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A New Strategy to Construct Embedded Cubature Formulae over Two-Dimensional Regions

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Abstract The purpose of this paper is to study a new strategy to construct embedded cubature formulae over two-dimensional regions. A new kind of embedded cubature formulae with some nodes along the selected algebraic curve is constructed. Some examples on the unit disk are presented to illustrate the validity of this strategy.

Keywords cubature formulae; embedded cubature formulae; polynomial ideal; moment.

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1. Introduction

Let $I[\omega;f]$ be a square positive integral functional with finite moments of all orders:

$$I[\omega;f]=\int_\Omega \omega(x,y)f(x,y)\mathrm{d}x\mathrm{d}y, \tag{1}$$

where $\Omega\subset\mathbb{R}^2$ and $\omega(x,y)$ is a nonnegative weight function over Ω. A cubature formula of degree m with respect to $I[\omega;f]$ is a linear functional

$$Q_m[\omega;f]=\sum_{i=1}^{N}a_i^{(m)}f(x_i^{(m)},y_i^{(m)}),\quad (x_i^{(m)},y_i^{(m)})\in\mathbb{R}^2,\quad a_i^{(m)}\in\mathbb{R}, \tag{2}$$

satisfying

$$Q_m[\omega;p]=I[\omega;p],\quad \forall p\in\Pi_m^2, \tag{3}$$

and $Q_m[\omega;q]\neq I[\omega;q]$ for at least one $q\in\Pi_{m+1}^2$, where Π_m^2 is the vector space of all polynomials in two variables of (total) degree $\le m$. Typically, $(x_i^{(m)},y_i^{(m)})$ and $a_i^{(m)}$ are called nodes and weights, respectively.

Usually, in order to approximate $I[\omega;f]$ together with an error estimate, we use two cubature formulae $Q_{m_j}[\omega;f]$,

$$Q_{m_j}[\omega;f]=\sum_{i=1}^{N_j}a_i^{(m_j)}f(x_i^{(m_j)},y_i^{(m_j)}),\quad j\in 1,2,\ m_1<m_2.$$

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The error for $Q_{m_1}[\omega; f]$ can be estimated by, for example, $|Q_{m_1}[\omega; f] - Q_{m_2}[\omega; f]|$ (see [3, 8]). The use of formula $Q_{m_2}[\omega; f]$ will result in a great deal of extra cost in the computation of the nodes and weights. One way to reduce the extra computation cost is to employ the embedded cubature formulae. We call $Q_{m_1}[\omega; f]$ and $Q_{m_2}[\omega; f]$ an embedded pair (of cubature formulae) of degrees (m_1, m_2), if $\{(x_i^{(m_1)}, y_i^{(m_1)})|i = 1, \ldots, N_1\} \subset \{(x_i^{(m_2)}, y_i^{(m_2)})|i = 1, \ldots, N_2\}$. Generally, there are three strategies for constructing the embedded pair of cubature formulae:

(i) Construct $Q_{m_1}[\omega; f]$ and $Q_{m_2}[\omega; f]$ simultaneously;

(ii) Construct $Q_{m_1}[\omega; f]$, and $Q_{m_2}[\omega; f]$ is obtained by adding nodes;

(iii) Construct $Q_{m_2}[\omega; f]$, and $Q_{m_1}[\omega; f]$ is obtained by using a subset of the set of the nodes of $Q_{m_2}[\omega; f]$ as the set of nodes of $Q_{m_1}[\omega; f]$.

The first strategy usually gives the embedded pair using the minimum number of nodes. However, it has to solve a large system of nonlinear equations for determining the nodes and weights. The second strategy was used first by Kronrod [8] for one-dimensional integration, named Kronrod quadrature formula, and by Cools and Haegemans [3, 4] for some two-dimensional regions. In [3] the embedded pair was constructed over some symmetric planar regions. In [4], due to a special structure of cubature formulae, a reduction of the number of equations and unknowns of the nonlinear system for determining the nodes and weights was obtained. The third strategy was introduced by Berntsen and Espelid [2] and Laurie [9] for one-dimensional integration and used by Rabinowitz et al. [12] for the n-cube. In a polynomial ideal point of view, it was used by Cools and Haegemans [5] to present a method for all the regions in $\mathbb{R}^d$, $d \geq 2$. All these strategies were used for the three-dimensional cube in [1].

The main purpose of this paper is to provide a new strategy to construct an embedded pair of cubature formulae over any two-dimensional region. Let $Z_n(x, y) = 0$ be an algebraic curve of degree n without multiple factors. There are two main steps in the strategy:

Step I. Construct the embedded pair $Q_{m_k}[\omega Z_n; f]$ of degrees (m_1, m_2);

Step II. Construct the embedded pair $Q_{m_k+n}[\omega; f]$ of degrees $(m_1 + n, m_2 + n)$ by adding nodes along $Z_n(x, y) = 0$ to $Q_{m_k}[\omega Z_n; f]$.

In our strategy, one embedded pair of cubature formulae is constructed in each step and two embedded pairs are constructed with respect to different weight functions. The theory and numerical examples show that the construction problem in $\mathbb{R}^2$ will degenerate into a one-dimensional construction problem in some special cases. To simplify the description, we omit the superscripts of nodes in the case of no confusion.

The paper is organized as follows. In Section 2, some notations and preliminaries are introduced. The main theory and algorithm for constructing the embedded pair of cubature formulae is given in Sections 3 and 4, respectively. Numerical examples are presented in Section 5. A brief conclusion is in Section 6.

2. Notations and preliminaries

The knowledge of ideals and varieties is needed in the rest of the paper. Here we introduce some notations and their meanings simply. For details, the readers may refer to [6] and [14].

