Journal of Mathematical Research with Applications
Mar., 2013, Vol. 33, No. 2, pp. 189–203
DOI:10.3770/j.issn:2095-2651.2013.02.006
Http://jmre.dlut.edu.cn

Existence of Positive Solutions for Singular One-Dimensional P-Laplace BVP of the Second-Order Difference Systems

Weimin HU[1,*], Guli Bahaer[1], Daqing JIANG[2]
1. *School of Mathematics and Statistics, Ili Normal University, Xinjiang* 835000*, P. R. China;*
2. *School of Mathematics and Statistics, Northeast Normal University, Jilin* 130024*, P. R. China*

Abstract In this paper we establish the existence of single and multiple positive solutions to the following singular discrete boundary value problem

$$\begin{cases} \Delta[\phi(\Delta x(i-1))] + q_1(i)f_1(i, x(i), y(i)) = 0, \quad i \in \{1, 2, \ldots, T\} \\ \Delta[\phi(\Delta y(i-1))] + q_2(i)f_2(i, x(i), y(i)) = 0, \\ x(0) = x(T+1) = y(0) = y(T+1) = 0, \end{cases} \tag{1.1}$$

where $\phi(s) = |s|^{p-2}s$, $p > 1$ and the nonlinear terms $f_k(i, x, y)$ $(k = 1, 2)$ may be singular at $(x, y) = (0, 0)$.

Keywords multiple solutions; singular; existence; discrete boundary value problem.

MR(2010) Subject Classification 34B15; 34B16

1. Introduction

In this paper, we establish the existence of single and multiple positive solutions to singular discrete boundary value problem

$$\begin{cases} \Delta[\phi(\Delta x(i-1))] + q_1(i)f_1(i, x(i), y(i)) = 0, \quad i \in \{1, 2, \ldots, T\} \\ \Delta[\phi(\Delta y(i-1))] + q_2(i)f_2(i, x(i), y(i)) = 0, \\ x(0) = x(T+1) = y(0) = y(T+1) = 0, \end{cases} \tag{1.1}$$

where $\phi(s) = |s|^{p-2}s$, $p > 1$ to $T \in \{1, 2, \ldots\}$, $N = \{1, \ldots, T\}$, $N^+ = \{0, 1, \ldots, T+1\}$ and $(x(i), y(i)) \in C(N^+, [0, \infty)^2 \setminus \{O\})$. Throughout this paper, we will assume $f_k : N \times ([0, \infty)^2 \setminus \{O\}) \to (0, \infty)$ is continuous. As a result, the nonlinear terms $f_k(i, x, y)$ may be singular at $O = (0, 0)$, $k = 1, 2$.

Remark 1.1 Recall a map $f : N \times ([0, \infty)^2 \setminus \{O\}) \to (0, \infty)$ is continuous if it is continuous as a map of the topological space $N \times ([0, \infty)^2 \setminus \{O\})$ into the topological space $(0, \infty)$. Moreover throughout this paper the topology on N will be the discrete topology.

Received December 15, 2011; Accepted October 12, 2012
Supported by the XJZDXK of China (Grant No. XJZDXK2011004) and the National Natural Science Foundation of China (Grant No. 10971021).
* Corresponding author
E-mail address: hwm680702@163.com (Weimin HU)

Let $C(N^+, \mathbf{R^2})$ denote the class of maps (x, y) continuous on N^+ (discrete topology), with the norm $\|(x, y)\| = \max_{i\in N^+}\{\|x\|, \|y\|\}$ for $(x, y) \in \mathbf{R^2}$, where $\|x\| = \max_{i\in N^+} |x(i)|$, $\|y\| = \max_{i\in N^+} |y(i)|$. For a solution to (1.1) we mean a $(x, y) \in C(N^+, [0, \infty)^2 \setminus \{O\})$ such that (x, y) satisfies (1.1) for $i \in N$ and the boundary (Dirichlet) conditions.

Here and henceforth, we denote $(x_1, y_1) > (x_2, y_2)$ $((x_1, y_1) \geq (x_2, y_2))$ if $(x_1 - x_2, y_1 - y_2) \in \bar{\mathbf{R}}^{\mathbf{2}}_+$ $((x_1 - x_2, y_1 - y_2) \in \mathbf{R}^{\mathbf{2}}_+$), $(\bar{\mathbf{R}}^{\mathbf{2}}_+ = [0, +\infty)^2\backslash\{O\}$, $\mathbf{R}^{\mathbf{2}}_+ = [0, +\infty)^2)$. Further, we say that a vector (x, y) is positive (nonnegative) if $(x, y) > (0, 0)((x, y) \geq (0, 0))$.

It is interesting to note here that the existence of single and multiple solutions to singular positive boundary value problems in the continuous case have been studied in great detail in the literature [5–8, 12] ($p = 2$). However, for the discrete case, ($p = 2$) was devoted to the existence of one solution for singular positive problems in almost all papers, for example, [1, 3, 10, 11, 15]. As far as we know, recently, in [13], the existence of one solution for singular discrete problems to the one-dimensional p-Laplacian has been discussed.

This paper discusses the existence of single and multiple positive solutions for singular discrete problems. The existence principles for nonsingular discrete Dirichlet problem to the one-dimensional p-Laplacian are presented in Section 2. Some general existence theorems will be presented in Section 3 and there we will show, for example, that the discrete boundary value problem

$$\begin{cases} \Delta(\phi(\Delta x(i-1))) + \delta[(\sqrt{x^2(i) + y^2(i)})^{-\alpha} + \gamma(\sqrt{x^2(i) + y^2(i)})^{\beta}] = 0, \ i \in N \\ \Delta(\phi(\Delta y(i-1))) + \delta[(\sqrt{x^2(i) + y^2(i)})^{-\alpha} + \gamma(\sqrt{x^2(i) + y^2(i)})^{\beta}] = 0, \\ x(0) = x(T+1) = y(0) = y(T+1) = 0, \end{cases}$$

has two nonnegative solutions, where $\alpha > 0$, $\beta > 1$, $\delta > 0$ small, $\gamma = (\frac{1}{\sqrt{2}})^{\alpha+\beta}$, $f_k(i, x, y) = \delta[(\sqrt{x^2(i) + y^2(i)})^{-\alpha} + \gamma(\sqrt{x^2(i) + y^2(i)})^{\beta}]$ $(k = 1, 2)$ at $O = (0, 0)$ singular. Existence in this paper will be established using a Leray-Schauder alternative [14] and a general cone fixed point theorem in [5, 9].

In this paper, we only consider the discrete Dirichlet boundary data. We should note that the Sturm-Liouville boundary data can be considered. However, the arguments are easy to follow, we leave the details to the readers.

2. Existence principles

Now, we consider the following discrete Dirichlet boundary value problem

$$\begin{cases} \Delta[\phi(\Delta x(i-1))] + f_1(i, x(i), y(i)) = 0, \quad i \in \{1, 2, \ldots, T\} \\ \Delta[\phi(\Delta y(i-1))] + f_2(i, x(i), y(i)) = 0, \\ x(0) = y(0) = A, \quad x(T+1) = y(T+1) = B, \end{cases} \tag{2.1}$$

where A and B are given real numbers, $\phi(s) = |s|^{p-2}s$, $p > 1$. Suppose the following two conditions are satisfied:

(A1) $f_k(i, x(i), y(i)) : N \times \mathbf{R^2} \to \mathbf{R}$ is continuous, $k = 1, 2$;

(A2) For each $r > 0$ there exists $h_r \in C(N, [0, \infty))$ such that $\|(x, y)\| \leq r$ implies

$|f_k(i,x(i),y(i))| \le h_r(i)$ for $i \in N$, $k = 1,2$.

