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The Existence of Solutions to a Class of Multi-point Boundary Value Problem of Fractional Differential Equation

Xiaohong HAO, Zongfu ZHOU*
Department of Mathematical Sciences, Anhui University, Anhui 230601, *P. R. China*

Abstract In this paper, we consider the following multi-point boundary value problem of fractional differential equation

$$
\begin{aligned}
&D_{0+}^{\alpha}u(t)=f(t,u(t),\ D_{0+}^{\alpha-1}u(t),D_{0+}^{\alpha-2}u(t),D_{0+}^{\alpha-3}u(t)),\quad t\in(0,1),\\
&I_{0+}^{4-\alpha}u(0)=0,\ D_{0+}^{\alpha-1}u(0)=\sum_{i=1}^{m}\alpha_i D_{0+}^{\alpha-1}u(\xi_i),\\
&D_{0+}^{\alpha-2}u(1)=\sum_{j=1}^{n}\beta_j D_{0+}^{\alpha-2}u(\eta_j),\ D_{0+}^{\alpha-3}u(1)-D_{0+}^{\alpha-3}u(0)=D_{0+}^{\alpha-2}u(\frac{1}{2}),
\end{aligned}
$$

where $3<\alpha\le 4$ is a real number. By applying Mawhin coincidence degree theory and constructing suitable operators, some existence results of solutions can be established.

Keywords fractional differential equation; multi-point boundary value problem; coincidence degree.

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1. Introduction

Recently, fractional differential equations have been of great interest due to the intensive development of the theory of fractional calculus itself as well as its applications. The fractional calculus has been applied to numerous and widespread fields of science and engineering, such as rheology, fluid flows, electrical networks, viscoelasticity, chemical physics, etc. It is really a useful tool for solving differential and integral equations and various other problems involving special functions. For details, see [1–11, 17–20] and the references therein.

There are some papers dealing with the solvability of fractional boundary value problems recently. In [5], Bai investigated the nonlinear nonlocal problem

$$
\begin{aligned}
&D_{0+}^{\alpha}u(t)=f(t,u(t)),\quad t\in(0,1),\\
&u(0)=0,\quad \beta u(\eta)=u(1),
\end{aligned}
$$

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* Corresponding author
E-mail address: haoxiaohong200866@163.com (Xiaohong HAO); zhouzf12@126.com (Zongfu ZHOU)

where $1 < \alpha \le 2$, $0 < \beta\eta^{\alpha-1} < 1$.

In [11], Jiang studied the following boundary value problem

$$D_{0+}^{\alpha}u(t) = f(t, u(t), D_{0+}^{\alpha-1}u(t)), \quad t \in (0,1),$$
$$u(0) = 0,\ D_{0+}^{\alpha-1}u(0) = \sum_{i=1}^{m} a_i D_{0+}^{\alpha-1}u(\xi_i),\ D_{0+}^{\alpha-2}u(1) = \sum_{j=1}^{n} b_j D_{0+}^{\alpha-2}u(\eta_j),$$

where $2 < \alpha \le 3$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_n < 1$, $\sum_{i=1}^{m} a_i = 1$, $\sum_{j=1}^{n} b_j = 1$, $\sum_{j=1}^{n} b_j\eta_j = 1$, $f : [0,1] \times R \times R \to R$ satisfies the Caratheodory condition.

However, no contributions exist, as far as we know, concerning the solvability of the following fractional boundary value problem with $3 < \alpha \le 4$. Motivated by the above works and recent studies on fractional differential equations, we fill the gap.

$$D_{0+}^{\alpha}u(t) = f(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-3}u(t)), \quad t \in (0,1), \tag{1}$$

$$I_{0+}^{4-\alpha}u(0) = 0,\ D_{0+}^{\alpha-1}u(0) = \sum_{i=1}^{m} \alpha_i D_{0+}^{\alpha-1}u(\xi_i),$$
$$D_{0+}^{\alpha-2}u(1) = \sum_{j=1}^{n} \beta_j D_{0+}^{\alpha-2}u(\eta_j),\ D_{0+}^{\alpha-3}u(1) - D_{0+}^{\alpha-3}u(0) = D_{0+}^{\alpha-2}u(\frac{1}{2}), \tag{2}$$

where $3 < \alpha \le 4$ is a real number, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_n < 1$, $\sum_{i=1}^{m} \alpha_i = 1$, $\sum_{j=1}^{n} \beta_j = 1$, $\sum_{j=1}^{n} \beta_j\eta_j = 1$, $f : [0,1] \times R^4 \to R$ satisfies the Caratheodory condition. D_{0+}^{α} and I_{0+}^{α} are the standard Riemann-Liouville fractional differential and integral, respectively.

When $\alpha = 4$, problem (1), (2) is reduced to four-order multi-point boundary value problem, which has been studied by many authors [12–16].

The purpose of this paper is to study the existence of solutions for boundary value problem (1), (2). Our method is based upon Mawhin coincidence degree theory [4].

The outline of the paper is as follows: in Section 2, we give some preliminaries, in Section 3, the existence of solutions for problem (1), (2) are presented. And at the end of this paper, we give an example to illustrate our main result.

Now, we briefly recall some notations and an abstract existence result.

Let Y, Z be real Banach spaces, $L : \mathrm{dom}(L) \subset Y \to Z$ be a Fredholm map of index zero and $P : Y \to Y, Q : Z \to Z$ be continuous projectors such that $\mathrm{Im}(P) = \mathrm{Ker}(L), \mathrm{Ker}(Q) = \mathrm{Im}(L)$ and $Y = \mathrm{Ker}(L) \oplus \mathrm{Ker}(P), Z = \mathrm{Im}(L) \oplus \mathrm{Im}(Q)$. It follows that $L|_{\mathrm{dom}(L)\cap\mathrm{Ker}(P)} : \mathrm{dom}(L) \cap \mathrm{Ker}(P) \to \mathrm{Im}(L)$ is invertible. We denote the inverse of the map by K_p. If Ω is an open bounded subset of Y such that $\mathrm{dom}(L) \cap \Omega \neq \emptyset$, the map $N : Y \to Z$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to Y$ is compact. $J : \mathrm{Im}(Q) \to \mathrm{Ker}(L)$ is the isomorphism.

Theorem 1.1 ([4]) *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\mathrm{dom}(L)\backslash\mathrm{Ker}(L)) \cap \partial\Omega] \times (0,1)$;

(ii) $Nx \notin \mathrm{Im}(L)$ for every $x \in \mathrm{Ker}(L) \cap \partial\Omega$;

(iii) $\deg(JQN|_{\mathrm{Ker}(L)}, \Omega \cap \mathrm{Ker}(L), 0) \neq 0$.

Then the equation $Lx = Nx$ *has at least one solution in* $\mathrm{dom}(L) \cap \overline{\Omega}$.

2. Preliminaries

For convenience, we present here some necessary basic knowledge about fractional calculus theory, which can be found in recent papers [1–3].

Definition 2.1 *The Riemann-Liouville fractional integral* $I_{0+}^{\alpha} y$ *of order* $\alpha (\alpha > 0)$ *is defined by*

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \mathrm{d}s, \quad t > 0,$$

provided the right side is pointwise defined on $(0, +\infty)$.

Definition 2.2 *The Riemann-Liouville fractional differential* $D_{0+}^{\alpha} y$ *of order* $\alpha (\alpha > 0)$ *is defined by*

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{\mathrm{d}}{\mathrm{d}t})^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s,$$

where $n = [\alpha] + 1$.

