{k_2}M_\varphi = W_{k_1}W_{k_2}M_{\psi(z^{k_2})\varphi}.$$

Since $W_{k_1}W_{k_2} = W_{k_1k_2}$, we can get that $U_\psi^{k_1}U_\varphi^{k_2} = U^{k_1k_2}_{\psi(z^{k_2})\varphi}$. $\square$

Lemma 2.1 *Let $\varphi = \sum_{l=-\infty}^{\infty} a_l z^l \in L^\infty(\mathbb{T})$. Then U^k_φ is a zero operator if and only if $\varphi = 0$.*

Proof Suppose that U^k_φ is a zero operator. Then for all i, j in $\mathbb{Z}$, we get that

$$\langle U^k_\varphi z^i, z^j\rangle = \langle \sum_{l=-\infty}^{\infty} a_{kl-i}z^l, z^j\rangle = a_{kj-i} = 0.$$

Thus $a_l = 0$ for all l in $\mathbb{Z}$, that is, $\varphi = 0$. The converse is obvious. $\square$

Theorem 2.1 *Let $\varphi(z) = \sum_{l=-\infty}^{\infty} a_l z^l \in L^\infty(\mathbb{T})$ and let m be an integer with $m \geq 2$ and $m \neq k$. Then U^k_φ is an m^{th}-order slant Toeplitz operator if and only if $\varphi = 0$.*

Proof Suppose that U^k_φ is an m^{th}-order slant Toeplitz operator. Then for all i, j in $\mathbb{Z}$, we get that $\langle U_\varphi^k z^i, z^j\rangle = \langle U_\varphi^k z^{i+mk}, z^{j+k}\rangle$, that is,

$$\langle \sum_{l=-\infty}^{\infty} a_{kl-i} z^l, z^j\rangle = \langle \sum_{l=-\infty}^{\infty} a_{kl-i-mk} z^l, z^{j+k}\rangle.$$

Therefore $a_{kj-i} = a_{k(j+k)-i-mk}$ for any integer i and j. From this we get that $a_0 = a_{lk|k-m|}$, $a_1 = a_{lk|k-m|+1}$, $a_2 = a_{lk|k-m|+2}, \ldots, a_{k|k-m|-1} = a_{lk|k-m|+k|k-m|-1}$. Since $a_l \to 0$ as $l \to \infty$, we get that $a_{lk|k-m|+i} \to 0$ as $l \to \infty$ for each $i = 0, 1, \ldots, k|k-m|-1$. Thus $a_0 = a_1 = \cdots = a_{k|k-m|-1} = 0$. Hence $a_l = 0$ for all integers l, which means that $\varphi = 0$. It is clear that the converse is true. $\Box$

Now we are in a position to state the properties for the product of slant Toeplitz operators.

Theorem 2.2 *Let $\varphi, \psi \in L^\infty(\mathbb{T})$. Then $U_\psi^{k_1} U_\varphi^{k_2}$ is a k^{th}-order slant Toeplitz operator if and only if one of the following statements holds:*

(1) $k = k_1 k_2$;

(2) $\psi(z^{k_2})\varphi = 0$, *if* $k \neq k_1 k_2$.

Proof By Proposition 2.1 we get that $U_\psi^{k_1} U_\varphi^{k_2} = U_{\psi(z^{k_2})\varphi}^{k_1 k_2}$. Then by the definition of U_φ^k and Theorem 2.1 we get the required results. $\Box$

Remark 2.1 From Theorem 2.2, it is obvious that the product of two k^{th}-order slant Toeplitz operators cannot be a k^{th}-order slant Toeplitz operator and k^{th}-order slant Toeplitz operators cannot be idempotent except for the zero operator [5, Theorem 2 and Corollary 3].

Remark 2.2 By the properties of W_k and U_φ^k, one can repeat the proof above and arrive at the conclusions analogous to those in Theorem 2.2 for the finite product of slant Toeplitz operators.

Recall that two operators A and B essentially commute if $AB - BA$ is compact; an operator A is said to be hyponormal and normal if its self-commutator $[A^*, A] := A^*A - AA^* \geq 0$ and $[A^*, A] = 0$, respectively.

Theorem 2.3 *Let $\varphi, \psi \in L^\infty(\mathbb{T})$. The following statements are equivalent:*

(1) $U_\psi^{k_1} U_\varphi^{k_2}$ *is compact;*

(2) $U_\psi^{k_1} U_\varphi^{k_2}$ *is hyponormal;*

(3) $U_\psi^{k_1} U_\varphi^{k_2}$ *is normal;*

(4) $U_\psi^{k_1} U_\varphi^{k_2} = 0$;

(5) $\psi(z^{k_2})\varphi = 0$.