Moreover, we also suppose that $D \subset E = E_1 \times E_1$ is a bounded set, and there exists a constant $r > 0$ such that $\|(x,y)\| \le r$ for $(x(i),y(i)) \in \bar{D}$. Thus $|f_k(i,x(i),y(i))| \le h_r(i)$ for $(x(i),y(i)) \in \bar{D}$, where $E_1 = C(N^+, \mathbf{R})$, $k = 1,2$.

For each fixed $(x,y) \in D$, we consider the discrete boundary value problem

$$\begin{cases} \Delta[\phi(\Delta w(i-1))] + f_1(i,x(i),y(i)) = 0, \quad i \in \{1,2,\dots,T\} \\ \Delta[\phi(\Delta u(i-1))] + f_2(i,x(i),y(i)) = 0, \\ w(0) = u(0) = A, \quad w(T+1) = u(T+1) = B. \end{cases} \tag{2.2}$$

Then (2.2) is equivalent to

$$(w(i),u(i)) = (\Phi(x,y))(i) = \Bigg(\begin{cases} A, & i = 0 \\ B + \sum_{s=i}^{T} \phi^{-1}(\tau + \sum_{r=1}^{s} f_1(r,x(r),y(r)) & i \in N \\ B, & i = T+1, \end{cases}$$

$$\begin{cases} A, & i = 0 \\ B + \sum_{s=i}^{T} \phi^{-1}(\tau^{'} + \sum_{r=1}^{s} f_2(r,x(r),y(r)) & i \in N\Big) \\ B, & i = T+1 \end{cases} \tag{2.3}$$

where $\tau = -\phi(\Delta w(0))$, $\tau' = -\phi(\Delta u(0))$ are, respectively, the solutions of the equations,

$$Z(\tau) := \phi^{-1}(\tau) + \sum_{s=1}^{T} \phi^{-1}(\tau + \sum_{r=1}^{s} f_1(r,x(r),y(r))) = A - B \tag{2.4}$$

$$Z(\tau') := \phi^{-1}(\tau^{'}) + \sum_{s=1}^{T} \phi^{-1}(\tau + \sum_{r=1}^{s} f_2(r,x(r),y(r))) = A - B. \tag{2.4$'$}$$

Similarly to the proofs of Lemmas 2.1, 2.2 and 2.3 in [15], we have the following results.

Lemma 2.1 *For each fixed $(x,y) \in D$, Eqs. (2.4) and (2.4)$'$ have unique solutions $\tau, \tau' \in \mathbf{R}$, and*

$$|\tau| \le C_r, \quad |\tau'| \le C_r,$$

where C_r is a positive constant independent of $(x,y) \in D$.

Lemma 2.2 *$\Phi : \bar{D} \to E$ is bounded and continuous.*

Lemma 2.3 *$\Phi : E \to E$ is completely continuous.*

We obtain the following general existence principles for (2.1) by using Schauder fixed point theorem and a nonlinear alternative of Leray-Schauder type.

Theorem 2.1 *Suppose (A1) and (A2) hold. In addition, suppose there exists a constant $M > |A| + |B|$, independent of λ with*

$$\|(x,y)\| = \max_{i\in N^+}\{\|x\|, \|y\|\} = \max_{i\in N^+}\{\max_{i\in N^+} |(x(i)|, \max_{i\in N^+} |(y(i)|\} \neq M \tag{2.5}$$

for any solution $(x(i), y(i)) \in C(N^+, \mathbf{R^2})$ *to*

$$\begin{cases} \Delta[\phi(\Delta x(i-1))] + \lambda^{p-1} f_1(i, x(i), y(i)) = 0, \quad i \in N \\ \Delta[\phi(\Delta y(i-1))] + \lambda^{p-1} f_2(i, x(i), y(i)) = 0, \\ x(0) = y(0) = \lambda A, \quad x(T+1) = y(T+1) = \lambda B, \end{cases} \tag{2.6$_\lambda$}$$

and $\lambda \in (0,1)$. *Then, (2.1) has a solution* (x, y) *with* $\|(x, y)\| \le M$.

Proof $(2.6)_\lambda$ is equivalent to the following fixed point problem

$$(x(i), y(i)) = \lambda(\Phi(x, y))(i), \quad i \in N^+ \tag{2.7$_\lambda$}$$

where Φ is as in (2.3). Set

$$U = \{(x, y) \in C(N^+, \mathbf{R^2}), \quad \|(x, y)\| < M\}.$$

Since $\Phi : C(N^+, \mathbf{R^2}) \to C(N^+, \mathbf{R^2})$ is continuous and completely continuous, the nonlinear alterative [14] guarantees that Φ has a fixed point, i.e., $(2.7)_1$ has a solution in $\bar{U}$. Thus (2.1) has a solution $(x, y) \in C(N^+, \mathbf{R^2})$ and $\|(x, y)\| < M$.

Theorem 2.2 *Suppose (A1) and (A2) hold. In addition, we also suppose there exists a constant* $M > |A| + |B|$, *independent of* λ *with*

$$\|(x, y)\| = \max_{i \in N^+}\{\max_{i \in N^+} |x(i)|, \max_{i \in N^+} |y(i)|\} \ne M$$

for any solution $(x, y) \in C(N^+, \mathbf{R^2})$ *to*

$$\begin{cases} \Delta[\phi(\Delta x(i-1) - (1-\lambda)(\frac{B-A}{T+1}))] + \lambda^{p-1} f_1(i, x(i), y(i)) = 0, \quad i \in N \\ \Delta[\phi(\Delta y(i-1) - (1-\lambda)(\frac{B-A}{T+1}))] + \lambda^{p-1} f_2(i, x(i), y(i)) = 0, \\ x(0) = y(0) = A, \quad x(T+1) = y(T+1) = B, \end{cases} \tag{2.8$_\lambda$}$$

and $\lambda \in (0,1)$. *Then (2.1) has a solution* (x, y) *with* $\|(x, y)\| \le M$.

Proof $(2.8)_\lambda$ is equivalent to the fixed point problem

$$(x, y) = (1-\lambda)(Q, Q) + \lambda\Phi(x, y) \quad \text{where} \quad Q = A + \frac{B-A}{T+1} i. \tag{2.9$_\lambda$}$$

Set

$$U = \{(x, y) \in C(N^+, \mathbf{R}^2), \quad \|(x, y)\| < M\}.$$

Since $\Phi : C(N^+, \mathbf{R}^2) \to C(N^+, \mathbf{R}^2)$ is continuous and completely continuous, the nonlinear alterative [14] guarantees that Φ has a fixed point, i.e., $(2.9)_1$ has a solution in $\bar{U}$. Thus (2.1) has a solution $(x, y) \in C(N^+, \mathbf{R}^2)$ and $\|(x, y)\| < M$.

Theorem 2.3 *Suppose that (A1) holds, and there exists* $h \in C(N, [0, \infty))$ *with* $|f_k(i, x(i), y(i))| \le h(i)$ *for* $i \in N, \ k = 1, 2$. *Then (2.1) has a solution* (x, y).

Proof Solving (2.1) is equivalent to the fixed point problem $(x, y) = \Phi(x, y)$. Since $\Phi : C(N^+, \mathbf{R}^2) \to C(N^+, \mathbf{R}^2)$ is continuous and compact, the result follows from Schauder's fixed point theorem.