We write the monomial $\mathbf{x}^\alpha = x^{\alpha_1}y^{\alpha_2}$ for $\mathbf{x} = (x, y)$ and $\alpha = (\alpha_1, \alpha_2)$. The monomial $\mathbf{x}^\alpha$ is of (total) degree $|\alpha| = \alpha_1 + \alpha_2$. We denote by Π^2 the set of all polynomials in two variables. For our purpose, we use the graded lexicographic order to order monomials in Π^2 throughout the paper. Denote by $\mathfrak{I}$ a polynomial ideal in Π^2 and $P_d(x, y)$ a polynomial of degree d. Some notations related to $\mathfrak{I}$ and their meanings that will be used hereafter are:

$$\begin{aligned} V(\mathfrak{I}) \quad & \text{the variety of } \mathfrak{I}, \\ |V(\mathfrak{I})| \quad & \text{the cardinality of } V(\mathfrak{I}), \\ S_{\mathfrak{I}} \quad & \text{the complement of } \mathfrak{I} \text{ in } \Pi^2, \\ \mathfrak{I}_{P_{d_1}, P_{d_2}, \ldots, P_{d_n}} \quad & \text{the ideal generated by } P_{d_1}, P_{d_2}, \ldots, P_{d_n} \text{ in } \Pi^2. \end{aligned}$$

Let m, n be two positive integers and

$$e_m(n) = \binom{m+2}{2} - \binom{m+2-n}{2} = \begin{cases} \frac{1}{2}n(2m+3-n), & \text{if } m \geq n; \\ \frac{1}{2}(m+1)(m+2), & \text{if } m \leq n-1. \end{cases}$$

We [11] presented a method to construct cubature formulae by choosing some nodes along certain selected algebraic curve.

Lemma 1 ([11]) *Let $Z_n(x, y) = 0$ be an algebraic curve of degree n without multiple factors. If there exists a cubature formula of degree m with respect to $I[\omega Z_n; f]$,*

$$Q_m[\omega Z_n; f] = \sum_{i=1}^{N} a_i^{(m)} f(x_i, y_i),$$

and none of the nodes $\{(x_i, y_i)\}_{i=1}^N$ lies on $Z_n(x, y) = 0$, then there exists a cubature formula of degree $m + n$ with respect to $I[\omega; f]$,

$$Q_{m+n}[\omega; f] = \sum_{i=1}^{N} \frac{a_i^{(m)}}{Z_n(x_i, y_i)} f(x_i, y_i) + \sum_{j=1}^{e_{m+n}(n)} b_j^{(m+n)} f(u_j, v_j),$$

where all the nodes (u_j, v_j), $j = 1, 2, \ldots, e_{m+n}(n)$, lie on $Z_n(x, y) = 0$.

We remark that the nodes $\{(u_j, v_j)\}_{j=1}^{e_{m+n}(n)}$ can be selected arbitrarily only if they form a properly posed set of nodes for the polynomial interpolation of degree $m + n$ along $Z_n(x, y) = 0$ (see [10]). Finally, we denote by $[x]$ the integer part of x.

3. Construction of embedded cubature formulae

3.1 Main result

Here is the main result.

Theorem 1 *Let $Z_n(x, y) = 0$ be an algebraic curve of degree n without multiple factors. If there* *exists an embedded pair of cubature formulae of degrees* (m_1, m_2) *with respect to* $I[\omega Z_n; f]$,

$$Q_{m_k}[\omega Z_n; f] = \sum_{i=1}^{N_k} a_i^{(m_k)} f(x_i, y_i), \quad k = 1, 2,\ m_1 < m_2,\ N_1 \leq N_2,$$

and none of the nodes $\{(x_i, y_i)\}_{i=1}^{N_2}$ *lies on* $Z_n(x, y) = 0$, *then there exists an embedded pair of cubature formulae of degrees* $(m_1 + n, m_2 + n)$ *with respect to* $I[\omega; f]$,

$$Q_{m_k+n}[\omega; f] = \sum_{i=1}^{N_k} \frac{a_i^{(m_k)}}{Z_n(x_i, y_i)} f(x_i, y_i) + \sum_{j=1}^{e_{m_k+n}(n)} b_j^{(m_k+n)} f(u_j, v_j), \tag{4}$$

where all the nodes (u_j, v_j), $j = 1, 2, \ldots, e_{m_2+n}(n)$, *lie on* $Z_n(x, y) = 0$.

Proof It is obvious that $e_{m_1+n}(n) < e_{m_2+n}(n)$. We can select $\{(u_j, v_j)\}_{j=1}^{e_{m_k+n}(n)}$ as properly posed sets of nodes for polynomial interpolation of degree $m_k + n$ along $Z_n(x, y) = 0$ for each k (see [11]). It follows from Lemma 1 that we obtain Eq. (4) for each k. This completes the proof. □

Replacing $e_{m_k+n}(n)$ by M_k in Eq. (4), we rewrite Eq. (4) as

$$Q_{m_k+n}[\omega; f] = \sum_{i=1}^{N_k} \frac{a_i^{(m_k)}}{Z_n(x_i, y_i)} f(x_i, y_i) + \sum_{j=1}^{M_k} b_j^{(m_k+n)} f(u_j, v_j). \tag{5}$$

In the next subsection, we give another method to choose the nodes (u_j, v_j) such that the number of nodes M_k may be less than $e_{m_k+n}(n)$. That is to say, Theorem 1 gives the upper bounds for the numbers M_k, $k = 1, 2$. To facilitate the description, we call the embedded pair $Q_{m_k}[\omega Z_n; f]$ and the embedded pair $Q_{m_k+n}[\omega; f]$ in Theorem 1 the initial embedded pair and the desired embedded pair, respectively.

3.2 Improvement of the method

For $k = 1$, define $I_{m_1+n}^{Z_n}[f] = I[\omega; f] - \sum_{i=1}^{N_1} \frac{a_i^{(m_1)}}{Z_n(x_i, y_i)} f(x_i, y_i)$. If we find

$$Q_{m_1+n}^{Z_n}[f] = \sum_{j=1}^{M_1} b_j^{(m_1+n)} f(u_j, v_j), \tag{6}$$

such that $I_{m_1+n}^{Z_n}[p] = Q_{m_1+n}^{Z_n}[p]$, $\forall p \in \Pi_{m_1+n}^2$, then

$$I[\omega; p] = \sum_{i=1}^{N_1} \frac{a_i^{(m_1)}}{Z_n(x_i, y_i)} p(x_i, y_i) + \sum_{j=1}^{M_1} b_j^{(m_1+n)} p(u_j, v_j), \quad \forall p \in \Pi_{m_1+n}^2.$$