Definition 2.3 *We say that the map* $f : [0,1] \times R^n \to R$ *satisfies the Caratheodory conditions with respect to* $L^1[0,1]$ *if the following conditions are satisfied:*

(i) for each $z \in R^n$, *the mapping* $t \to f(t,z)$ *is Lebesgue measurable;*

(ii) for almost every $t \in [0,1]$, *the mapping* $z \to f(t,z)$ *is continuous on* R^n*;*

(iii) for each $r > 0$, *there exists* $\rho_r \in L^1([0,1], R)$ *such that, for a.e.* $t \in [0,1]$ *and every* $|z| \leq r$, *we have* $|f(t,z)| \leq \rho_r(t)$.

Lemma 2.1 ([11]) *Assume* $f \in C[0,1]$, $q \geq p \geq 0$, *then*

$$D_{0+}^{p} I_{0+}^{q} f(t) = I_{0+}^{q-p} f(t).$$

Lemma 2.2 ([11]) *Assume* $\alpha > 0$, *then* $D_{0+}^{\alpha} u(t) = 0$ *if and only if*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

for some $c_i \in R$, $i = 1, 2, \ldots, n$, *where* n *is the smallest integer greater than or equal to* α.

Lemma 2.3 ([19,20]) *Let* $\alpha > 0$, $n = [\alpha] + 1$. *Assume that* $u \in L^1(0,1)$ *with a fractional integration of order* $n - \alpha$ *that belongs to* $AC^n[0,1]$. *Then the equality*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) - \sum_{i=1}^{n} \frac{((I_{0+}^{n-\alpha} u)(t))^{(n-i)}|_{t=0}}{\Gamma(\alpha - i + 1)} t^{\alpha-i}$$

holds almost everywhere on $[0,1]$.

Most time, we use the following form:

Assume that $u \in C[0,1] \cap L^1[0,1]$ with a fractional differential of order $\alpha > 0$ that belongs to $C[0,1] \cap L^1[0,1]$. Then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N},$$

for some $c_i \in R$, $i = 1, 2, \ldots, N$, where N is the smallest integer greater than or equal to α.

We use the classical Banach space $C[0,1]$ with the norm $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$. Given $\mu > 0$ and $N = [\mu] + 1$, we can define a linear space

$$C^\mu[0,1] := \{u(t)|u(t) = I_{0+}^\mu x(t) + c_1 t^{\mu-1} + c_2 t^{\mu-2} + \cdots + c_{N-1} t^{\mu-(N-1)},\ t \in [0,1]\},$$

where $x \in C[0,1]$ and $c_i \in R$, $i = 1, 2, \ldots, N-1$. By means of the linear functional analysis theory, we can prove that with the norm $\|u\|_{C^\mu} = \|D_{0+}^\mu u\|_\infty + \cdots + \|D_{0+}^{\mu-(N-1)} u\|_\infty + \|u\|_\infty$, $C^\mu[0,1]$ is a Banach space [10].

Lemma 2.4 ([9]) *$F \subset C^\mu[0,1]$ is a sequentially compact set if and only if F is uniformly bounded and equicontinuous. Here uniformly bounded means there exists $M > 0$, such that for every $u \in F$,*

$$\|u\|_{C^\mu} = \|D_{0+}^\mu u\|_\infty + \cdots + \|D_{0+}^{\mu-(N-1)} u\|_\infty + \|u\|_\infty < M,$$

and equicontinuous means that $\forall \varepsilon > 0$, $\exists \delta > 0$, such that

$$|u(t_1) - u(t_2)| < \varepsilon,\quad \forall t_1, t_2 \in [0,1], |t_1 - t_2| < \delta,\ \forall u \in F,$$

and

$$D_{0+}^{\alpha-i} u(t_1) - D_{0+}^{\alpha-i} u(t_2) < \varepsilon,\quad \forall t_1, t_2 \in [0,1], |t_1 - t_2| < \delta,\ \forall u \in F,\ \forall i \in 0, \ldots, N-1.$$

Let $Z = L^1[0,1]$ with the norm $\|g\|_1 = \int_0^1 |g(s)|\mathrm{d}s$. $Y = C^{\alpha-1}[0,1] = \{u(t)|u(t) = I_{0+}^{\alpha-1} x(t) + c_1 t^{\alpha-2} + c_2 t^{\alpha-3},\ t \in [0,1]\}$, where $x \in C[0,1]$, $c_i \in R$, $i = 1, 2$, with the norm $\|u\|_{C^{\alpha-1}} = \|D_{0+}^{\alpha-1} u\|_\infty + \|D_{0+}^{\alpha-2} u\|_\infty + \|D_{0+}^{\alpha-3} u\|_\infty + \|u\|_\infty$. Then Y is a Banach space. Define L to be the linear operator from $\mathrm{dom}(L) \cap Y$ to Z with

$$\mathrm{dom}(L) = \{C^{\alpha-1}[0,1] | D_{0+}^\alpha u \in L^1[0,1], u \text{ satisfies } (2)\} \tag{3}$$

and

$$Lu = D_{0+}^\alpha u,\quad u \in \mathrm{dom}(L). \tag{4}$$

Define $N : Y \to Z$ by

$$Nu(t) = f(t, u(t), D_{0+}^{\alpha-1} u(t), D_{0+}^{\alpha-2} u(t), D_{0+}^{\alpha-3} u(t)). \tag{5}$$

Then boundary value problem (1), (2) can be written as

$$Lu = Nu. \tag{6}$$

3. Existence results

In order to simplify the calculation process, let

$$A_1 = \sum_{i=1}^m \alpha_i \xi_i^\alpha,\ A_2 = \sum_{i=1}^m \alpha_i \xi_i^{\alpha-1},\ A_3 = \sum_{i=1}^m \alpha_i \xi_i^{\alpha-2},$$

$$B_1 = 1 - \sum_{j=1}^n \beta_j \eta_j^{\alpha+1},\ B_2 = 1 - \sum_{j=1}^n \beta_j \eta_j^\alpha,\ B_3 = 1 - \sum_{j=1}^n \beta_j \eta_j^{\alpha-1},$$

$$\begin{aligned}\Delta_1 =&\frac{\alpha-2}{\alpha-1}\frac{A_2}{A_3}[B_3\frac{\Gamma(\alpha-2)}{\Gamma(\alpha+3)}(1-\frac{1}{2^{\alpha+1}})-B_1\frac{\Gamma(\alpha-2)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha)}{\Gamma(\alpha+2)}(1-\frac{1}{2^{\alpha-1}})]+\\&\frac{\alpha-2}{\alpha}\frac{A_1}{A_3}[B_2\frac{\Gamma(\alpha-2)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)}(1-\frac{1}{2^{\alpha-1}})-B_3\frac{\Gamma(\alpha-2)}{\Gamma(\alpha)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+2)}(1-\frac{1}{2^{\alpha}})]+\\&B_1\frac{\Gamma(\alpha)}{\Gamma(\alpha+2)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+2)}(1-\frac{1}{2^{\alpha}})-B_2\frac{\Gamma(\alpha)}{\Gamma(\alpha+3)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)}(1-\frac{1}{2^{\alpha+1}})\neq 0,\\ \Delta_2 =&\frac{\alpha}{\alpha-2}\frac{A_3}{A_1}[B_2\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)}\frac{\alpha}{\alpha+3}(1-\frac{1}{2^{\alpha+1}})-B_1\frac{\Gamma(\alpha)}{\Gamma(\alpha+2)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+2)}(1-\frac{1}{2^{\alpha}})]+\\&\frac{\alpha}{\alpha-1}\frac{A_2}{A_1}[B_1\frac{\Gamma(\alpha)}{\Gamma(\alpha+2)}\frac{\Gamma(\alpha-2)}{\Gamma(\alpha+1)}(1-\frac{1}{2^{\alpha-1}})-B_1\frac{\Gamma(\alpha-2)}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{\Gamma(\alpha+3)}(1-\frac{1}{2^{\alpha+1}})]+\\&B_3\frac{\Gamma(\alpha-2)}{\Gamma(\alpha)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+2)}(1-\frac{1}{2^{\alpha}})-B_2\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha-2)}{\Gamma(\alpha+1)}(1-\frac{1}{2^{\alpha-1}})\neq 0.\end{aligned}$$