Proof By Proposition 2.1 we get that $U_\psi^{k_1} U_\varphi^{k_2} = U_{\psi(z^{k_2})\varphi}^{k_1 k_2}$. Then by Theorems 5, 9 ([5]) and Lemma 2.1 we can obtain that (1), (2), (4) and (5) are equivalent.

Now we start to show that (3) and (5) are equivalent. Suppose that $U_\psi^{k_1} U_\varphi^{k_2} = U_{\psi(z^{k_2})\varphi}^{k_1 k_2}$ is normal. Since (2) and (5) are equivalent and the normal operator is hyponormal, we can get that $\psi(z^{k_2})\varphi = 0$. The converse is clear. Hence (3) and (5) are equivalent. $\Box$

Remark 2.3 One can obtain the conclusions analogous to those in Theorem 2.3 for the finite product of slant Toeplitz operators.

Theorem 2.4 *Let $\varphi, \psi \in L^\infty(\mathbb{T})$. The following statements are equivalent:*

(1) $U_\psi^{k_1}$ and $U_\varphi^{k_2}$ essentially commute;

(2) $U_\psi^{k_1}$ and $U_\varphi^{k_2}$ commute;

(3) $\psi(z^{k_2})\varphi - \psi\varphi(z^{k_1}) = 0$.

Proof By Proposition 2.1 we get that $U_\psi^{k_1}U_\varphi^{k_2} = U_{\psi(z^{k_2})\varphi}^{k_1k_2}$ and $U_\varphi^{k_2}U_\psi^{k_1} = U_{\psi\varphi(z^{k_1})}^{k_1k_2}$, so by the properties of U_φ^k we have

$$U_\psi^{k_1}U_\varphi^{k_2} - U_\varphi^{k_2}U_\psi^{k_1} = U_{\psi(z^{k_2})\varphi}^{k_1k_2} - U_{\psi\varphi(z^{k_1})}^{k_1k_2} = U_{\psi(z^{k_2})\varphi-\psi\varphi(z^{k_1})}^{k_1k_2}.$$

Then by Theorem 9 ([5]) and Lemma 2.1 we obtain that $U_{\psi(z^{k_2})\varphi-\psi\varphi(z^{k_1})}^{k_1k_2}$ is compact if and only if $U_{\psi(z^{k_2})\varphi-\psi\varphi(z^{k_1})}^{k_1k_2} = 0$ if and only if $\psi(z^{k_2})\varphi - \psi\varphi(z^{k_1}) = 0$. Thus the required results hold. □

Proposition 2.2 *Let $\varphi \in L^\infty(\mathbb{T})$ and $\psi(z) = z^m$, where m is a nonnegative integer. Then*

$$\varphi(z^{k_1})\psi(z) = \varphi(z)\psi(z^{k_2})$$

if and only if one of the following statements holds:

(1) If $m = 0$, φ is a constant;

(2) If $m \geq 1$ and $(k_2-1)m$ is not divisible by $k_1 - 1$, $\varphi = 0$;

(3) If $m \geq 1$ and $(k_2-1)m$ is divisible by $k_1 - 1$, $\varphi = Cz^{\frac{(k_2-1)m}{k_1-1}}$, where C is a constant.

Proof Suppose that $\varphi(z^{k_1})\psi(z) = \varphi(z)\psi(z^{k_2})$. Since m is a nonnegative integer, we continue the proof in two cases: $m = 0$ and $m \geq 1$.

If $m = 0$, since $\varphi(z^{k_1})\psi(z) = \varphi(z)\psi(z^{k_2})$ and $\psi(z) = z^m$, we have that $\varphi(z^{k_1}) = \varphi(z)$. Then by Lemma 2.9 ([10]) we get that φ is a constant.