Thus, the construction of $Q_{m_1+n}[\omega; f]$ is transformed to the construction of $Q_{m_1+n}^{Z_n}[f]$. If all the nodes (u_j, v_j), $j = 1, 2, \ldots, M_1$, are chosen along $Z_n(x, y) = 0$, then

$$I_{m_1+n}^{Z_n}[p] = 0 = Q_{m_1+n}^{Z_n}[p], \quad \forall p \in \Im_{Z_n} \bigcap \Pi_{m_1+n}^2.$$

Hence, $Q_{m_1+n}^{Z_n}[f]$ will have degree $m_1 + n$ if for $\forall \widetilde{p} \in S_{\Im_{Z_n}} \bigcap \Pi_{m_1+n}^2$,

$$I_{m_1+n}^{Z_n}[\widetilde{p}] = Q_{m_1+n}^{Z_n}[\widetilde{p}]. \tag{7}$$

Suppose that $P_{d_i}, i=1,2,\ldots,e_{m_1+n}(n)$, are a basis of $S_{\Im_{Z_n}}\bigcap\Pi^2_{m_1+n}$. Then Eq. (7) implies that $I^{Z_n}_{m_1+n}[P_{d_i}]=Q^{Z_n}_{m_1+n}[P_{d_i}],\ i=1,2,\ldots,e_{m_1+n}(n)$.

Similarly, for $k=2$, let

$$I^{Z_n}_{m_2+n}[f]=I[\omega;f]-\sum_{i=1}^{N_2}\frac{a_i^{(m_2)}}{Z_n(x_i,y_i)}f(x_i,y_i), \tag{8}$$

and

$$Q^{Z_n}_{m_2+n}[f]=\sum_{j=1}^{M_2}b_j^{(m_2+n)}f(u_j,v_j),\quad M_1\le M_2, \tag{9}$$

where the former M_1 nodes of $\{(u_j,v_j)\}_{j=1}^{M_2}$ are those used in Eq. (7). Then $Q_{m_2+n}[\omega;f]$ is obtained by choosing $\{(u_j,v_j)\}_{j=M_1+1}^{M_2}$ along $Z_n(x,y)=0$ such that $I^{Z_n}_{m_2+n}[\widetilde{q}]=Q^{Z_n}_{m_2+n}[\widetilde{q}]$ holds for any $\widetilde{q}\in S_{\Im_{Z_n}}\bigcap\Pi^2_{m_2+n}$.

In fact, it is not easy to construct the embedded pair of Eqs. (6) and (9). However, if $Z_n(x,y)$ is selected carefully, the constructions of Eqs. (6) and (9) will be reduced to one-dimensional moment problems. We will show this below for some special $Z_n(x,y)$'s. In the rest of this section, we suppose that all the nodes $\{(u_j,v_j)\}_{j=1}^{M_2}$ are on the corresponding algebraic curve $Z_n(x,y)=0$.

Case 1 $n=1$.

Without loss of generality, let $Z_1(x,y)=x+\text{lower terms}$. Let $Q_{m_k}[\omega Z_1;f]$ be the initial embedded pair. Now we choose $\{(u_j,v_j)\}_{j=1}^{M_2}$ such that the desired embedded pair $Q_{m_k+1}[\omega;f]$ has degrees (m_1+1,m_2+1).

It is easy to verify that $1,y,\ldots,y^{m_k+1}$ are a basis of $S_{\Im_{Z_1}}\bigcap\Pi^2_{m_k+1}$. Let

$$I^{Z_1}_{m_k+1}[f]=I[\omega Z_1;f]-\sum_{i=1}^{N_k}\frac{a_i^{(m_k)}}{Z_1(x_i,y_i)}f(x_i,y_i). \tag{10}$$

Then $Q_{m_k+1}[\omega;f]$ will have degree m_k+1 for each k if $\{v_j\}_{j=1}^{M_2}$ satisfies

$$I^{Z_1}_{m_k+1}[y^{h_k}]=\sum_{j=1}^{M_k}b_j^{(m_k+1)}v_j^{h_k},\quad k=1,2,\ h_k=0,1,2,\ldots,m_k+1. \tag{11}$$

Suppose that $I^{Z_1}_{m_1+1}[y^{h_1}]$ and $I^{Z_1}_{m_2+1}[y^{h_2}]$ are nonzero for some h_1 and h_2, respectively. Then for $k=1$, Eq. (11) is a one-dimensional moment problem. It may be solved [13] when degree M_1 is non-degenerate by taking $\{v_j\}_{j=1}^{M_1}$ as zeros of the orthogonal polynomial $P_{M_1}(y)$ with respect to the weight function $\rho_1(y)$, where $M_1=[(m_1+3)/2]$ and

$$\int_a^b\rho_1(y)y^{h_1}\mathrm{d}y=I^{Z_1}_{m_1+1}[y^{h_1}],\quad h_1=0,1,2,\ldots,m_1+1. \tag{12}$$

Meanwhile, there exists a weight function $\rho_2(y)$ such that

$$\int_a^b\rho_2(y)y^{h_2}\mathrm{d}y=I^{Z_1}_{m_2+1}[y^{h_2}],\quad h_2=0,1,2,\ldots,m_2+1. \tag{13}$$

Let $M=[(m_2-M_1+3)/2]$. For any polynomial $P_{m_2+1}(y)$,

$$P_{m_2+1}(y)=P_{M_1}(y)P_M(y)q(y)+r(y),$$

where $\deg(r(y))\leq M_1+M-1$ or $r(y)=0$. For $k=2$, Eq. (11) holds by taking $\{v_j\}_{j=M_1+1}^{M_1+M}$ as zeros of $P_M(y)$ which satisfies

$$\int_a^b \rho_2(y)P_{M_1}(y)P_M(y)y^h\mathrm{d}y=0,\quad h=0,1,2\dots,M-1. \tag{14}$$

Similarly to $P_{M_1}(y)$, $P_M(y)$ exists. Finally, we denote $M_2=M_1+M$. In absence of degeneracy, $M_2=[(m_2+M_1+3)/2]$.

Case 2 $n=2$.

For $n=2$, we will deal with the algebraic curves $xy=0$ and $Z_2(x,y)=x^2$+lower terms $=0$.

Case 2.1 The algebraic curve $xy=0$.