Lemma 3.1 *The mapping $L:\mathrm{dom}(L)\cap Y\to Z$ is a Fredholm operator of index zero.*

Proof By Lemma 2.3, $D_{0+}^{\alpha}u(t)=0$ has solution

$$\begin{aligned}u(t)=&\frac{1}{\Gamma(\alpha)}((I_{0+}^{4-\alpha}u)(t))^{(3)}|_{t=0}t^{\alpha-1}+\frac{1}{\Gamma(\alpha-1)}((I_{0+}^{4-\alpha}u)(t))^{(2)}|_{t=0}t^{\alpha-2}+\\&\frac{1}{\Gamma(\alpha-2)}((I_{0+}^{4-\alpha}u)(t))^{(1)}|_{t=0}t^{\alpha-3}+\frac{1}{\Gamma(\alpha-3)}((I_{0+}^{4-\alpha}u)(t))|_{t=0}t^{\alpha-4}.\end{aligned}$$

Combining with (2) gives

$$\mathrm{Ker}(L)=\{at^{\alpha-1}+bt^{\alpha-2}+ct^{\alpha-3}|a,b,c\in R\}\cong R^3.$$

Let $g\in\mathrm{Im}(L)$. Then there exists $u\in\mathrm{dom}(L)$ s.t. $g=D_{0+}^{\alpha}u$. From Lemma 2.3, we have

$$u(t)=I_{0+}^{\alpha}g(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}+c_4t^{\alpha-4},$$

which, due to the boundary value conditions (2), implies that g satisfies

$$\sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}g(s)\mathrm{d}s=0, \tag{7}$$

$$\int_0^1(1-s)g(s)\mathrm{d}s-\sum_{j=1}^{n}\beta_j\int_0^{\eta_j}(\eta_j-s)g(s)\mathrm{d}s=0, \tag{8}$$

$$\frac{1}{2}\int_0^1(1-s)^2g(s)\mathrm{d}s-\int_0^{\frac{1}{2}}(\frac{1}{2}-s)g(s)\mathrm{d}s=0. \tag{9}$$

Hence

$$\mathrm{Im}(L)\subseteq\{g\in Z|g\text{ satisfies (7), (8) and (9)}\}.$$

Let $g\in Z$ and

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s)\mathrm{d}s+c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}=I_{0+}^{\alpha}g(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}.$$

Then $D_{0+}^{\alpha}u(t)=g(t)$ a.e. $t\in(0,1)$ and if (7)–(9) hold, then $u(t)$ satisfies the boundary conditions (2). That is, $u\in\mathrm{dom}(L)$, then we have

$$\{g\in Z|g\text{ satisfies (7), (8) and (9)}\}\subseteq\mathrm{Im}(L).$$

Therefore,

$$\mathrm{Im}(L)=\{g\in Z|g \text{ satisfies } (7), (8) \text{ and } (9)\}.$$

Define the following continuous linear mapping $Q_1:Z\to Z$, $Q_2:Z\to Z$ and $Q_3:Z\to Z$

$$Q_1g=\sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}g(s)\mathrm{d}s, \tag{10}$$

$$Q_2g=\int_0^1(1-s)g(s)\mathrm{d}s-\sum_{j=1}^{n}\beta_j\int_0^{\eta_j}(\eta_j-s)g(s)\mathrm{d}s, \tag{11}$$

$$Q_3g=\frac{1}{2}\int_0^1(1-s)^2g(s)\mathrm{d}s-\int_0^{\frac{1}{2}}(\frac{1}{2}-s)g(s)\mathrm{d}s. \tag{12}$$

Using the above definitions, we construct three auxiliary maps $R_1:Z\to Z$, $R_2:Z\to Z$ and $R_3:Z\to Z$

$$\begin{aligned}R_1g=&\frac{1}{\Delta_1}\{\frac{\alpha-2}{A_3}[B_2\frac{\Gamma(\alpha-2)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)}(1-\frac{1}{2^{\alpha-1}})-B_3\frac{\Gamma(\alpha-2)}{\Gamma(\alpha)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+2)}(1-\frac{1}{2^{\alpha}})]Q_1g-\\&[\frac{\alpha-2}{\alpha-1}\frac{A_2}{A_3}\frac{\Gamma(\alpha-2)}{\Gamma(\alpha+1)}(1-\frac{1}{2^{\alpha-1}})-\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+2)}(1-\frac{1}{2^{\alpha}})]Q_2g+\\&[B_3\frac{\alpha-2}{\alpha-1}\frac{A_2}{A_3}\frac{\Gamma(\alpha-2)}{\Gamma(\alpha)}-B_2\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)}]Q_3g\},\end{aligned} \tag{13}$$

$$\begin{aligned}R_2g=&\frac{1}{-\Delta_1}\{\frac{\alpha-2}{A_3}[B_1\frac{\Gamma(\alpha-2)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha)}{\Gamma(\alpha+2)}(1-\frac{1}{2^{\alpha-1}})-B_3\frac{\Gamma(\alpha-2)}{\Gamma(\alpha+3)}(1-\frac{1}{2^{\alpha+1}})]Q_1g-\\&[\frac{\alpha-2}{\alpha}\frac{A_1}{A_3}\frac{\Gamma(\alpha-2)}{\Gamma(\alpha+1)}(1-\frac{1}{2^{\alpha-1}})-\frac{\Gamma(\alpha)}{\Gamma(\alpha+3)}(1-\frac{1}{2^{\alpha+1}})]Q_2g+\\&[B_3\frac{\alpha-2}{\alpha}\frac{A_1}{A_3}\frac{\Gamma(\alpha-2)}{\Gamma(\alpha)}-B_1\frac{\Gamma(\alpha)}{\Gamma(\alpha+2)}]Q_3g\},\end{aligned} \tag{14}$$

$$\begin{aligned}R_3g=&\frac{1}{\Delta_2}\{\frac{\alpha}{A_1}[B_2\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha)}{\Gamma(\alpha+3)}(1-\frac{1}{2^{\alpha+1}})-B_1\frac{\Gamma(\alpha)}{\Gamma(\alpha+2)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+2)}(1-\frac{1}{2^{\alpha}})]Q_1g-\\&[\frac{\alpha}{\alpha-1}\frac{A_2}{A_1}\frac{\Gamma(\alpha)}{\Gamma(\alpha+3)}(1-\frac{1}{2^{\alpha+1}})-\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+2)}(1-\frac{1}{2^{\alpha}})]Q_2g+\\&[B_1\frac{\alpha}{\alpha-1}\frac{A_2}{A_1}\frac{\Gamma(\alpha)}{\Gamma(\alpha+2)}-B_2\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)}]Q_3g\}.\end{aligned} \tag{15}$$