3. Singular discrete boundary value problems

In this section, we examine the singular Dirichlet boundary value problem

$$\begin{cases} \Delta[\phi(\Delta x(i-1))] + q_1(i) f_1(i, x(i), y(i)) = 0, \quad i \in N \\ \Delta[\phi(\Delta y(i-1))] + q_2(i) f_2(i, x(i), y(i)) = 0, \\ x(0) = y(0) = 0, \quad x(T+1) = y(T+1) = 0, \end{cases} \tag{3.1}$$

where $\phi(s) = |s|^{p-2}s,\ p > 1$, and the nonlinear terms $f_k (k = 1, 2)$ may be singular at $(x, y) = (0, 0)$. We begin by showing that (3.1) has a solution. To do so, we first establish, via Theorem 2.2, the existence of a solution, for each sufficiently large n, to the "modified" problem

$$\begin{cases} \Delta[\phi(\Delta x(i-1))] + q_1(i) f_1(i, x(i), y(i)) = 0, \quad i \in N \\ \Delta[\phi(\Delta y(i-1))] + q_2(i) f_2(i, x(i), y(i)) = 0, \\ x(0) = y(0) = \frac{1}{n}, \quad x(T+1) = y(T+1) = \frac{1}{n}. \end{cases} \tag*{$(3.1)^n$}$$

To show that (3.1) has a solution, we let $n \to \infty$. The key idea in this step is Arzela-Ascoli theorem.

Before we prove our main results, we first state a well known result in [4].

Lemma 3.1 ([4]) *Let $y \in C(N^+, \mathbf{R})$ satisfy $y(i) \geq 0$ for $i \in N^+$. If $u \in C(N^+, \mathbf{R})$ satisfies*

$$\begin{cases} \Delta^2 u(i-1) + y(i) = 0, \quad i \in N \\ u(0) = u(T+1) = 0, \end{cases}$$

then

$$u(i) \geq \mu(i)||u|| \quad \text{for} \quad i \in N^+,$$

here

$$\mu(i) = \min\{\frac{T+1-i}{T+1}, \frac{i}{T}\}.$$

Theorem 3.1 *Suppose the following conditions are satisfied:*

(H_1) $q_k \in C(N, (0, +\infty)),\ k = 1, 2$;

(H_2) $f_k \in C(N \times ([0, \infty)^2 \setminus \{O\})),\ k = 1, 2$;

(H_3) $f_k(i, x, y) \leq g_k(x, y) + h_k(x, y)$ *on* $N \times ([0, \infty)^2 \setminus \{O\})$ *with* $g_k > 0$ *continuous and nonincreasing on* $[0, \infty)^2 \setminus \{O\}, h_k \geq 0$ *continuous on* $[0, \infty)^2$, *and* $\frac{h_k}{g_k}$ *nondecreasing on* $[0, \infty)^2 \setminus \{O\}$, $k = 1, 2$;

(H_4) For each constant $H > 0$ there exists a function ψ_H which is continuous on N^+ and positive on N such that $f_k(i, x, y) \geq \psi_H^{(k)}(i)$ on $N \times (0, H]^2$, $k = 1, 2$;

(H_5) There exists a constant $r > 0$ such that

$$\frac{1}{\phi^{-1}\Big(1 + \frac{h_1(r,r)}{g_1(r,r)}\Big)} \int_0^r \frac{\mathrm{d}u}{\phi^{-1}(g_1(u, 0))} > b_{10}, \quad \frac{1}{\phi^{-1}\Big(1 + \frac{h_2(r,r)}{g_2(r,r)}\Big)} \int_0^r \frac{\mathrm{d}v}{\phi^{-1}(g_2(0, v))} > b_{20}, \tag{3.2}$$

where

$$b_{k0} = \max_{i \in N} \Big(\sum_{s=1}^{i} \phi^{-1}(\sum_{z=s}^{i} q_k(z)),\ \sum_{s=i}^{T} \phi^{-1}(\sum_{z=i}^{s} q_k(z)) \Big) \quad (k = 1, 2).$$

Then (3.1) has a solution $(x, y) \in C(N^+, [0, \infty)^2 \setminus \{O\})$ with $(x, y) > (0, 0)$ on N and $\|(x, y)\| < r$.

Proof Choose $\varepsilon > 0$, and $\varepsilon < r$ with

$$\frac{1}{\phi^{-1}\Big(1+\frac{h_1(r,r)}{g_1(r,r)}\Big)}\int_\varepsilon^r \frac{\mathrm{d}u}{\phi^{-1}(g_1(u,0))} > b_{10}, \quad \frac{1}{\phi^{-1}\Big(1+\frac{h_2(r,r)}{g_2(r,r)}\Big)}\int_\varepsilon^r \frac{\mathrm{d}v}{\phi^{-1}(g_2(0,v))} > b_{20}. \tag{3.3}$$

Let $n_0 \in \{1,2,\ldots\}$ be chosen so that $\frac{1}{n_0} < \varepsilon$ and let $N_0 = \{n_0, n_0+1, \ldots\}$. We will show that the following boundary value problem

$$\begin{cases} \Delta[\phi(\Delta x(i-1))] + q_1(i)f_1(i,x(i),y(i)) = 0, \quad i \in N \\ \Delta[\phi(\Delta y(i-1))] + q_2(i)f_2(i,x(i),y(i)) = 0, \\ x(0)=y(0)=\frac{1}{n}, \quad x(T+1)=y(T+1)=\frac{1}{n}, \quad n \in N_0 \end{cases} \tag*{$(3.1)^n$}$$

has a solution $(x_n(i), y_n(i))$ for $n \in N_0$ such that for $i \in N$, $(x_n(i), y_n(i)) > (\frac{1}{n}, \frac{1}{n})$ and $\|(x_n, y_n)\| < r$.

To see this, we will deal with the modified boundary value problem

$$\begin{cases} \Delta[\phi(\Delta x(i-1))] + q_1(i)F_1(i,x(i),y(i)) = 0, \quad i \in N \\ \Delta[\phi(\Delta y(i-1))] + q_2(i)F_2(i,x(i),y(i)) = 0, \\ x(0)=y(0)=\frac{1}{n}, \quad x(T+1)=y(T+1)=\frac{1}{n}, \quad n \in N_0 \end{cases} \tag*{$(3.4)^n$}$$

where $(\forall i \in N)$

$$F_1(i,x,y) = f_1(i, \max\{x, \frac{1}{n}\}, \max\{0,y\}), \quad F_2(i,x,y) = f_2(i, \max\{x,0\}, \max\{y, \frac{1}{n}\}).$$