A basis of $S_{\mathfrak{I}_{xy}}\bigcap\Pi^2_{m_k+2}$ is $1,x,\dots,x^{m_k+2},y,\dots,y^{m_k+2}$. Let $Q_{m_k}[\omega xy;f]$ be the initial embedded pair, $X_k=\{1,x,\dots,x^{m_k+2}\}$ and $Y_k=\{1,y,\dots,y^{m_k+2}\}$. In this case, we construct the desired embedded pair $Q_{m_k+2}[\omega;f]$ with the following form

$$\begin{aligned}Q_{m_k+2}[\omega;f]=&\sum_{i=1}^{N_k}\frac{a_i^{(m_k)}}{x_iy_i}f(x_i,y_i)+\sum_{j=1}^{M_k}b_j^{(m_k+2)}f(u_j,0)+\\&\sum_{j=1}^{M_k}c_j^{(m_k+2)}f(0,v_j)+d^{(m_k+2)}f(0,0),\quad M_1\leq M_2.\end{aligned} \tag{15}$$

Let

$$I^{xy}_{m_k}[f]=I[\omega;f]-\sum_{i=1}^{N_k}\frac{a_i^{(m_k)}}{x_iy_i}f(x_i,y_i),$$

and

$$Q^{xy}_{m_k}[f]=\sum_{j=1}^{M_k}b_j^{(m_k+2)}f(u_j,0)+\sum_{j=1}^{M_k}c_j^{(m_k+2)}f(0,v_j)+\widetilde{d}^{(m_k+2)}f(0,0).$$

For each k, by the similar process in Case 1, we first choose $\{(u_j,0)\}_{j=1}^{M_2}$ and $\{(0,v_j)\}_{j=1}^{M_2}$ such that $Q^{xy}_{m_k}[f]=I^{xy}_{m_k}[f]$ holds for $f\in X_k$ and for $f\in Y_k$, respectively. Next, we obtain $d^{(m_k+2)}$ by substituting $f(x,y)=1$ into Eq. (15). In absence of degeneracy, $M_1=[(m_1+4)/2]$ and $M_2=[(m_2+M_1+4)/2]$.

Case 2.2 The algebraic curve $Z_2(x,y)=x^2+\text{lower terms}=0$.

A basis of $S_{\mathfrak{I}_{Z_2}}\bigcap\Pi^2_{m_k+2}$ is $1,y,\dots,y^{m_k+2},xy,\dots,xy^{m_k+1}$. Let $Q_{m_k}[\omega Z_2;f]$ be the initial embedded pair. Let

$$I^{Z_2}_{m_k}[f]=I[\omega;f]-\sum_{i=1}^{N_k}\frac{a_i^{(m_k)}}{Z_2(x_i,y_i)}f(x_i,y_i).$$

Under some assumption on $I^{Z_2}_{m_k}[f]$, we will construct two kinds of desired embedded pairs.

First, we assume that $I_{m_k}^{Z_2}[f]=0$, for $\forall f(x,y)=-f(-x,y)$, $k=1,2$. Then for any non-negative integer h, $I_{m_k}^{Z_2}[xy^h]=0$. Let

$$Q_{m_k}^{Z_2}[f]=\sum_{j=1}^{M_k} b_j^{(m_k+2)}(f(u_j,v_j)+f(-u_j,v_j)).$$

Thus, the desired embedded pair

$$Q_{m_k+2}[\omega;f]=\sum_{i=1}^{N_k}\frac{a_i^{(m_k)}}{Z_2(x_i,y_i)}f(x_i,y_i)+\sum_{j=1}^{M_k} b_j^{(m_k+2)}(f(u_j,v_j)+f(-u_j,v_j)) \tag{16}$$

will have degrees (m_1+2,m_2+2) only if $Q_{m_k}^{Z_2}[f]=I_{m_k}^{Z_2}[f]$ holds for $f\in Y_k$, where $M_1=[(m_1+4)/2]$, $M_2=[(m_2+M_1+4)/2]$ and all the nodes $\{(u_j,v_j)\}_{j=1}^{M_2}$ lie on $Z_2(x,y)=0$.

Secondly, we assume that $I_{m_k}^{Z_2}[f]=0$, for $\forall f(x,y)=f(-x,y)$, $k=1,2$. Similarly, the desired embedded pair could be of the form

$$Q_{m_k+2}[\omega;f]=\sum_{i=1}^{N_k}\frac{a_i^{(m_k)}}{Z_2(x_i,y_i)}f(x_i,y_i)+\sum_{j=1}^{M_k} b_j^{(m_k+2)}(f(u_j,v_j)-f(-u_j,v_j)), \tag{17}$$

where $M_1=[(m_1+4)/2]$ and $M_2=[(m_2+M_1+4)/2]$.

The nodes $\{(u_j,v_j)\}_{j=1}^{M_2}$ in Eqs. (16) and (17) can be chosen by the similar method in Case 1.

Based on all the cases above, we will introduce an algorithm for constructing an embedded pair of cubature formulae in the next section.

4. Algorithm for constructing an embedded pair of cubature formulae

Suppose that $Z_n(x,y)=\prod_{i=1}^N R_{d_i}(x,y)$, where $R_{d_i}(x,y)$ is a polynomial of degree d_i with the form $Z_1(x,y)$, $Z_2(x,y)$ or xy and $\sum_{i=1}^N d_i=n$. If there exists an initial embedded pair $Q_{m_k}[\omega Z_n;f]$ of degrees (m_1,m_2) $(m_1<m_2)$ with respect to $I[\omega Z_n;f]$ over the region Ω and none of the nodes lies on $Z_n(x,y)=0$, then there exists a desired embedded pair $Q[\omega;f]$ of degrees (m_1+n,m_2+n) with respect to $I[\omega;f]$ by adding nodes along $R_{d_i}(x,y)=0$.

Taking $\omega(x,y)=1$, $Z_{2n}(x,y)=xy\prod_{i=1}^{n-1}S_i$ and $\Omega=B^2=\{(x,y)|x^2+y^2\le 1\}$ as an example, we construct the embedded pair of cubature formulae of degrees (m_1+2n,m_2+2n), where $S_i=x^2+y^2-s_i^2$ and $0\le s_i\le 1$.

Step 1. Construct $Q_{m_k}[Z_{2n};f]$, $k=1,2$.