Consider continuous linear mapping $Q:Z\to Z$ defined by

$$Qg=(R_1g)t^{\alpha-1}+(R_2g)t^{\alpha-2}+(R_3g)t^{\alpha-3}. \tag{16}$$

It is well-defined. Recall $\Delta_1\neq 0$ and $\Delta_2\neq 0$, and note that

$$\begin{aligned}R_1(R_1gt^{\alpha-1})=&\frac{1}{\Delta_1}\{\frac{\alpha-2}{A_3}[B_2\frac{\Gamma(\alpha-2)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)}(1-\frac{1}{2^{\alpha-1}})-B_3\frac{\Gamma(\alpha-2)}{\Gamma(\alpha)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+2)}(1-\frac{1}{2^{\alpha}})]\cdot\\&Q_1(R_1gt^{\alpha-1})-[\frac{\alpha-2}{\alpha-1}\frac{A_2}{A_3}\frac{\Gamma(\alpha-2)}{\Gamma(\alpha+1)}(1-\frac{1}{2^{\alpha-1}})-\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+2)}(1-\frac{1}{2^{\alpha}})]\cdot\\&Q_2(R_1gt^{\alpha-1})+[B_3\frac{\alpha-2}{\alpha-1}\frac{A_2}{A_3}\frac{\Gamma(\alpha-2)}{\Gamma(\alpha)}-B_2\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)}]Q_3(R_1gt^{\alpha-1})\}\\=&R_1g\frac{1}{\Delta_1}\{\frac{\alpha-2}{A_3}[B_2\frac{\Gamma(\alpha-2)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)}(1-\frac{1}{2^{\alpha-1}})-B_3\frac{\Gamma(\alpha-2)}{\Gamma(\alpha)}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+2)}\cdot\end{aligned}$$

$$
\begin{aligned}
&(1-\frac{1}{2^{\alpha}})]\frac{1}{\alpha}A_1-[\frac{\alpha-2}{\alpha-1}\frac{A_2}{A_3}\frac{\Gamma(\alpha-2)}{\Gamma(\alpha+1)}(1-\frac{1}{2^{\alpha-1}})-\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+2)}(1-\frac{1}{2^{\alpha}})]\cdot\\
&\frac{\Gamma(\alpha)}{\Gamma(\alpha+2)}B_1+[B_3\frac{\alpha-2}{\alpha-1}\frac{A_2}{A_3}\frac{\Gamma(\alpha-2)}{\Gamma(\alpha)}-B_2\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+1)}]\frac{\Gamma(\alpha)}{\Gamma(\alpha+3)}(1-\frac{1}{2^{\alpha+1}})\}\\
=&R_1g\frac{\Delta_1}{\Delta_1}=R_1g,
\end{aligned}
$$

and similarly we can derive that

$$
\begin{aligned}
&R_1(R_2gt^{\alpha-2})=0,\ R_1(R_3gt^{\alpha-3})=0,\ R_2(R_1gt^{\alpha-1})=0,\ R_2(R_2gt^{\alpha-2})=R_2g,\\
&R_2(R_3gt^{\alpha-3})=0,\ R_3(R_1gt^{\alpha-1})=0,\ R_3(R_2gt^{\alpha-2})=0,\ R_3(R_3gt^{\alpha-3})=R_3g.
\end{aligned}
$$

Therefore, for $g\in Z$, it follows from the nine relations above that

$$
\begin{aligned}
Q^2g=&R_1(R_1gt^{\alpha-1}+R_2gt^{\alpha-2}+R_3gt^{\alpha-3})t^{\alpha-1}+R_2(R_1gt^{\alpha-1}+R_2gt^{\alpha-2}+R_3gt^{\alpha-3})t^{\alpha-2}+\\
&R_3(R_1gt^{\alpha-1}+R_2gt^{\alpha-2}+R_3gt^{\alpha-3})t^{\alpha-3}\\
=&(R_1g)t^{\alpha-1}+(R_2g)t^{\alpha-2}+(R_3g)t^{\alpha-3}=Qg.
\end{aligned}
\tag{17}
$$

That is, the map Q is a continuous linear projector.

Note that $g\in \operatorname{Im}(L)$ implies $Qg=0$. Conversely, if $Qg=0$, then we must have $R_1g=R_2g=R_3g=0$; this can only be the case if $Q_1g=Q_2g=Q_3g=0$, that is, $g\in\operatorname{Im}(L)$, in fact $\operatorname{Im}(L)=\operatorname{Ker}(Q)$.

Take $g\in Z$ in the form $g=(g-Qg)+Qg$, so that $g-Qg\in\operatorname{Im}(L)$ and $Qg\in\operatorname{Im}(Q)$. Thus, $Z=\operatorname{Im}(L)+\operatorname{Im}(Q)$. Let $g\in\operatorname{Im}(L)\cap\operatorname{Im}(Q)$ and assume that $g(s)=as^{\alpha-1}+bs^{\alpha-2}+cs^{\alpha-3}$ is not identically zero on $[0,1]$. Then, since $g\in Im(L)$, from (7),(8),(9) and the condition $\Delta_1\neq 0$ and $\Delta_2\neq 0$, we derive $a=b=c=0$, which is a contradiction. Hence, $\operatorname{Im}(L)\cap\operatorname{Im}(Q)=\{0\}$; thus $Z=\operatorname{Im}(L)\oplus\operatorname{Im}(Q)$.

Now, $\dim\operatorname{Ker}(L)=3=\operatorname{codim}\operatorname{Im}(L)$ and so L is a Fredholm operator of index zero.

Let $P:Y\to Y$ be defined by

$$
Pu(t)=\frac{1}{\Gamma(\alpha)}D_{0+}^{\alpha-1}u(0)t^{\alpha-1}+\frac{1}{\Gamma(\alpha-1)}D_{0+}^{\alpha-2}u(0)t^{\alpha-2}+\frac{1}{\Gamma(\alpha-2)}D_{0+}^{\alpha-3}u(0)t^{\alpha-3}.
$$

Note that P is a continuous linear projector and

$$
\operatorname{Ker}(P)=\{u\in Y|D_{0+}^{\alpha-1}u(0)=D_{0+}^{\alpha-2}u(0)=D_{0+}^{\alpha-3}u(0)=0\}.
$$

It is clear that $Y=\operatorname{Ker}(L)\oplus\operatorname{Ker}(P)$.

Note that the projectors P and Q are exact. Define $K_p:\operatorname{Im}(L)\to\operatorname{dom}(L)\cap\operatorname{Ker}(P)$ by

$$
K_pg(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s)\mathrm{d}s=I_{0+}^{\alpha}g(t).
$$

Then

$$
\begin{aligned}
&\|K_pg\|_\infty\le\frac{1}{\Gamma(\alpha)}\|g\|_1,\ \|D_{0+}^{\alpha-1}(K_pg)\|_\infty\le\|g\|_1,\\
&\|D_{0+}^{\alpha-2}(K_pg)\|_\infty\le\|g\|_1,\ \|D_{0+}^{\alpha-3}(K_pg)\|_\infty\le\frac{1}{2}\|g\|_1.
\end{aligned}
$$

Hence

$$\begin{aligned}\|K_pg\|_{C^{\alpha-1}} &= \|K_pg\|_\infty + \|D_{0+}^{\alpha-1}(K_pg)\|_\infty + \|D_{0+}^{\alpha-2}(K_pg)\|_\infty + \|D_{0+}^{\alpha-3}(K_pg)\|_\infty \\ &\le (\frac{5}{2} + \frac{1}{\Gamma(\alpha)})\|g\|_1. \end{aligned} \tag{18}$$