If $m \geq 1$, let $\varphi(z) = \sum_{p=-\infty}^{\infty} a_p z^p$. Since $\varphi(z^{k_1})\psi(z) = \varphi(z)\psi(z^{k_2})$ and $\psi(z) = z^m$, we have that

$$z^{(k_2-1)m} \sum_{p=-\infty}^{\infty} a_p z^p = \sum_{p=-\infty}^{\infty} a_p z^{k_1 p}.$$

Let $(k_2 - 1)m = k_1 m_1 + r_1$, where m_1 and r_1 are nonnegative integers with $0 \leq r_1 \leq k_1 - 1$. Then

$$\sum_{p=-\infty}^{\infty} a_{k_1p-r_1} z^{k_1p} + \sum_{i=1}^{k_1-1} \sum_{p=-\infty}^{\infty} a_{k_1p+i-r_1} z^{k_1p+i} = \sum_{p=-\infty}^{\infty} a_{p+m_1} z^{k_1p}.$$

So $a_{k_1p+i-r_1} = 0$ for any integer p and any integer i with $1 \leq i \leq k_1 - 1$, and $a_{k_1p-r_1} = a_{p+m_1}$ for any integer p. Therefore for any integer p, we have

$$a_{k_1p-r_1} = a_{k_1^n[p-\frac{r_1+m_1}{k_1-1}]+\frac{(k_2-1)m}{k_1-1}}$$

for any nonnegative integer n. Here are two cases: $\frac{r_1+m_1}{k_1-1}$ is an integer and $\frac{r_1+m_1}{k_1-1}$ is not an integer.

If $\frac{r_1+m_1}{k_1-1}$ is not an integer, that is, $(k_2-1)m$ is not divisible by $k_1 - 1$, then for any positive integer r and any integer p,

$$(r+1)(|a_{k_1p-r_1}|^2) = \sum_{n=0}^{r} |a_{k_1^n[p-\frac{r_1+m_1}{k_1-1}]+\frac{(k_2-1)m}{k_1-1}}|^2 \leq \sum_{p=-\infty}^{\infty} |a_p|^2 < +\infty,$$

which implies that $a_{k_1 p-r_1}=0$ for any integer p. So $\varphi(z)=0$.

If $\frac{r_1+m_1}{k_1-1}$ is an integer, that is, $(k_2-1)m$ is divisible by k_1-1, then for any positive integer r and any integer p with $p\neq\frac{r_1+m_1}{k_1-1}$,

$$(r+1)(|a_{k_1 p-r_1}|^2)=\sum_{n=0}^{r}|a_{k_1^n[p-\frac{r_1+m_1}{k_1-1}]+\frac{(k_2-1)m}{k_1-1}}|^2\leq\sum_{p=-\infty}^{\infty}|a_p|^2<+\infty,$$

which implies that $a_{k_1 p-r_1}=0$ for any integer p with $p\neq\frac{r_1+m_1}{k_1-1}$. So $\varphi(z)=Cz^{\frac{(k_2-1)m}{k_1-1}}$, where $C=a_{\frac{(k_2-1)m}{k_1-1}}$.

Now we start to show the other direction. If (1) and (2) hold, then it is obvious that $\varphi(z^{k_1})\psi(z)=\varphi(z)\psi(z^{k_2})$. If (3) holds, then $\varphi(z^{k_1})\psi(z)=Cz^{\frac{k_1(k_2-1)m}{k_1-1}}\cdot z^m=Cz^{\frac{(k_1k_2-1)m}{k_1-1}}$ and $\varphi(z)\psi(z^{k_2})=Cz^{\frac{(k_2-1)m}{k_1-1}}\cdot z^{k_2m}=Cz^{\frac{(k_1k_2-1)m}{k_1-1}}$. Hence $\varphi(z^{k_1})\psi(z)=\varphi(z)\psi(z^{k_2})$. $\square$

Remark 2.4 If $\varphi(z^{k_1})\psi(z)=\varphi(z)\psi(z^{k_2})$, then $\overline{\varphi(z^{k_1})\psi(z)}=\overline{\varphi(z)\psi(z^{k_2})}$. Therefore, one can repeat the proof above and get the same conclusions as Proposition 2.2 for any negative integer m.

From the preceding analysis it is obvious that the following theorem holds.

Theorem 2.5 *Let $\varphi\in L^\infty(\mathbb{T})$ and $\psi(z)=z^m$, where m is an integer. Then the following statements are equivalent:*

(1) $U_\psi^{k_1}$ and $U_\varphi^{k_2}$ essentially commute;

(2) $U_\psi^{k_1}$ and $U_\varphi^{k_2}$ commute;

(3) $\psi(z^{k_2})\varphi-\psi\varphi(z^{k_1})=0$;

(4) if $(k_2-1)m$ is not divisible by k_1-1, $\varphi=0$,

if $(k_2-1)m$ is divisible by k_1-1, $\varphi=Cz^{\frac{(k_2-1)m}{k_1-1}}$, where C is a constant.

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