Let $T(x,y)=\prod_{i=1}^{n'}T_i(x,y)$, $T_i(x,y)=x^2+y^2-t_i^2$ and $0\le t_i\le 1$. Then

$$\int_{B^2} Z_{2n}(x,y)p(x,y)\mathrm{d}x\mathrm{d}y=0, \tag{18}$$

$$\int_{B^2} Z_{2n}(x,y)T(x,y)p(x,y)\mathrm{d}x\mathrm{d}y=0, \tag{19}$$

where $\deg(p(x,y))\le 1$. Thus it is easy to obtain a cubature formula of degree 1 with respect to $I[Z_{2n};f]$ by choosing zero node since Eq. (18) holds, that is,

$$Q_{m_1}[Z_{2n};f]=0. \tag{20}$$

Suppose that $Z_{2n}(x, y)$ is a quasi-orthogonal polynomial of degree $2n$ and order $2n - m_1 - 1$ with respect to the weight function 1 (a polynomial $p(x, y)$ of degree n is called a quasi-orthogonal polynomial of degree n and order s, if it is orthogonal to all polynomials of degree at most $n - s - 1$). From Eq. (18), we see that $m_1 \geq 1$. We obtain a cubature formula $Q_{m_1}[Z_{2n}; f]$ of degree m_1 with the form Eq. (20). Furthermore, suppose that $T(x, y)$ is a quasi-orthogonal polynomial of degree $2n'$ and order $2n' - m' - 1$ with respect to the weight function $Z_{2n}(x, y)$ and that $m' + 2n' > m_1$. We obtain a cubature formula of degree m' with zero node with respect to $I[Z_{2n}T; f]$. Since $I[Z_{2n}T; f] = 0$, for $\forall f(-x, y) = f(x, y)$, we obtain a cubature formula of degree $m' + 2n'$ with respect to $I[Z_{2n}; f]$ by choosing nodes along $T_i(x, y) = 0$, which is denoted by $Q_{m_2}[Z_{2n}; f]$ as follows ($m_2 = m' + 2n'$),

$$Q_{m_2}[Z_{2n}; f] = \sum_{i=1}^{M_{2,1}} a_{i,1}^{(m_2)} (f(u_{i,1}, v_{i,1}) - f(-u_{i,1}, v_{i,1})), \tag{21}$$

where $\{(u_{i,1}, v_{i,1})\}_{i=1}^{M_{2,1}}$ are on $T(x, y) = 0$. Clearly, formulae $Q_{m_1}[Z_{2n}; f]$ and $Q_{m_2}[Z_{2n}; f]$ are an embedded pair since there are no nodes in $Q_{m_1}[Z_{2n}; f]$. Step 1 is completed.

Let $N_{1,0} = N_{2,0} = M_{1,1} = N_{1,1} = 0$, $N_{2,1} = M_{2,1}$, $b_{i,1}^{(m_2)} = a_{i,1}^{(m_2)}$ and $\mathfrak{S}(\gamma) = \prod_{i=\gamma}^{n-1} S_i$. Denote $\mathfrak{S}(n) = 1$.

Step γ. Construct $Q_{m_k+2(\gamma-1)}[xy\mathfrak{S}(\gamma); f]$, $k = 1, 2$, $\gamma = 2, \ldots, n$.

After the $(\gamma - 1)$th step, the embedded pair of degrees $(m_1 + 2(\gamma - 2), m_2 + 2(\gamma - 2))$ is obtained as follows,

$$\begin{aligned} Q_{m_k+2(\gamma-2)}[xy\mathfrak{S}(\gamma-1); f] = &\sum_{i=1}^{N_{k,\gamma-2}} \frac{a_{i,\gamma-2}^{(m_k+2(\gamma-3))}}{S_{\gamma-2}(x_{i,\gamma-2}, y_{i,\gamma-2})} (f(x_{i,\gamma-2}, y_{i,\gamma-2}) - f(-x_{i,\gamma-2}, y_{i,\gamma-2})) + \\ &\sum_{i=1}^{M_{k,\gamma-1}} b_{i,\gamma-1}^{(m_k+2(\gamma-2))} (f(u_{i,\gamma-1}, v_{i,\gamma-1}) - f(-u_{i,\gamma-1}, v_{i,\gamma-1})), \end{aligned}$$

where $k = 1, 2$ and $M_{1,\gamma-1} \leq M_{2,\gamma-1}$. Choosing nodes along $S_{\gamma-1}(x, y) = 0$, the embedded pair of degrees $(m_1 + 2(\gamma - 1), m_2 + 2(\gamma - 1))$ is obtained as follows,

$$\begin{aligned} Q_{m_k+2(\gamma-1)}[xy\mathfrak{S}(\gamma); f] = &\sum_{i=1}^{N_{k,\gamma-1}} \frac{a_{i,\gamma-1}^{(m_k+2(\gamma-2))}}{S_{\gamma-1}(x_{i,\gamma-1}, y_{i,\gamma-1})} (f(x_{i,\gamma-1}, y_{i,\gamma-1}) - f(-x_{i,\gamma-1}, y_{i,\gamma-1})) + \\ &\sum_{i=1}^{M_{k,\gamma}} b_{i,\gamma}^{(m_k+2(\gamma-1))} (f(u_{i,\gamma}, v_{i,\gamma}) - f(-u_{i,\gamma}, v_{i,\gamma})), \end{aligned} \tag{22}$$

where $k = 1, 2$, $M_{1,\gamma} \leq M_{2,\gamma}$, $N_{k,\gamma} = N_{k,\gamma-1} + M_{k,\gamma}$ and

$$\begin{aligned} (x_{i,\gamma}, y_{i,\gamma}) &= \begin{cases} (x_{i,\gamma-1}, y_{i,\gamma-1}), & i = 1, 2, \ldots, N_{2,\gamma-1}, \\ (u_{i-N_{2,\gamma-1}}, v_{i-N_{2,\gamma-1}}), & i = N_{2,\gamma-1}+1, N_{2,\gamma-1}+2, \ldots, N_{2,\gamma-1}+M_{2,\gamma}, \end{cases} \\ a_{i,\gamma}^{(m_k+2(\gamma-1))} &= \begin{cases} \dfrac{a_{i,\gamma-1}^{(m_k+2(\gamma-2))}}{S_{\gamma-1}(x_{i,\gamma-1}, y_{i,\gamma-1})}, & i = 1, 2, \ldots, N_{k,\gamma-1}, \\ b_{i-N_{k,\gamma-1}}^{(m_k+2(\gamma-1))}, & i = N_{k,\gamma-1}+1, N_{k,\gamma-1}+2, \ldots, N_{k,\gamma-1}+M_{k,\gamma}, \end{cases} \end{aligned} \tag{23}$$