So $K_pg \in C^{\alpha-1}[0,1]$. It is clear that $K_pg \in \mathrm{dom}(L)$ and $K_pg \in \mathrm{Ker}(P)$. Therefore, $K_p(\mathrm{Im}(L)) \subset \mathrm{dom}(L) \cap \mathrm{Ker}(P)$. And if $g \in \mathrm{Im}(L)$, then $(LK_p)g = D_{0+}^\alpha I_{0+}^\alpha g = g$. If $u \in \mathrm{dom}(L) \cap \mathrm{Ker}(P)$,

$$(K_pL)u(t) = I_{0+}^\alpha D_{0+}^\alpha u(t) = u(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + c_3t^{\alpha-3} + c_4t^{\alpha-4},$$

from the boundary value conditions (2) and the fact that $u \in \mathrm{dom}(L) \cap \mathrm{Ker}(P)$, we have $c_1 = c_2 = c_3 = c_4 = 0$. Thus

$$K_p = (L|_{\mathrm{dom}(L)\cap\mathrm{Ker}(P)})^{-1}. \tag{19}$$

Lemma 3.2 *Assume $\Omega \subset X$ is an open bounded subset and* $\mathrm{dom}(L) \cap \overline{\Omega} \neq \emptyset$, *then N is L-compact on $\overline{\Omega}$.*

Proof By Definition 2.3, we can get $QN(\overline{\Omega})$ is bounded. Now we show that $K_p(I-Q)N : \overline{\Omega} \to X$ is compact. $\Omega \subset X$ is bounded, i.e., there exists a positive constant $N > 0$, s.t $\|u\|_\infty \le N$ for all $u \in \Omega$. Denote

$$M = \max_{t\in[0,1]} \|f(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-3}u(t)) - Qf(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-3}u(t))\|_1.$$

For $u \in \Omega$,

$$\begin{aligned}\|K_p(I-Q)Nu\|_\infty =&\|I_{0+}^\alpha[f(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-3}u(t)) - \\ &Qf(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-3}u(t))]\|_\infty \\ \le&\frac{1}{\Gamma(\alpha)}\|f(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-3}u(t)) - Qf(t, u(t), D_{0+}^{\alpha-1}u(t), \\ &D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-3}u(t))\|_1 \le \frac{1}{\Gamma(\alpha)}M, \\ \|D_{0+}^{\alpha-1}K_p(I-Q)Nu\|_\infty =&\|I_{0+}^1[f(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-3}u(t)) - Qf(t, u(t), \\ &D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-3}u(t))]\|_\infty \\ \le&\|f(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-3}u(t)) - Qf(t, u(t), D_{0+}^{\alpha-1}u(t), \\ &D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-3}u(t))\|_1 \le M.\end{aligned}$$

Hence $K_p(I-Q)N(\overline{\Omega}) \subset X$ is bounded.

It follows from the Lebesgue dominated convergence theorem that $K_p(I-Q)N : \overline{\Omega} \to X$ is continuous. For $0 \le t_1 \le t_2 \le 1$, $u \in \overline{\Omega}$, we have

$$\begin{aligned}&|K_p(I-Q)Nu(t_2) - K_p(I-Q)Nu(t_1)| \\ &= \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}(I-Q)Nu(s)\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}(I-Q)Nu(s)\mathrm{d}s\Big| \\ &\le \frac{M}{\Gamma(\alpha)}\Big[\int_0^{t_1}(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\mathrm{d}s + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathrm{d}s\Big]\end{aligned}$$

$$= \frac{M}{\alpha\Gamma(\alpha)}(t_2^\alpha - t_1^\alpha)$$

and

$$\begin{aligned}&|D_{0+}^{\alpha-1}K_p(I-Q)Nu(t_2) - D_{0+}^{\alpha-1}K_p(I-Q)Nu(t_1)| \\ &= \Big|\int_0^{t_2}(I-Q)Nu(s)\mathrm{d}s - \int_0^{t_1}(I-Q)Nu(s)\mathrm{d}s\Big| \le M(t_2-t_1).\end{aligned}$$

Since t^α and t are uniformly continuous on $[0,1]$, we can get that $K_p(I-Q)N(\overline{\Omega})$ and $D_{0+}^{\alpha-1}K_p(I-Q)N(\overline{\Omega})$ are equicontinuous. By the Ascoli-Arzela theorem, $K_p(I-Q)N : \overline{\Omega} \to X$ is compact. Then the map $N : X \to Y$ is L-compact on $\overline{\Omega}$.

Theorem 3.1 *Let $f : [0,1] \times R^4 \to R$ be continuous and assume the following conditions are satisfied:*

(A_1) For all $(x, y, z, s) \in R^4$ and a.e. $t \in [0,1]$, there exist functions $a, b, c, d, e, f, g, h \in L^1[0,1]$ and constants $\theta, \varphi, \tau \in [0,1)$ such that one of the following inequalities is satisfied:

$$|f(t,x,y,z,s)| \le h(t) + a(t)|x| + b(t)|y| + c(t)|z| + d(t)|s| + e(t)|y|^\theta + f(t)|z|^\varphi + g(t)|s|^\tau,$$
$$|f(t,x,y,z,s)| \le h(t) + a(t)|x| + b(t)|y| + c(t)|z| + d(t)|s| + e(t)|z|^\theta + f(t)|s|^\varphi + g(t)|x|^\tau,$$
$$|f(t,x,y,z,s)| \le h(t) + a(t)|x| + b(t)|y| + c(t)|z| + d(t)|s| + e(t)|s|^\theta + f(t)|x|^\varphi + g(t)|y|^\tau,$$
$$|f(t,x,y,z,s)| \le h(t) + a(t)|x| + b(t)|y| + c(t)|z| + d(t)|s| + e(t)|x|^\theta + f(t)|y|^\varphi + g(t)|z|^\tau.$$

(A_2) There exists a constant $A > 0$ such that for $u \in \mathrm{dom}(L)\backslash\mathrm{Ker}(L)$ satisfying $|D_{0+}^{\alpha-1}u(t) + D_{0+}^{\alpha-2}u(t) + D_{0+}^{\alpha-3}u(t)| > A$ for all $t \in [0,1]$, then we have

$$Q_1Nu(t) \ne 0 \text{ or } Q_2Nu(t) \ne 0 \text{ or } Q_3Nu(t) \ne 0.$$

(A_3) There exists a constant $B > 0$ such that for every $l, m, n \in R$ satisfying $l^2 + m^2 + n^2 > B$, then

$$lR_1N(lt^{\alpha-1}+mt^{\alpha-2}+nt^{\alpha-3})+mR_2N(lt^{\alpha-1}+mt^{\alpha-2}+nt^{\alpha-3})+nR_3N(lt^{\alpha-1}+mt^{\alpha-2}+nt^{\alpha-3}) > 0$$

or

$$lR_1N(lt^{\alpha-1}+mt^{\alpha-2}+nt^{\alpha-3})+mR_2N(lt^{\alpha-1}+mt^{\alpha-2}+nt^{\alpha-3})+nR_3N(lt^{\alpha-1}+mt^{\alpha-2}+nt^{\alpha-3}) < 0.$$

Then, the boundary value problem (1), (2) has at least one solution in $C^{\alpha-1}[0,1]$ provided that

$$\|a\|_1 + \|b\|_1 + \|c\|_1 + \|d\|_1 < \frac{1}{\Lambda}.$$

Proof Set

$$\Omega_1 = \{u \in \mathrm{dom}(L) \setminus \mathrm{Ker}(L) | Lu = \lambda Nu \text{ for some } \lambda \in [0,1]\}.$$