To show that $(3.4)^n$ has a solution for $n \in N_0$, we will apply Theorem 2.2. Consider the family of problems

$$\begin{cases} -\Delta[\phi(\Delta x(i-1))] = \lambda^{p-1}q_1(i)F_1(i,x(i),y(i)), \quad i \in N \\ -\Delta[\phi(\Delta y(i-1))] = \lambda^{p-1}q_2(i)F_2(i,x(i),y(i)), \\ x(0)=y(0)=\frac{1}{n}, \quad x(T+1)=y(T+1)=\frac{1}{n}, \quad n \in N_0 \end{cases} \tag*{$(3.5)^n_\lambda$}$$

where $\lambda \in (0,1)$. Let (x,y) be a solution of $(3.5)^n_\lambda$. Since

$$\begin{cases} \Delta[\phi(\Delta x(i-1))] \le 0, \quad i \in N \\ \Delta[\phi(\Delta y(i-1))] \le 0, \end{cases}$$

i.e.,

$$\begin{cases} \Delta^2 x(i-1) \le 0, \quad i \in N \\ \Delta^2 y(i-1) \le 0, \end{cases}$$

then $(x(i), y(i)) \ge (\frac{1}{n}, \frac{1}{n})$ on N^+ and there exists $i_0 \in N$ with $\Delta x(i) \ge 0$ on $[0, i_0) = \{0,1,\ldots,i_0-1\}$ and $\Delta x(i) \le 0$ on $[i_0, T+1) = \{i_0, i_0+1, \ldots, T\}$, and $x(i_0) = ||x||$; there exists $i_0' \in N$ with $\Delta y(i) \ge 0$ on $[0, i_0') = \{0,1,\ldots,i_0'-1\}$, $\Delta y(i) \le 0$ on $[i_0', T+1) = \{i_0', i_0'+1, \ldots, T\}$, and $y(i_0') = \|y\|$. Also notice that

$$F_1(i,x(i),(y(i)) = f_1(i,x(i),(y(i)) \le g_1(x(i),(y(i)) + h_1(x(i),(y(i)), \quad i \in N$$

so for $z \in N$, we have

$$-\Delta[\phi(\Delta x(z-1))] \le g_1(x(z),y(z))\Big(1+\frac{h_1(x(i_0),y(i_0'))}{g_1(x(i_0),y(i_0'))}\Big)q_1(z). \tag{3.6}$$

Summing the equation (3.6) from $s+1$ $(0 \le s < i_0)$ to i_0, we obtain

$$\phi[\Delta x(s)] \le \phi[\Delta x(i_0)] + \Big(1 + \frac{h_1(x(i_0), y(i_0'))}{g_1(x(i_0), y(i_0'))}\Big) \sum_{z=s+1}^{i_0} g_1(x(z), y(z)) q_1(z).$$

Since $\Delta x(i_0) \le 0$, and $(x(z), y(z)) \ge (x(s+1), 0)$ when $s+1 \le z \le i_0$, we have

$$\phi[\Delta x(s)] \le g_1(x(s+1), 0) \Big(1 + \frac{h_1(x(i_0), y(i_0'))}{g_1(x(i_0), y(i_0'))}\Big) \sum_{z=s+1}^{i_0} q_1(z), \quad s < i_0$$

i.e.,

$$\frac{\Delta x(s)}{\phi^{-1}(g_1(x(s+1), 0))} \le \phi^{-1}\Big(1 + \frac{h_1(x(i_0), y(i_0'))}{g_1(x(i_0), y(i_0'))}\Big) \phi^{-1}\Big(\sum_{z=s+1}^{i_0} q_1(z)\Big), \quad s < i_0. \tag{3.7}$$

Since $g_1(x(s+1), 0) \le g_1(u, 0) \le g_1(x(s), 0)$ for $(x(s), 0) \le (u, 0) \le (x(s+1), 0)$ when $s < i_0$, we have

$$\int_{x(s)}^{x(s+1)} \frac{\mathrm{d}u}{\phi^{-1}(g_1(u, 0))} \le \frac{\Delta x(s)}{\phi^{-1}(g_1(x(s+1), 0))}, \quad s < i_0. \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\int_{x(s)}^{x(s+1)} \frac{\mathrm{d}u}{\phi^{-1}(g_1(u, 0))} \le \phi^{-1}\Big(1 + \frac{h_1(x(i_0), y(i_0'))}{g_1(x(i_0), y(i_0'))}\Big) \phi^{-1}\Big(\sum_{z=s+1}^{i_0} q_1(z)\Big), \quad s < i_0$$

and then, we sum the above from 0 to $i_0 - 1$ to obtain

$$\begin{aligned} \int_{\frac{1}{n}}^{x(i_0)} \frac{\mathrm{d}u}{\phi^{-1}(g_1(u, 0))} &\le \phi^{-1}\Big(1 + \frac{h_1(x(i_0), y(i_0'))}{g_1(x(i_0), y(i_0'))}\Big) \sum_{s=0}^{i_0-1} \phi^{-1}\Big(\sum_{z=s+1}^{i_0} q_1(z)\Big) \\ &= \phi^{-1}\Big(1 + \frac{h_1(x(i_0), y(i_0'))}{g_1(x(i_0), y(i_0'))}\Big) \sum_{s=1}^{i_0} \phi^{-1}\Big(\sum_{z=s}^{i_0} q_1(z)\Big). \end{aligned} \tag{3.9}$$

Similarly, we sum the equation (3.6) from i_0 to s $(i_0 \le s < T+1)$ to obtain

$$-\phi[\Delta x(s)] \le -\phi[\Delta x(i_0 - 1)] + \Big(1 + \frac{h_1(x(i_0), y(i_0'))}{g_1(x(i_0), y(i_0'))}\Big) \sum_{z=i_0}^{s} g_1(x(z), y(z)) q_1(z), \quad s \ge i_0.$$

Since $\Delta x(i_0 - 1) \ge 0$, and $(x(z), y(z)) \ge (x(s), 0)$ when $i_0 \le z \le s$, we have

$$-\phi[\Delta x(s)] \le g_1(x(s), 0) \Big(1 + \frac{h_1(x(i_0), y(i_0'))}{g_1(x(i_0), y(i_0'))}\Big) \sum_{z=i_0}^{s} q_1(z), \quad s \ge i_0$$

i.e.,

$$\frac{-\Delta x(s)}{\phi^{-1}(g_1(x(s), 0))} \le \phi^{-1}\Big(1 + \frac{h_1(x(i_0), y(i_0'))}{g_1(x(i_0), y(i_0'))}\Big) \phi^{-1}\Big(\sum_{z=i_0}^{s} q_1(z)\Big), \quad s \ge i_0.$$

So we have

$$\int_{x(s+1)}^{x(s)} \frac{\mathrm{d}u}{\phi^{-1}(g_1(u, 0))} \le \frac{-\Delta x(s)}{\phi^{-1}(g_1(x(s), 0))} \le \phi^{-1}\Big(1 + \frac{h_1(x(i_0), y(i_0'))}{g_1(x(i_0), y(i_0'))}\Big) \phi^{-1}\Big(\sum_{z=i_0}^{s} q_1(z)\Big), \quad s \ge i_0$$

and then we sum the above from i_0 to T to obtain

$$\int_{\frac{1}{n}}^{x(i_0)} \frac{\mathrm{d}u}{\phi^{-1}(g_1(u, 0))} \le \phi^{-1}\Big(1 + \frac{h_1(x(i_0), y(i_0'))}{g_1(x(i_0), y(i_0'))}\Big) \sum_{s=i_0}^{T} \phi^{-1}\Big(\sum_{z=i_0}^{s} q_1(z)\Big), \quad s \ge i_0. \tag{3.10}$$

Now (3.9) and (3.10) imply

$$\int_{\varepsilon}^{x(i_0)} \frac{\mathrm{d}u}{\phi^{-1}(g_1(u,0))} \le \int_{\frac{1}{n}}^{x(i_0)} \frac{\mathrm{d}u}{\phi^{-1}(g_1(u,0))} \le b_{10}\phi^{-1}\Big(1+\frac{h_1(x(i_0),y(i_0'))}{g_1(x(i_0),y(i_0'))}\Big).$$