Obviously, $N_{1,\gamma} \leq N_{2,\gamma}$. When $\gamma = n$, the embedded pair of degrees $(m_1+2(n-1), m_2+2(n-1))$

with respect to $I[xy;f]$ is obtained as follows,

$$\begin{aligned}&Q_{m_k+2(n-1)}[xy;f]\\ &=\sum_{i=1}^{N_{k,n-1}}\frac{a_{i,n-1}^{(m_k+2(n-2))}}{S_{n-1}(x_{i,n-1},y_{i,n-1})}(f(x_{i,n-1},y_{i,n-1})-f(-x_{i,n-1},y_{i,n-1}))+\\ &\sum_{i=1}^{M_{k,n}}b_{i,n}^{(m_k+2(n-1))}(f(u_{i,n},v_{i,n})-f(-u_{i,n},v_{i,n})),\end{aligned}\tag{24}$$

where $M_{1,n}\le M_{2,n}$.

Step $n+1$. Construct $Q_{m_k+2n}[1;f]$, $k=1,2$.

In this step, the embedded pair of degree (m_1+2n,m_2+2n) is obtained by choosing nodes along $xy=0$ based on Eq. (24). That is,

$$\begin{aligned}Q_{m_k+2n}[1;f]=&\sum_{i=1}^{N_{k,n}}\frac{a_{i,n}^{(m_k+2(n-1))}}{x_{i,n}y_{i,n}}(f(x_{i,n},y_{i,n})-f(x_{i,n},y_{i,n}))+\\ &\sum_{i=1}^{M_{k,n+1}}b_{i,n+1}^{(m_k+2n)}f(u_{i,n+1},0)+\sum_{i=1}^{M_{k,n+1}}c_{i,n+1}^{(m_k+2n)}f(0,v_{i,n+1})+d_{n+1}^{(m_k+2n)}f(0,0),\\ &M_{1,n+1}\le M_{2,n+1}.\end{aligned}\tag{25}$$

In this algorithm, only one-dimensional problems are solved, which are very convenient in practice.

5. Examples of embedded cubature formulae on the unit disk

In this section, we give two numerical examples of constructing embedded cubature formulae by the methods in Sections 3 and 4 on the unit disk.

Example 1 An embedded pair of cubature formulae of degrees $(5,7)$

The formulae are constructed in two steps:

Step 1. Construct the initial embedded pair of degrees $(3,5)$ with respect to $I[xy;f]$.

Step 2. Construct the desired embedded pair of degrees $(5,7)$ with respect to $I[1;f]$ by choosing nodes along $xy=0$.

Let $F_j(x,y)=y^2-l_j^2x^2$, $G_j(x,y)=x^2+y^2-r_j^2$ and $\mathfrak{I}_j=\mathfrak{I}_{F_j,G_j}$, $j=1,2$. Here we demand $0<r_1,r_2\le 1$, $l_1,l_2>0$, and $\frac{1+l_1^2}{1+l_2^2}\ne\frac{l_1^2r_1^2}{l_2^2r_2^2}$. Let $\mathfrak{I}_3=\mathfrak{I}_1\cdot\mathfrak{I}_2$. Then $\mathfrak{I}_3=\mathfrak{I}_{F_iG_j,i,j=1,2}$ and $V(\mathfrak{I}_3)=V(\mathfrak{I}_1)\bigcup V(\mathfrak{I}_2)$.

In Step 1, it is easy to verify that $F_1(x,y)$ and $G_1(x,y)$ are orthogonal polynomials of degree 2 and that $F_i(x,y)G_j(x,y)$, $1\le i,j\le 2$, are quasi-orthogonal polynomials of degree 4 and order 2 with respect to the weight function xy. Since $\dim S_{\mathfrak{I}_3}=|V(\mathfrak{I}_3)|=8$, according to the theory in [14] there exist $Q_3[xy;f]$ and $Q_5[xy;f]$ as follows,

$$\int_{B^2}xyp(x,y)\mathrm{d}x\mathrm{d}y=Q_3[xy;p]=\sum_{i=1}^{4}a_i^{(3)}p(x_i,y_i),\quad p\in\Pi_3^2,\tag{26}$$

$$\int_{B^2} xyq(x,y)\mathrm{d}x\mathrm{d}y = Q_5[xy;q] = \sum_{i=1}^{8} a_i^{(5)} q(x_i,y_i), \quad q \in \Pi_5^2, \tag{27}$$

where $\{(x_i,y_i)\}_{i=1}^4 \in V(\mathfrak{I}_1), \{(x_i,y_i)\}_{i=5}^8 \in V(\mathfrak{I}_2)$ and $a_i^{(3)}$ and $a_i^{(5)}$ are unknowns. In order to reduce the number of equations and unknowns in Eqs. (26) and (27), we set $\alpha = \frac{a_i^{(3)}}{x_i y_i}$, $\beta_1 = \frac{a_i^{(5)}}{x_i y_i}$, $i = 1,2,3,4$ and $\beta_2 = \frac{a_i^{(5)}}{x_i y_i}$, $i = 5,6,7,8$. Thus, we have $\alpha = \frac{(1+l_1^2)^2\pi}{96 l_1^2 r_1^4}$ and

$$A\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = b, \quad A = \begin{pmatrix} \frac{l_1^2 r_1^4}{(1+l_1^2)^2} & \frac{l_2^2 r_2^4}{(1+l_2^2)^2} \\ \frac{l_1^4 r_1^6}{(1+l_1^2)^3} & \frac{l_2^4 r_2^6}{(1+l_2^2)^3} \\ \frac{l_1^2 r_1^6}{(1+l_1^2)^3} & \frac{l_2^2 r_2^6}{(1+l_2^2)^3} \end{pmatrix}, \quad b = \begin{pmatrix} \frac{\pi}{96} \\ \frac{\pi}{256} \\ \frac{\pi}{256} \end{pmatrix}.$$