Then for $u \in \Omega_1$, $Lu = \lambda Nu$, thus $\lambda \ne 0$, $Nu \in \mathrm{Im}(L)$, hence $QNu = 0$ for all $t \in [0,1]$. By the definition of Q, we have $Q_1Nu(t) = Q_2Nu(t) = Q_3Nu(t) = 0$. Then it follows from ($A_2$) that there exists $t_0 \in [0,1]$ s.t.

$$|D_{0+}^{\alpha-1}u(t_0) + D_{0+}^{\alpha-2}u(t_0) + D_{0+}^{\alpha-3}u(t_0)| \le A.$$

Note that

$$D_{0+}^{\alpha-1}u(t)=D_{0+}^{\alpha-1}u(t_0)+\int_{t_0}^{t}D_{0+}^{\alpha}u(s)\mathrm{d}s,\ D_{0+}^{\alpha-2}u(t)=D_{0+}^{\alpha-2}u(t_0)+\int_{t_0}^{t}D_{0+}^{\alpha-1}u(s)\mathrm{d}s,$$

$$D_{0+}^{\alpha-3}u(t)=D_{0+}^{\alpha-3}u(t_0)+\int_{t_0}^{t}D_{0+}^{\alpha-2}u(s)\mathrm{d}s,$$

and then

$$\begin{aligned}|D_{0+}^{\alpha-1}u(0)|&\le\|D_{0+}^{\alpha-1}u(t)\|_\infty\le|D_{0+}^{\alpha-1}u(t_0)|+\|D_{0+}^{\alpha}u(t)\|_1\le A+\|Lu\|_1\le A+\|Nu\|_1,\\ |D_{0+}^{\alpha-2}u(0)|&\le\|D_{0+}^{\alpha-2}u(t)\|_\infty\le|D_{0+}^{\alpha-2}u(t_0)|+|D_{0+}^{\alpha-1}u(t_0)|+\|D_{0+}^{\alpha}u(t)\|_1\\ &\le A+\|Lu\|_1\le A+\|Nu\|_1,\\ |D_{0+}^{\alpha-3}u(0)|&\le\|D_{0+}^{\alpha-3}u(t)\|_\infty\le|D_{0+}^{\alpha-3}u(t_0)|+|D_{0+}^{\alpha-2}u(t_0)|+|D_{0+}^{\alpha-1}u(t_0)|+\|D_{0+}^{\alpha}u(t)\|_1\\ &\le A+\|Nu\|_1.\end{aligned}$$

By the above three inequalities, we have

$$\begin{aligned}\|Pu\|_{C^{\alpha-1}}=&\|\frac{1}{\Gamma(\alpha)}D_{0+}^{\alpha-1}u(0)t^{\alpha-1}+\frac{1}{\Gamma(\alpha-1)}D_{0+}^{\alpha-2}u(0)t^{\alpha-2}+\frac{1}{\Gamma(\alpha-2)}D_{0+}^{\alpha-3}u(0)t^{\alpha-3}\|_\infty+\\ &\|D_{0+}^{\alpha-1}u(0)\|_\infty+\|D_{0+}^{\alpha-1}u(0)t+D_{0+}^{\alpha-2}u(0)\|_\infty+\\ &\|\frac{1}{2}D_{0+}^{\alpha-1}u(0)t^2+D_{0+}^{\alpha-2}u(0)t+D_{0+}^{\alpha-3}u(0)\|_\infty\\ \le&(\frac{5}{2}+\frac{1}{\Gamma(\alpha)})|D_{0+}^{\alpha-1}u(0)|+(2+\frac{1}{\Gamma(\alpha-1)})|D_{0+}^{\alpha-2}u(0)|+(1+\frac{1}{\Gamma(\alpha-2)})|D_{0+}^{\alpha-3}u(0)|\\ \le&(\frac{5}{2}+\frac{1}{\Gamma(\alpha)})(A+\|Nu\|_1)+(2+\frac{1}{\Gamma(\alpha-1)})(A+\|Nu\|_1)+\\ &(1+\frac{1}{\Gamma(\alpha-2)})(A+\|Nu\|_1).\end{aligned}\tag{20}$$

Note that $(I-P)u\in\mathrm{Im}(K_p)=\mathrm{dom}(L)\cap\mathrm{Ker}(P)$ for $u\in\Omega_1$. Then by (18) and (19) we have

$$\begin{aligned}\|(I-P)u\|_{C^{\alpha-1}}&=\|K_pL(I-P)u\|_{C^{\alpha-1}}\le(\frac{5}{2}+\frac{1}{\Gamma(\alpha)})\|L(I-P)u\|_1\\ &=(\frac{5}{2}+\frac{1}{\Gamma(\alpha)})\|Lu\|_1\le(\frac{5}{2}+\frac{1}{\Gamma(\alpha)})\|Nu\|_1.\end{aligned}\tag{21}$$

Combining (20) and (21) gives

$$\begin{aligned}\|u\|_{C^{\alpha-1}}&\le\|Pu\|_{C^{\alpha-1}}+\|(I-P)u\|_{C^{\alpha-1}}\\ &\le(8+\frac{2}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}+\frac{1}{\Gamma(\alpha-2)})\|Nu\|_1+(\frac{11}{2}+\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}+\frac{1}{\Gamma(\alpha-2)})A\\ &=\Lambda\|Nu\|_1+D,\end{aligned}$$

where $D=(\frac{11}{2}+\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}+\frac{1}{\Gamma(\alpha-2)})A$ is a constant. That is, for all $u\in\Omega_1$,

$$\|u\|_{C^{\alpha-1}}\le\Lambda\|Nu\|_1+D.$$

If the first condition of (A_1) is satisfied, then we have

$$\max(\|u\|_\infty,\|D_{0+}^{\alpha-1}u\|_\infty,\|D_{0+}^{\alpha-2}u\|_\infty,D_{0+}^{\alpha-3}u\|_\infty)$$

$$\leq \|u\|_{C^{\alpha-1}} \leq \Lambda(\|h\|_1 + \|a\|_1\|u\|_\infty + \|b\|_1\|D_{0+}^{\alpha-1}u\|_\infty + \|c\|_1\|D_{0+}^{\alpha-2}u\|_\infty + \|d\|_1\|D_{0+}^{\alpha-3}u\|_\infty + \|e\|_1\|D_{0+}^{\alpha-1}u\|_\infty^\theta + \|f\|_1\|D_{0+}^{\alpha-2}u\|_\infty^\varphi + \|g\|_1\|D_{0+}^{\alpha-3}u\|_\infty^\tau) + D,$$