Similarly, we also have

$$\int_{\varepsilon}^{y(i_0')} \frac{\mathrm{d}v}{\phi^{-1}(g_2(0,v))} \le b_{20}\phi^{-1}\Big(1+\frac{h_2(x(i_0),y(i_0'))}{g_2(x(i_0),y(i_0'))}\Big).$$

This togethers with (3.3) implies $\|(x,y)\| \neq r$. In fact, if $\|(x,y)\| = r$, without loss of generality, we assume that $\|x\| = r$, i.e., $x(i_0) = r$, then

$$\int_{\varepsilon}^{r} \frac{\mathrm{d}u}{\phi^{-1}(g_1(u,0))} \le b_{10}\phi^{-1}\Big(1+\frac{h_1(r,r)}{g_1(r,r)}\Big).$$

If we assume that $\|y\| = r$, we have also

$$\int_{\varepsilon}^{r} \frac{\mathrm{d}u}{\phi^{-1}(g_2(0,v))} \le b_{20}\phi^{-1}\Big(1+\frac{h_2(r,r)}{g_2(r,r)}\Big).$$

This contradicts (3.3). Then Theorem 2.2 implies that $(3.4)^n$ has a solution (x_n, y_n) with $\|(x_n,y_n)\| \le r$. In fact (as above)

$$\frac{1}{n} \le x_n(i) < r,\ \frac{1}{n} \le y_n(i) < r, \quad \text{for} \ \ i \in N^+$$

i.e.,

$$(\frac{1}{n},\frac{1}{n}) \le (x_n(i), y_n(i)) < (r,r), \quad \text{for} \ \ i \in N^+.$$

Thus, $(x_n(i), y_n(i))$ is also a solution of $(3.1)^n$.

Next, we obtain a sharper lower bound on $(x_n(i), y_n(i))$, namely, we will show that there exists a constant $C > 0$, independent of n, with

$$x_n(i) \ge C\mu(i), y_n(i) \ge C\mu(i), \quad \text{for} \ \ i \in N^+ \tag{3.11}$$

where $\mu(i)$ is as in Lemma 3.1.

To see this, notice (H_4) guarantees the existence of functions $\psi_r^{(k)}(i)$ continuous on N^+ and positive on N with $f_k(i,x,y) \ge \psi_r^{(k)}(i)$ for $(i,x,y) \in N \times (0,r]^2$, $k = 1,2$. Let $(u(i), v(i)) \in C(N^+, \mathbf{R}^2)$ be a unique solution to the problem

$$\begin{cases} \Delta[\phi(\Delta u(i-1))] + q_1(i)\psi_r^{(1)}(i) = 0, \quad i \in N \\ \Delta[\phi(\Delta v(i-1))] + q_2(i)\psi_r^{(2)}(i) = 0, \\ u(0) = v(0) = 0, \quad u(T+1) = v(T+1) = 0. \end{cases} \tag{3.12}$$

Since $\Delta[\phi(\Delta u(i-1))] \le 0$ on N, with $u(0) = u(T+1) = 0$, we have $\Delta^2 u(i-1) \le 0$ on N, and so Lemma 3.1 implies

$$u(i) \ge \mu(i)||u||, v(i) \ge \mu(i)||v||, \quad \forall i \in N^+. \tag{3.13}$$

Since $f_k(i,x,y) \ge \psi_r^{(k)}(i)$ for $(i,x,y) \in N \times (0,r]^2$, $k = 1,2$, similarly to the proof in [15], we have

$$(x_n(i), y_n(i)) \ge (u(i), v(i)), \quad i \in N^+. \tag{3.14}$$

Now (3.14) togethers with (3.13) implies that (3.11) holds for $C = \min_{i\in N^+}\{\|u\|, \|v\|\}$.

The Arzela-Ascoli theorem guarantees the existence of a subsequence $N_1 \subset N_0$ and functions $(x, y) \in C(N^+, \mathbf{R}^2)$ with $(x_n, y_n) \to (x, y)$ in $C(N^+, \mathbf{R}^2)$ as $n \to \infty$ through N_1.

Also, for $x(0) = x(T+1) = 0$, $y(0) = y(T+1) = 0$, $\|(x, y)\| \le r$ for $i \in N^+$. In particular $x(i) \ge C\mu(i) \ge \frac{C}{T+1}$ and $y(i) \ge C\mu(i) \ge \frac{C}{T+1}$ on N.

Fix $i \in N$, we obtain

$$\begin{aligned}\Delta[\phi(\Delta x_n(i-1))] &= \phi(\Delta x_n(i)) - \phi(\Delta x_n(i-1))\\ &= \phi(x_n(i+1) - x_n(i)) - \phi(x_n(i) - x_n(i-1))\\ &\to \Delta[\phi(\Delta x(i-1))],\quad i \in N,\ n \in N_1,\ n \to \infty,\end{aligned}$$

$$\Delta[\phi(\Delta y_n(i-1))] \to \Delta[\phi(\Delta y(i-1))],\quad i \in N,\ n \in N_1,\ n \to \infty$$

and

$$f_k(i, x_n(i), y_n(i)) \to f_k(i, x(i), y(i)),\quad i \in N,\ n \in N_1,\ n \to \infty,\quad k = 1, 2,$$

i.e.,

$$\begin{cases}\Delta[\phi(\Delta x(i-1))] + q_1(i) f_1(i, x(i), y(i)) = 0, & i \in N\\ \Delta[\phi(\Delta y(i-1))] + q_2(i) f_2(i, x(i), y(i)) = 0, & \\ x(0) = y(0) = 0, \quad x(T+1) = y(T+1) = 0. &\end{cases}$$

Finally, it is easy to see that $\|(x, y)\| < r$ (note if $\|(x, y)\| = r$, then following essentially the same argument from (3.6)–(3.10) will yield a contradiction).

This completes the proof of Theorem 3.1. □

Example 3.1 Consider the singular boundary value problem

$$\begin{cases}\Delta[\phi(\Delta x(i-1))] + \delta((\sqrt{x(i)^2 + y(i)^2})^{-\alpha} + (\sqrt{x(i)^2 + y(i)^2})^{\beta}) = 0, \quad i \in N,\\ \Delta[\phi(\Delta y(i-1))] + \delta((\sqrt{x(i)^2 + y(i)^2})^{-\alpha} + (\sqrt{x(i)^2 + y(i)^2})^{\beta}) = 0,\\ x(0) = y(0) = 0, \quad x(T+1) = y(T+1) = 0, \quad \alpha > 0, \quad \beta \ge 0, \quad \gamma = (\frac{1}{\sqrt{2}})^{\alpha+\beta}\end{cases} \tag{3.15}$$

with $\delta > 0$ and also

$$\delta < \left[\frac{p-1}{b_1(\alpha + p - 1)}\right]^{p-1} \sup_{c\in(0,\infty)} \frac{c^{\alpha+p-1}}{1 + c^{\alpha+\beta}}, \tag{3.16}$$

here

$$b_1 = \max_{i\in N}\Big(\sum_{s=1}^{i}(i-s+1)^{\frac{1}{p-1}},\ \sum_{s=i}^{T}(s-i+1)^{\frac{1}{p-1}}\Big) = \sum_{i=1}^{T} i^{\frac{1}{p-1}}. \tag{3.17}$$

Then (3.15) has a solutions (x, y) with $(x(i), y(i)) > (0, 0)$ for $i \in N$.