There exists a solution of β_i if and only if $r(A) = r(A,b)$, where $r(\cdot)$ denotes the rank of matrix. Since $r(A) \le 2$, we have $\det(A,b) = 0$. Hence the necessary condition of the existence of β_i is

$$3(1 - l_1^2 + l_2^2 - l_1^2 l_2^2) r_1^2 - 3(1 + l_1^2 - l_2^2 - l_1^2 l_2^2) r_2^2 + 8(l_1^2 - l_2^2) r_1^2 r_2^2 = 0. \tag{28}$$

Since $\frac{1+l_1^2}{1+l_2^2} \ne \frac{l_1^2 r_1^2}{l_2^2 r_2^2}$, β_1 and β_2 are solved by the following equations

$$A_1\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = b_1, \quad A_1 = \begin{pmatrix} \frac{l_1^2 r_1^4}{(1+l_1^2)^2} & \frac{l_2^2 r_2^4}{(1+l_2^2)^2} \\ \frac{l_1^4 r_1^6}{(1+l_1^2)^3} & \frac{l_2^4 r_2^6}{(1+l_2^2)^3} \end{pmatrix}, \quad b_1 = \begin{pmatrix} \frac{\pi}{96} \\ \frac{\pi}{256} \end{pmatrix}.$$

Therefore,

$$\beta_i = (-1)^{(i-1)} \frac{(1+l_i^2)^3 \sigma(l_1,l_2,r_1,r_2)}{l_i^2 r_i^4 (3 + 3l_i^2 - 8 l_i^2 r_i^2)}, \quad i = 1,2,$$

where $\sigma(l_1,l_2,r_1,r_2) = \frac{(3+3l_1^2-8l_1^2r_1^2)(3+3l_2^2-8l_2^2r_2^2)\pi}{768(l_1^2r_1^2 - l_2^2r_2^2 + l_1^2l_2^2(r_1^2-r_2^2))}$.

In Step 2, the nodes are chosen by the method in Section 3.2. The desired embedded pair is

$$Q_5[1;f] = \sum_{i=1}^{4} \alpha f(x_i,y_i) + \sum_{j=1}^{2} b_j^{(5)} f(u_j,0) + \sum_{j=1}^{2} c_j^{(5)} f(0,v_j) + d^{(5)} f(0,0)$$

and

$$Q_7[1;f] = \sum_{i=1}^{8} \beta_{1+[(i-1)/4]} f(x_i,y_i) + \sum_{j=1}^{4} b_j^{(7)} f(u_j,0) + \sum_{j=1}^{4} c_j^{(7)} f(0,v_j) + d^{(7)} f(0,0).$$

The expressions of $u_j, v_j, b_j^{(k)}, c_j^{(k)}$, and $d^{(k)}$, $k = 5,7$, are functions of l_1, l_2, r_1 and r_2 and complicated, and we omit them here. Two special cases are shown as follows.

Case 1 $l_1 = 0.596673042488015$, $l_2 = 9.589366183178306$, $r_1 = 0.902004008737817$, $r_2 = 0.803707572904092$.

The desired embedded pair was constructed in [3] by using a different method.

Case 2 $l_1 = l_2 = 1$.

The condition Eq. (28) is satisfied. Through the computation, we conclude that

(i) There does not exist such desired embedded pair of degrees (5,7) that its nodes are located inside the disk symmetrically and invariant by rotation of $\pi/2$.

(ii) All the weights are positive if one of the following conditions is satisfied:

(a) $r_1 = 0.759909177984682$, $r_2 = 1$;

(b) $r_1 = 0.866025403784439$, $r_2 \neq r_1$;

(c) $0.650115167343736 \leq r_1 \leq 0.746381949494463$, $0.866025403784439 \leq r_2 \leq 1$;

(d) $0.759909177984682 < r_1 < 0.746381949494463$, $\eta(r_1) \leq r_2 \leq 1$;

(e) $0.866025403784439 < r_1 \leq 0.910995803644429$, $\eta(r_1) \leq r_2 \leq 0.866025403784439$;

(f) $0.910995803644429 < r_1 \leq 1$, $\xi(r_1) \leq r_2 \leq 0.866025403784439$,

where $\xi(r_1) = \frac{1}{2}\sqrt{\frac{9-12r_1^2}{3-5r_1^2}}$ and

$$\eta(r_1) = \frac{1}{2}\sqrt{\frac{18-81r_1^2+121r_1^4-60r_1^6+\sqrt{3}\sqrt{108-972r_1^2+3543r_1^4-6678r_1^6+6899r_1^8-3768r_1^{10}+880r_1^{12}}}{12-56r_1^2+78r_1^4-30r_1^6}}.$$

In Table 1 we present the desired embedded pair of degrees (5,7) which is constructed by choosing $r_1 = 0.650115167343736$ and $l_1 = l_2 = r_2 = 1$ and has the following properties:

(i) All the nodes are located symmetrically and invariant by rotation of $\pi/2$;

(ii) The nodes in the fourth row in Table 1 are on the boundary;

(iii) All the weights are positive.

$\pm x_i$	$\pm y_i$	ω_{5i}	ω_{7i}
0.459700843380983	0.459700843380983	0.732786462492640	0.317305846033979
1.255926060399109	0.0	0.052611700904808	0.004253968490140
0.0	1.255926060399109	0.052611700904808	0.004253968490140
0.707106781186548	0.707106781186548		0.074218463767255
0.822933195591952	0.0		0.262339623560236
0.0	0.822933195591952		0.262339623560236
0.0	0.0		0.509121046183350

Table 1 The embedded pair of cubature formulae of degrees $(5,7)$ over B^2 with $l_1 = l_2 = r_2 = 1$ and $r_1 = 0.650115167343736$

In this example, if $r_1 = 0.650115167343736$ and $r_2 = 0.524576772205617$, then $d^{(5)} = d^{(7)} = 0$. Thus the numbers of nodes of the cubature formulae of degree 5 and 7 are 8 and 16, respectively. The corresponding formulae are omitted here.