and consequently, we have

$$\begin{aligned}
&\|u\|_\infty \\
&\leq \frac{\Lambda}{1-\|a\|_1\Lambda}(\|h\|_1 + \|b\|_1\|D_{0+}^{\alpha-1}u\|_\infty + \|c\|_1\|D_{0+}^{\alpha-2}u\|_\infty + \|d\|_1\|D_{0+}^{\alpha-3}u\|_\infty + \|e\|_1\|D_{0+}^{\alpha-1}u\|_\infty^\theta \\
&\quad \|f\|_1\|D_{0+}^{\alpha-2}u\|_\infty^\varphi + \|g\|_1\|D_{0+}^{\alpha-3}u\|_\infty^\tau) + \frac{D}{1-\|a\|_1\Lambda}, \\
&\|D_{0+}^{\alpha-1}u\|_\infty \\
&\leq \frac{\Lambda\|e\|_1\|D_{0+}^{\alpha-1}u\|_\infty^\theta}{1-\|a\|_1\Lambda-\|b\|_1\Lambda} + \frac{\Lambda}{1-\|a\|_1\Lambda-\|b\|_1\Lambda}(\|h\|_1 + \|c\|_1\|D_{0+}^{\alpha-2}u\|_\infty + \|d\|_1\|D_{0+}^{\alpha-3}u\|_\infty + \\
&\quad \|f\|_1\|D_{0+}^{\alpha-2}u\|_\infty^\varphi + \|g\|_1\|D_{0+}^{\alpha-3}u\|_\infty^\tau) + \frac{D}{1-\|a\|_1\Lambda-\|b\|_1\Lambda}, \\
&\|D_{0+}^{\alpha-2}u\|_\infty \\
&\leq \frac{\Lambda\|f\|_1\|D_{0+}^{\alpha-2}u\|_\infty^\varphi}{1-\|a\|_1\Lambda-\|b\|_1\Lambda-\|c\|_1\Lambda} + \frac{\Lambda}{1-\|a\|_1\Lambda-\|b\|_1\Lambda-\|c\|_1\Lambda}(\|h\|_1 + \|e\|_1\|D_{0+}^{\alpha-1}u\|_\infty^\theta + \\
&\quad \|d\|_1\|D_{0+}^{\alpha-3}u\|_\infty + \|g\|_1\|D_{0+}^{\alpha-3}u\|_\infty^\tau) + \frac{D}{1-\|a\|_1\Lambda-\|b\|_1\Lambda-\|c\|_1\Lambda}, \\
&\|D_{0+}^{\alpha-3}u\|_\infty \\
&\leq \frac{\Lambda\|g\|_1\|D_{0+}^{\alpha-3}u\|_\infty^\tau}{1-\|a\|_1\Lambda-\|b\|_1\Lambda-\|c\|_1\Lambda-\|d\|_1\Lambda} + \frac{\Lambda}{1-\|a\|_1\Lambda-\|b\|_1\Lambda-\|c\|_1\Lambda-\|d\|_1\Lambda}(\|h\|_1 + \\
&\quad \|e\|_1\|D_{0+}^{\alpha-1}u\|_\infty^\theta + \|f\|_1\|D_{0+}^{\alpha-2}u\|_\infty^\varphi + \frac{D}{1-\|a\|_1\Lambda-\|b\|_1\Lambda-\|c\|_1\Lambda-\|d\|_1\Lambda}.
\end{aligned}$$

As $\theta, \varphi, \tau \in [0,1)$ and $\|a\|_1 + \|b\|_1 + \|c\|_1 + \|d\|_1 < \frac{1}{\Lambda}$ holds, then there exist $M_1, M_2, M_3, M_4 > 0$ such that for all $u \in \Omega_1$ $\|u\|_\infty \leq M_1, \|D_{0+}^{\alpha-1}u\|_\infty \leq M_2, \|D_{0+}^{\alpha-2}u\|_\infty \leq M_3, \|D_{0+}^{\alpha-3}u\|_\infty \leq M_4$. Therefore, for all $u \in \Omega_1$,

$$\|u\|_{C^{\alpha-1}} = \|u\|_\infty + \|D_{0+}^{\alpha-1}u\|_\infty + \|D_{0+}^{\alpha-2}u\|_\infty + \|D_{0+}^{\alpha-3}u\|_\infty \leq M_1 + M_2 + M_3 + M_4.$$

So Ω_1 is bounded given the first condition of (A_1).

If the other conditions of (A_1) hold, similarly to the above, we can prove that Ω_1 is also bounded.

Let

$$\Omega_2 = \{u \in \mathrm{Ker}(L) | Nu \in \mathrm{Im}(L)\}.$$

For $u \in \Omega_2$, $u \in \mathrm{Ker}(L) = \{lt^{\alpha-1} + mt^{\alpha-2} + nt^{\alpha-3} | l, m, n \in R, t \in [0,1]\}$, from $Nu \in \mathrm{Im}(L)$ and $\mathrm{Im}(L) = \mathrm{Ker}(Q)$, we have $QN(lt^{\alpha-1} + mt^{\alpha-2} + nt^{\alpha-3}) = 0$. Thus

$$R_1N(lt^{\alpha-1}+mt^{\alpha-2}+nt^{\alpha-3}) = R_2N(lt^{\alpha-1}+mt^{\alpha-2}+nt^{\alpha-3}) = R_3N(lt^{\alpha-1}+mt^{\alpha-2}+nt^{\alpha-3}) = 0.$$

By (A_3), we have $l^2 + m^2 + n^2 \leq B$. Therefore, Ω_2 is bounded.

We define the isomorphism $J : \mathrm{Im}(Q) \to \mathrm{Ker}(L)$ by

$$J(lt^{\alpha-1} + mt^{\alpha-2} + nt^{\alpha-3}) = lt^{\alpha-1} + mt^{\alpha-2} + nt^{\alpha-3}, \quad l, m, n \in R.$$

If the first part of (A_3) is satisfied, let

$$\Omega_3 = \{u \in \mathrm{Ker}(L) | \lambda J^{-1}u + (1-\lambda)QNu = 0, \lambda \in [0,1]\}.$$

For every $lt^{\alpha-1} + mt^{\alpha-2} + nt^{\alpha-3} \in \Omega_3$,

$$\begin{aligned}
&\lambda(lt^{\alpha-1} + mt^{\alpha-2} + nt^{\alpha-3}) \\
&\quad = -(1-\lambda)[R_1N(lt^{\alpha-1} + mt^{\alpha-2} + nt^{\alpha-3})t^{\alpha-1} + R_2N(lt^{\alpha-1} + mt^{\alpha-2} + nt^{\alpha-3})t^{\alpha-2} + \\
&\qquad R_3N(lt^{\alpha-1} + mt^{\alpha-2} + nt^{\alpha-3})t^{\alpha-3}],
\end{aligned}$$

if $\lambda = 1$, then $l = m = n = 0$; if $l^2 + m^2 + n^2 > B$, then by (A_3),

$$\begin{aligned}
\lambda(l^2 + m^2 + n^2) = &-(1-\lambda)[lR_1N(lt^{\alpha-1} + mt^{\alpha-2} + nt^{\alpha-3}) + mR_2N(lt^{\alpha-1} + mt^{\alpha-2} + nt^{\alpha-3}) + \\
&nR_3N(lt^{\alpha-1} + mt^{\alpha-2} + nt^{\alpha-3})] > 0,
\end{aligned}$$

which, in either case, is a contradiction. Thus, for all $u \in \Omega_3$,

$$\begin{aligned}
\|u\|_{C^{\alpha-1}} = &\|lt^{\alpha-1} + mt^{\alpha-2} + nt^{\alpha-3}\|_\infty + \|l\Gamma(\alpha)\|_\infty + \|l\Gamma(\alpha)t + m\Gamma(\alpha-1)\|_\infty + \\
&\|\frac{1}{2}l\Gamma(\alpha)t^2 + m\Gamma(\alpha-1)t + n\Gamma(\alpha-2)\|_\infty + \\
\le &(1 + \frac{5}{2}\Gamma(\alpha))|l| + (1 + 2\Gamma(\alpha-1))|m| + (1 + \Gamma(\alpha-2))|n| \\
\le &(3 + \frac{5}{2}\Gamma(\alpha) + 2\Gamma(\alpha-1) + \Gamma(\alpha-2))B.
\end{aligned}$$

So Ω_3 is bounded.