To see this, we will apply Theorem 3.1 with

$$q_k(s) = \delta,\ g_k(x, y) = (\sqrt{(x^2 + y^2)})^{-\alpha},\ h_k(x, y) = \gamma(\sqrt{(x^2 + y^2)})^{\beta},\quad k = 1, 2.$$

Clearly (H_1)–(H_4) hold. Also notice

$$\sum_{s=1}^{i}\phi^{-1}(\sum_{z=s}^{i} q_k(z)) = \delta^{\frac{1}{p-1}}\sum_{s=1}^{i}(i-s+1)^{\frac{1}{p-1}},$$

$$\sum_{s=i}^{T}\phi^{-1}(\sum_{z=i}^{s}q_k(z))=\delta^{\frac{1}{p-1}}\sum_{s=1}^{i}(s-i+1)^{\frac{1}{p-1}},$$

and so

$$b_{k0}=\max_{i\in N}\Big(\delta^{\frac{1}{p-1}}\sum_{s=1}^{i}(i-s+1)^{\frac{1}{p-1}},\ \delta^{\frac{1}{p-1}}\sum_{s=i}^{T}(s-i+1)^{\frac{1}{p-1}}\Big)=\delta^{\frac{1}{p-1}}b_1,\quad k=1,2.$$

Consequently (H_5) holds since (3.16) implies there exists $r>0$ such that

$$\delta<\Big[\frac{p-1}{b_1(\alpha+p-1)}\Big]^{p-1}\frac{r^{\alpha+p-1}}{1+r^{\alpha+\beta}},$$

and so

$$\frac{1}{\phi^{-1}(1+\frac{h_1(r,r)}{g_1(r,r)})}\int_0^r\frac{\mathrm{d}u}{\phi^{-1}(g_1(u,0))}=\frac{p-1}{p-1+\alpha}\phi^{-1}\Big(\frac{r^{\alpha+p-1}}{1+r^{\alpha+\beta}}\Big)>b_{10}.$$

Similarly, we also have

$$\frac{1}{\phi^{-1}(1+\frac{h_2(r,r)}{g_2(r,r)})}\int_0^r\frac{\mathrm{d}v}{\phi^{-1}(g_2(0,v))}\frac{p-1}{p-1+\alpha}\phi^{-1}\Big(\frac{r^{\alpha+p-1}}{1+r^{\alpha+\beta}}\Big)>b_{20}.$$

Thus all the conditions of Theorem 3.1 are satisfied, and then, the existence is guaranteed.

Remark 3.1 If $\beta<p-1$, then (3.16) is automatically satisfied.

Next we establish the existence of two positive solutions to (3.1). First, we state the fixed point result we will use to establish multiplicity.

Lemma 3.2 ([5]) *Let $E=(E,\|\cdot\|)$ be a Banach space, $K\subset E$ be a cone in E, and $\|\cdot\|$ be increasing with respect to K. Moreover, let r,R be constants with $0<r<R$. Suppose $\Phi:\bar{\Omega}_R\cap K\to K$ (here $\Omega_R=\{x\in E,\|x\|<R\}$) is a continuous, compact map and assume the conditions*

$$x\neq\lambda\Phi(x),\ \text{for }\lambda\in[0,1)\text{ and }x\in\partial\Omega_r\cap K\tag{3.18}$$

and also,

$$\|\Phi x\|>\|x\|,\quad\text{for }\ x\in\partial\Omega_R\cap K\tag{3.19}$$

hold. Then Φ has a fixed point in $K\cap\{x\in E:r\le\|x\|\le R\}$.

Remark 3.2 In Lemma 3.2 if (3.18) and (3.19) are replaced by

$$x\neq\lambda\Phi(x),\ \text{for }\lambda\in[0,1)\text{ and }x\in\partial\Omega_R\cap K\tag*{(3.18)*}$$

and

$$\|\Phi x\|>\|x\|,\quad\text{for }\ x\in\partial\Omega_r\cap K.\tag*{(3.19)*}$$

Then Φ has a fixed point in $K\cap\{x\in E:r\le\|x\|\le R\}$.

In this paper, let $\|u\|=\max_{i\in N^+}|u(i)|,u(i)\in C(N^+,R)$. Then $E_1=(C(N,R),\|\cdot\|)$ is a Banach space. Let $K_1=\{u\in C(N^+,[0,+\infty)):u(i)\ge\mu(i)||u||,\ i\in N^+\}$.

Let $E=E_1\times E_1$, $K=K_1\times K_1$, and $\|z\|=\|(x,y)\|=\max\{||x||,||y||\}$, $\forall z=(x,y)\in E$. Then $(E,\|\cdot\|)$ is a Banach space and K is a cone in E.

Theorem 3.2 *Assume that* (H_1), (H_2), (H_3) *and* (H_5) *hold. In addition, suppose*

$$\begin{cases} (H_6):\ \text{Let } f_k(i,x,y) \ge \bar{g}_k(x,y) + \bar{h}_k(x,y) \text{ on } N \times ([0,+\infty)^2 \backslash \{O\}), \\ \quad \text{with } \bar{g}_k > 0 \text{ continuous and nonincreasing on } [0,\infty)^2 \backslash \{O\}, \\ \quad \bar{h}_k \ge 0 \text{ continuous on } [0,\infty)^2, \\ \quad \frac{\bar{h}_k}{\bar{g}_k} \text{ nondecreasing on } [0,\infty)^2 \backslash \{O\}, \quad k = 1,2. \end{cases} \tag{3.20}$$

(H_7) *There exists a constant* $R > r$ *such that*

$$\frac{R}{\phi^{-1}\Big(\bar{g}_1(R,R)(1+\frac{\bar{h}_1(\frac{R}{T+1},\,0)}{\bar{g}_1(\frac{R}{T+1},\,0)})\Big)} < \|v\|, \quad \frac{R}{\phi^{-1}\Big(\bar{g}_2(R,R)(1+\frac{\bar{h}_2(0,\frac{R}{T+1})}{\bar{g}_2(0,\frac{R}{T+1})})\Big)} < \|v\|, \tag{3.21}$$

where v *satisfies*

$$\begin{cases} \Delta[\phi(\Delta v(i-1))] + q(i) = 0, \quad i \in N \\ v(0) = v(T+1) = 0. \end{cases} \tag{3.22}$$

Then (3.1) has a solution $(x(i),y(i)) \in C(N^+,\mathbf{R}^2)$ *with* $(x,y) > (0,0)$ *on* N *and* $r < \|(x,y)\| \le R$.

Proof To show the existence of the solution described in the statement of Theorem 3.2, we will apply Lemma 3.2. First, we choose $\varepsilon > 0$ ($\varepsilon < r$) with

$$\frac{1}{\phi^{-1}\Big(1+\frac{h_1(r,r)}{g_1(r,r)}\Big)} \int_\varepsilon^r \frac{\mathrm{d}u}{\phi^{-1}(g_1(u,0))} > b_{10}, \quad \frac{1}{\phi^{-1}\Big(1+\frac{h_2(r,r)}{g_2(r,r)}\Big)} \int_\varepsilon^r \frac{\mathrm{d}v}{\phi^{-1}(g_2(0,v))} > b_{20}. \tag{3.23}$$

Let $n_0 \in \{1,2,\ldots\}$ be chosen so that $\frac{1}{n_0} < \frac{\varepsilon}{2}$ and $\frac{1}{n_0} < \frac{r}{T+1}$ and let $N_0 = \{n_0, n_0+1, \ldots\}$.