Example 2 Another embedded pair of cubature formulae of degrees $(5,7)$

The example is constructed by the algorithm in Section 4. For any r_1 and r_2 $(0 \leq r_1, r_2 \leq 1)$,

$$\int_{B^2} xy(x^2+y^2-r_1^2)p(x,y)\mathrm{d}x\mathrm{d}y = 0, \tag{29}$$

$$\int_{B^2} xy(x^2+y^2-r_1^2)(x^2+y^2-r_2^2)p(x,y)\mathrm{d}x\mathrm{d}y = 0, \tag{30}$$

where $\deg(p(x,y)) \leq 1$. The detail process is as follows.

Step 1. Based on Eq. (29), there exists a cubature formula of degree 1 with zero node with respect to $I[xy(x^2+y^2-r_1^2); f]$. Based on Eq. (30), construct a cubature formula of degree 3 with respect to $I[xy(x^2+y^2-r_1^2);f]$ by choosing nodes along $(x^2+y^2-r_2^2)=0$. Thus, an embedded pair of cubature formulae of degrees (1,3) with respect to $I[xy(x^2+y^2-r_1^2);f]$ is obtained.

Step 2. First construct the embedded pair of cubature formulae of degrees (3,5) with respect to $I[xy;f]$ by choosing nodes along $x^2+y^2-r_1^2=0$. Then construct the embedded pair of cubature formulae of degrees $(5,7)$ with respect to $I[1;f]$ by choosing nodes along $xy=0$.

In this example, all the nodes and weights are parameterized by r_1 and r_2. Here we give some conditions for special embedded pairs of cubature formulae.

$\pm x_i$	$\pm y_i$	ω_{5i}	ω_{7i}
0.653674609031777	0.612372435695795	0.204232164591664	0.130016551340444
0.747106153371610	0.0	0.390861373496535	-1.011700001352971
0.0	0.761535561006202	0.413019348061539	0.066960213050551
0.0	0.0	0.716902552106991	0.074330402067366
0.706072094640341	0.128164835786863		0.727260315941574
0.259900178979615	0.857171156455472		0.119609150980555
0.983076911315562	0.0		0.108641290974499
0.0	0.369197652664678		0.415957586563990

Table 2 The embedded pair of cubature formulae of degrees (5,7) over B^2 with $r_1=0.895706701154371$ and $r_2=0.717609941376283$

Condition 1 All the nodes are inside the disk:

(i) $0.894427190999916 < r_1 < 0.895706701154371$,
$\sqrt{\frac{-12-30r_1^2+60r_1^4}{-75+100r_1^2}} < r_2 \leq \sqrt{\frac{234-409r_1^2-420r_1^4+720r_1^6}{50(-3+4r_1^2)(-7+12r_1^2)} + \frac{1}{50}\phi(r_1)}$;

(ii) $r_1 = 0.895706701154371$, $0.698530447975331 < r_2 < 0.720721950181746$;

(iii) $0.895706701154371 < r_1 < 0.896561075287405$,
$\sqrt{\frac{-12-30r_1^2+60r_1^4}{-75+100r_1^2}} < r_2 < \sqrt{\frac{315-555r_1^2-116r_1^4+400r_1^6}{50(-3+4r_1^2)^2} - \frac{1}{50}\varphi(r_1)}$,

where

$$\phi(r_1) = \sqrt{\frac{4356-33612r_1^2+220861r_1^4-840120r_1^6+1603440r_1^8-1468800r_1^{10}+518400r_1^{12}}{(-3+4r_1^2)^2(-7+12r_1^2)^2}},$$

$$\varphi(r_1) = \sqrt{\frac{75249-503334r_1^2+1411965r_1^4-2125032r_1^6+1810096r_1^8-828800r_1^{10}+160000r_1^{12}}{(-3+4r_1^2)^4}}.$$

Condition 2 All the weights of the cubature formula of degree 5 are positive:

$$0.711276861706101 < r_1 \leq 1,\ r_1 \neq 0.735980072193987,\ 0 < r_2 \leq 1.$$

Condition 3 All the weights of the cubature formula of degree 7 are positive:

(i) $0.962594280847148 < r_1 < 0.970736323872751$,
$\sqrt{\frac{1}{50}(\frac{459-1009r_1^2-20r_1^4+720r_1^6}{21-64r_1^2+48r_1^4} + \zeta(r_1^2))} < r_2 \leq \sqrt{\frac{57-75r_1^2}{75-100r_1^2}}$;

(ii) $0.899219894623581 < r_1 \leq 0.899255633514933$,
$\sqrt{\frac{459-1009r_1^2-20r_1^4+720r_1^6}{21-64r_1^2+48r_1^4} + \zeta(r_1^2)} < \sqrt{50}r_2 \leq \sqrt{\frac{225-315r_1^2-276r_1^4+400r_1^6}{(3-4r_1^2)^2} + \theta(r_1^2)}$,

where

$$\zeta(r_1) = \sqrt{\frac{51606-437262r_1^2+1572661r_1^4-3064120r_1^6+3405040r_1^8-2044800r_1^{10}+518400r_1^{12}}{(21-64r_1^2+48r_1^4)^2}},$$

$$\theta(r_1)=\sqrt{\frac{34344-251694r_1^2+787005r_1^4-1340712r_1^6+1312176r_1^8-700800r_1^{10}+160000r_1^{12}}{(-3+4r_1^2)^4}}.$$

In Table 2 we present the embedded pair of cubature formulae of degrees (5,7) which is constructed by choosing $r_1=0.895706701154371$ and $r_2=0.717609941376283$.

6. Conclusion

In this paper, we introduce a new strategy to construct an embedded pair of cubature formulae in $\mathbb{R}^2$. The methods in Section 3 show that the problem of constructing embedded cubature formulae in $\mathbb{R}^2$ can be simplified and for some special cases it is only a one-dimensional problem. The algorithm in Section 4 shows that an embedded pair of cubature formulae can be iteratively constructed and during all the process only one-dimensional moment problems are dealt with. The main result in Section 3 and the algorithm in Section 4 can be also used for constructing an embedded sequence of cubature formulae. Examples 1 and 2 show that the positions of the nodes and the signs of the weights of embedded cubature formulae can be controlled by some parameters, which will be much more convenient in practice.

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