Similarly, if the second part of (A_3) is satisfied, let

$$\Omega_3 = \{u \in \mathrm{Ker}(L) | -\lambda J^{-1}u + (1-\lambda)QNu = 0,\ \lambda \in [0,1]\},$$

where J is as above. Similarly to above arguments, we can show that Ω_3 is bounded too.

Note that $\Omega_1, \Omega_2, \Omega_3$ are all bounded. So there exist $H_i > 0$, such that for all $u \in \Omega_i$, $\|u\|_{C^{\alpha-1}} \le H_i$, $i = 1,2,3$. Let

$$H = \max\{H_1, H_2, H_3\}$$

and

$$\Omega = \{u | u \in Y, \|u\|_{C^{\alpha-1}} < H\}.$$

In the following, we shall prove that the conditions of Theorem 1.1 are satisfied. Ω is a bounded open set of Y defined as above. By the above argument, we have

(i) $Lx \ne \lambda Nx$ for every $(x,\lambda) \in [(\mathrm{dom}(L)\backslash \mathrm{Ker}(L)) \cap \partial\Omega] \times (0,1)$;

(ii) $Nx \notin \mathrm{Im}(L)$ for every $x \in \mathrm{Ker}(L) \cap \partial\Omega$;

Finally, we will prove that (iii) of Theorem 1.1 is satisfied.

Let

$$H(u,\lambda) = \pm\lambda\, \mathrm{id}\, u + (1-\lambda)JQNu,$$

where id is the identity operator in the Banach space Y. According to the above argument, we know that

$$H(u,\lambda) \ne 0,\ \text{for all } u \in \mathrm{Ker}(L) \cap \partial\Omega,$$

and thus, by the homotopy property of degree,

$$\begin{aligned}\deg(JQN|_{\mathrm{Ker}(L)}, \Omega \cap \mathrm{Ker}(L), 0) =& \deg(H(\cdot, 0), \Omega \cap \mathrm{Ker}(L), 0) = \deg(H(\cdot, 1), \Omega \cap \mathrm{Ker}(L), 0)\\ =& \deg(\pm \mathrm{id}, \Omega \cap \mathrm{Ker}(L), 0) = \pm 1 \neq 0,\end{aligned}$$

Then by Theorem 1.1, $Lu = Nu$ has at least one solution in $\mathrm{dom}(L) \cap \overline{\Omega}$.

Therefore, the boundary value problem (1), (2) has at least one solution in the space $C^{\alpha-1}[0,1]$. □

4. Example

Example 4.1 Consider the boundary value problem

$$\begin{aligned}D_{0+}^{\frac{7}{2}}u(t) =& \frac{1}{48}\sin(u(t)) + \frac{1}{48}(D_{0+}^{\frac{5}{2}}u(t)) + \frac{1}{48}(D_{0+}^{\frac{3}{2}}u(t)) + \frac{1}{48}(D_{0+}^{\frac{1}{2}}u(t)) +\\ & \sin(D_{0+}^{\frac{5}{2}}u(t))^{\frac{1}{4}} + \cos(D_{0+}^{\frac{3}{2}}u(t))^{\frac{1}{2}} + \sin^2(D_{0+}^{\frac{1}{2}}u(t))^{\frac{1}{5}}), \quad t \in (0,1),\end{aligned} \tag{22}$$

$$\begin{aligned}& I_{0+}^{\frac{1}{2}}u(0) = 0,\ D_{0+}^{\frac{5}{2}}u(0) = D_{0+}^{\frac{5}{2}}u(\frac{1}{2}),\ D_{0+}^{\frac{3}{2}}u(1) = -D_{0+}^{\frac{3}{2}}u(\frac{1}{3}) + 2D_{0+}^{\frac{3}{2}}u(\frac{2}{3}),\\ & D_{0+}^{\frac{1}{2}}u(1) - D_{0+}^{\frac{1}{2}}u(0) = D_{0+}^{\frac{3}{2}}u(\frac{1}{2}).\end{aligned} \tag{23}$$

That is $\alpha = \frac{7}{2}, m = 1, n = 2, \alpha_i = 1, \xi_i = \frac{1}{2}, \beta_1 = -1, \beta_2 = 2, \eta_1 = \frac{1}{3}, \eta_2 = \frac{2}{3}$, and

$$\begin{aligned}f(t,x,y,z,s) =& \frac{1}{48}\sin x + \frac{1}{48}y + \frac{1}{48}z + \frac{1}{48}s + \sin y^{\frac{1}{4}} + \cos z^{\frac{1}{2}} + \sin^2 s^{\frac{1}{5}}\\ \leq& \frac{|x|}{48} + \frac{|y|}{48} + \frac{|z|}{48} + \frac{|s|}{48} + |y|^{\frac{1}{4}} + |z|^{\frac{1}{2}} + |s|^{\frac{1}{5}}.\end{aligned}$$

Taking $a = b = c = d = \frac{1}{48}$, we have

$$\|a\|_1 + \|b\|_1 + \|c\|_1 + \|d\|_1 = \frac{1}{12} < \frac{1}{\Lambda} = \frac{1}{8 + \frac{2}{\Gamma(\frac{7}{2})} + \frac{1}{\Gamma(\frac{5}{2})} + \frac{1}{\Gamma(\frac{3}{2})}} = \frac{1}{8 + \frac{66}{15\sqrt{\pi}}} \approx \frac{2}{21}.$$

Let $A = 146$. For any $u \in C^{\frac{5}{2}}[0,1] \cap I_{0+}^{\frac{7}{2}}(L^1[0,1])$, assume $|D_{0+}^{\frac{5}{2}}u(t) + D_{0+}^{\frac{3}{2}}u(t) + D_{0+}^{\frac{1}{2}}u(t)| > A$ for any $t \in [0,1]$. If $(D_{0+}^{\frac{5}{2}}u(t) + D_{0+}^{\frac{3}{2}}u(t) + D_{0+}^{\frac{1}{2}}u(t)) > A$ holds for any $t \in [0,1]$, then

$$f(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-3}u(t)) \geq \frac{A - 97}{48} > 0,$$

so

$$\int_0^{\frac{1}{2}} f(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-3}u(t))\mathrm{d}s \geq \frac{A - 97}{48}\int_0^{\frac{1}{2}} \mathrm{d}s = \frac{A - 97}{96} > 0.$$

If $(D_{0+}^{\frac{5}{2}}u(t) + D_{0+}^{\frac{3}{2}}u(t) + D_{0+}^{\frac{1}{2}}u(t)) < -A$ holds for any $t \in [0,1]$, then

$$f(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-3}u(t)) \leq \frac{145 - A}{48} < 0,$$

so

$$\int_0^{\frac{1}{2}} f(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-3}u(t))\mathrm{d}s \leq \frac{145 - A}{48}\int_0^{\frac{1}{2}} \mathrm{d}s = \frac{145 - A}{96} < 0.$$

Thus, the condition (A_2) holds. Again, taking $B = 200$, then for any $l, m, n \in R$ satisfying $l^2 + m^2 + n^2 > B$, we have

$$lR_1N(lt^{\alpha-1}+mt^{\alpha-2}+nt^{\alpha-3})+mR_2N(lt^{\alpha-1}+mt^{\alpha-2}+nt^{\alpha-3})+nR_3N(lt^{\alpha-1}+mt^{\alpha-2}+nt^{\alpha-3}) > 0.$$

So, the condition (A_3) holds.

Thus, according to Theorem 3.1, the boundary value problem (22), (23) has at least one solution in $C^{\frac{5}{2}}[0,1]$.

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