First we will show that

$$\begin{cases} \Delta[\phi(\Delta x(i-1))] + q_1(i) f_1(i,x(i),y(i)) = 0, \quad i \in N, \\ \Delta[\phi(\Delta y(i-1))] + q_2(i) f_2(i,x(i),y(i)) = 0, \\ x(0) = y(0) = \frac{1}{n}, \quad x(T+1) = y(T+1) = \frac{1}{n}, \quad n \in N_0 \end{cases} \tag*{$(3.24)^n$}$$

has a solution (x_n,y_n) for each $n \in N_0$ with $(x_n(i),y_n(i)) > (\frac{1}{n},\frac{1}{n})$ on N and $r < \|(x_n,y_n)\| \le R$.

To show $(3.24)^n$ has such a solution for each $n \in N_0$, we will deal with the modified boundary value problem

$$\begin{cases} \Delta[\phi(\Delta x(i-1))] + q_1(i) F_1(i,x(i),y(i)) = 0, \quad i \in N \\ \Delta[\phi(\Delta y(i-1))] + q_2(i) F_2(i,x(i),y(i)) = 0, \\ x(0) = y(0) = \frac{1}{n}, \ x(T+1) = y(T+1) = \frac{1}{n}, \quad n \in N_0 \end{cases} \tag*{$(3.25)^n$}$$

where $\forall i \in N$, and

$$F_1(i,x(i),(y(i)) = f_1(i,\max\{x,\frac{1}{n}\},\max\{0,y\}) \text{ and } F_2(i,x,y) = f_2(i,\max\{x,0\},\max\{y,\frac{1}{n}\}).$$

Fix $n \in N_0$. Let $\Phi : K \to C(N^+,\mathbf{R^2})$ be defined by

$$(w(i),u(i)) = (\Phi(x,y))(i) = \Bigg(\begin{cases} \frac{1}{n}, \quad i = 0 \text{ or } T+1 \\ B + \sum_{s=i}^T \phi^{-1}(\tau + \sum_{r=1}^s F_1(r,x(r),y(r)), \quad i \in N, \end{cases}$$

$$\begin{cases} \frac{1}{n}, \quad i = 0 \text{ or } T+1 \\ B + \sum_{s=i}^T \phi^{-1}(\tau' + \sum_{r=1}^s F_2(r,x(r),y(r)), \quad i \in N \end{cases} \Bigg), \tag{3.26}$$

where τ and τ' are, respectively, the solutions of the equations

$$\phi^{-1}(\tau)+\sum_{s=1}^{T}\phi^{-1}(\tau+\sum_{r=1}^{s}F_1(r,x(r),y(r)))=0 \tag{3.27}$$

and

$$\phi^{-1}(\tau')+\sum_{s=1}^{T}\phi^{-1}(\tau'+\sum_{r=1}^{s}F_2(r,x(r),y(r)))=0. \tag*{(3.27)$'$}$$

From Section 2, $\Phi:K\to C(N^+,\mathbf{R}^2)$ is completely continuous. Moreover, we have

$$\begin{cases}\Delta[\phi(\Delta w(i-1))]+q_1(i)F_1(i,x(i),y(i))=0, & i\in N,\\ \Delta[\phi(\Delta v(i-1))]+q_2(i)F_2(i,x(i),y(i))=0, & \\ w(0)=v(0)=\frac{1}{n},\ w(T+1)=v(T+1)=\frac{1}{n}, & n\in N_0.\end{cases} \tag{3.28}$$

This implies that $\Delta[\phi(\Delta w(i-1))\leq 0,\ i\in N$. Thus $\Delta^2 w(i-1)\leq 0,\ i\in N$, and $w(i)\geq\frac{1}{n}$. Consequently, $w(i)-\frac{1}{n}\geq\mu(i)||w-\frac{1}{n}||$ (from Lemma 3.1), thus $w(i)\geq\frac{1}{n}+\mu(i)(||w||-\frac{1}{n})\geq\mu(i)||w||,\ i\in N^+$. Similarly, we also have $v(i)\geq\mu(i)||v||,\ i\in N^+$, and so $\Phi:K\to K$.

First, we show

$$(x,y)\neq\lambda\Phi(x,y)\ \ \text{for}\ \ \lambda\in[0,1),\ (x,y)\in\partial\Omega_r\cap K, \tag{3.29}$$

where $\Omega_r=\{(x,y)\in E:\|(x,y)\|<r\}$.

Suppose this is false, i.e., there exists $(x,y)\in\partial\Omega_r$ and $\lambda\in[0,1)$ with $(x,y)=\lambda\Phi(x,y)$. We can assume $\lambda\neq 0$. Now since $(x,y)=\lambda\Phi(x,y)$, we have

$$\begin{cases}-\Delta[\phi(\Delta x(i-1))]=\lambda^{p-1}q_1(i)F_1(i,x(i),y(i)), & i\in N,\\ -\Delta[\phi(\Delta y(i-1))]=\lambda^{p-1}q_2(i)F_2(i,x(i),y(i)), & \\ x(0)=y(0)=\frac{\lambda}{n},\ x(T+1)=y(T+1)=\frac{\lambda}{n}, & n\in N_0.\end{cases} \tag{3.30}$$

Since $\|(x,y)\|=r$, without loss of generality, we assume that $\|x\|=r$. Clearly, there exists $i_0\in N$ with $\Delta x(i)\geq 0$ on $[0,i_0)=\{0,1,\dots,i_0-1\}$, $\Delta x(i)\leq 0$ on $[i_0,T+1)=\{i_0,i_0+1,\dots,T\}$ and $x(i_0)=\|x\|=r$.

Also notice $x(i)\geq\mu(i)\|x(i)\|=\mu(i)r\geq\frac{r}{T+1}>\frac{1}{n_0}$, for $i\in N$, and so

$$F_1(i,x(i),y(i))=f_1(i,x(i),y(i))\leq g_1(x(i),y(i))+h_1(x(i),y(i)),\quad i\in N.$$

Fix $z\in N$, and then, we have

$$-\Delta[\phi(\Delta x(z-1))]\leq g_1(x(z),y(z))\{1+\frac{h_1(r,r)}{g_1(r,r)}\}q_1(z). \tag{3.31}$$

By the arguments which were used to obtain (3.9) and (3.10) in Theorem 3.1, we have

$$\int_{\frac{\lambda}{n}}^{x(i_0)}\frac{\mathrm{d}u}{\phi^{-1}(g_1(u,0)}\leq\phi^{-1}\Big(1+\frac{h_1(r,r)}{g_1(r,r)}\Big)\sum_{s=1}^{i_0}\phi^{-1}(\sum_{r=s}^{i_0}q_1(r)) \tag{3.32}$$

and

$$\int_{\frac{\lambda}{n}}^{x(i_0)}\frac{\mathrm{d}u}{\phi^{-1}(g_1(u,0)}\leq\phi^{-1}\Big(1+\frac{h_1(r,r)}{g_1(r,r)}\Big)\sum_{s=i_0}^{T}\phi^{-1}(\sum_{r=i_0}^{s}q_1(r)). \tag{3.33}$$

Now (3.32) and (3.33) imply

$$\int_{\varepsilon}^{r} \frac{\mathrm{d}u}{\phi^{-1}(g_1(u,0))} \leq b_{10}\phi^{-1}\Big(1+\frac{h_1(r,r)}{g_1(r,r)}\Big). \tag{3.34}$$

Similarly, we also have

$$\int_{\varepsilon}^{r} \frac{\mathrm{d}v}{\phi^{-1}(g_2(0,v))} \leq b_{20}\phi^{-1}\Big(1+\frac{h_2(r,r)}{g_2(r,r)}\Big). \tag{3.34$'$}$$

This contradicts (3.23) and consequently (3.29) is true.

Next, we show

$$\|(w,u)\| = \|\Phi(x,y)\| > \|(x,y)\|, \quad \forall (x,y) \in \partial\Omega_R \cap K,$$

where $\Omega_R = \{(x,y) \in E : \|(x,y)\| < R\}$.

To see this, let $(x,y) \in \partial\Omega_R \cap K$ such that $\|(x,y)\| = R$. Since $\|(x,y)\| = \max_{i\in N^+}\{\|x\|,\|y\|\} = R$, without loss of generality, we assume that $\|x\| = R$.

Also, since $(x,y) \in K$, we have

$$x(i) \geq \mu(i)\|x\| \geq \mu(i)R \geq \frac{R}{T+1} > \frac{1}{n_0}, \quad \forall i \in N.$$

Thus

$$F_1(i,x(i),y(i)) = f_1(i,x(i),y(i)) \geq \bar{g}_1(x(i),y(i)) + \bar{h}_1(x(i),y(i)), \quad \forall i \in N,$$

so we have

$$\begin{aligned} -\Delta[\phi(\Delta w(i-1))] &= q_1(i)F_1(i,x(i),y(i)) = q_1(i)f_1(x(i),y(i)) \\ &\geq \bar{g}_1(x(i),y(i))\Big(1+\frac{\bar{h}_1(x(i),y(i))}{\bar{g}_1(x(i),y(i))}\Big)q_1(i) \\ &\geq \bar{g}_1(R,R)\Big(1+\frac{\bar{h}_1(\frac{R}{T+1},\ 0)}{\bar{g}_1(\frac{R}{T+1},\ 0)}\Big)q_1(i) := C_1(R)q_1(i), \end{aligned} \tag{3.35}$$

and

$$-\Delta[\phi(\Delta u(i-1))] \geq \bar{g}_2(R,R)\Big(1+\frac{\bar{h}_2(0,\frac{R}{T+1})}{\bar{g}_2(0,\frac{R}{T+1})}\Big)q_2(i) := C_2(R)q_2(i).$$

Then, we obtain

$$-\Delta\Big(\phi\Big(\Delta\frac{w(i-1)}{\phi^{-1}(C_1(R))}\Big)\Big) \geq q_1(i), \quad w(0) = w(T+1) = \frac{\lambda}{n} \geq 0. \tag{3.36}$$

The argument used to get (3.22) yields

$$\frac{w(i)}{\phi^{-1}(C_1(R))} \geq v(i), \quad i \in N^+. \tag{3.37}$$

Similarly, we have also

$$\frac{u(i)}{\phi^{-1}(C_2(R))} \geq v(i), \quad i \in N^+.$$

Now (3.21) and (3.37) yield

$$\|w\| \geq \|v\|\phi^{-1}(C_1(R)) > R, \quad \|u\| \geq \|v\|\phi^{-1}(C_2(R)) > R,$$

i.e.,

$$\|(w,u)\| = \|\Phi(x,y)\| > R = \|(x,y)\|, \quad \forall (x,y) \in \partial\Omega_R \cap K.$$

This implies Φ has a fixed point $(x_n,y_n) \in K \cap (\bar{\Omega}_R \setminus \Omega_r)$ i.e., $r < \|(x_n,y_n)\| \le R$. In fact $\|(x_n,y_n)\| \ne r$ (note if $\|(x_n,y_n)\| = r$, then following essentially the same argument from (3.31)–(3.34) will yield a contradiction). Consequently $(3.25)^n$ (and also $(3.24)^n$) has a solution $(x_n,y_n) \in C(N^+,\mathbf{R}^2)$, $(x_n,y_n) \in K$, with

$$x_n \ge r\mu(i),\ y_n \ge r\mu(i), \quad i \in N,\ r < \|(x_n,y_n)\| \le R. \tag{3.38}$$

Essentially the same reason as before guarantees that there exists a subsequence N_1 of N_0, and a function $(x,y) \in C(N^+,\mathbf{R}^2)$ with $(x_n(i),y_n(i))$ converging to $(x(i),y(i))$ as $n \to \infty$ through N_1. It is easy to show that $(x(i),y(i)) \in C(N^+,\mathbf{R}^2)$ is a solution of (3.1) and $r < \|(x,y)\| \le R$.

Thus, the proof of Theorem 3.2 is completed. □

Remark 3.3 In (H$_7$), if we have $R < r$, then (3.1) has a solution $(x(i),y(i)) \in C(N^+,\mathbf{R}^2)$ with $(x,y) > (0,0)$ on N and $R \le \|(x,y)\| < r$. The argument is similar to that in Theorem 3.2 except for that here we use Remark 3.2.

Theorem 3.3 *Assume (H$_1$)–(H$_7$) hold. Then (3.1) has two solutions $(x_k,y_k) \in C(N^+,\mathbf{R}^2)$, $k = 1,2$ with $(x_k,y_k) > (0,0)$, $k=1,2$ on N and $0 < \|(x_1,y_1)\| < r < \|(x_2,y_2)\| \le R$.*

Proof The existence of (x_1,y_1) follows from Theorem 3.1, and the existence of (x_2,y_2) follows from Theorem 3.2.

Example 3.2 The singular boundary value problem

$$\begin{cases} \Delta[\phi(\Delta x(i-1))] + \delta((\sqrt{x(i)^2+y(i)^2})^{-\alpha} + (1+(\sqrt{x(i)^2+y(i)^2})^{\beta})) = 0, \quad i \in N, \\ \Delta[\phi(\Delta y(i-1))] + \delta((\sqrt{x(i)^2+y(i)^2})^{-\alpha} + (1+(\sqrt{x(i)^2+y(i)^2})^{\beta})) = 0, \\ x(0) = y(0) = 0, \quad x(T+1) = y(T+1) = 0, \end{cases} \tag{3.39}$$

has two solutions (x_1,y_1), $(x_2,y_2) \in C(N^+,\mathbf{R}^2)$ with $(x_1,y_1) > (0,0)$, $(x_2,y_2) > (0,0)$ on N and $\|(x_1,y_1)\| < r < \|(x_2,y_2)\|$. Here $\alpha > 0$, $\beta \ge p-1$ and

$$0 < \delta < \frac{r^{p-1+\alpha}}{1+(\sqrt{2}r)^{\alpha}+(\sqrt{2}r)^{\alpha+\beta}}\Big(\frac{p-1}{b_1(p-1+\alpha)}\Big)^{p-1},\ b_1 := \sum_{t=1}^{T} t^{\frac{1}{p-1}}. \tag{3.40}$$

To see this, we will apply Theorem 3.3 with $q_k(i) = \delta$,

$$g_k(x,y) = \bar{g}_k(x,y) = (\sqrt{x^2+y^2})^{-\alpha} \quad \text{and} \quad h_k(x,y) = \bar{h}_1(x,y) = 1+(\sqrt{x^2+y^2})^{\beta},\ k=1,2.$$

Clearly (H$_1$)–(H$_4$), (H$_6$) hold. Also notice (see Example 3.1)

$$b_{k0} = \max_{i\in N}\Big(\delta^{\frac{1}{p-1}}\sum_{s=1}^{i}(i-s+1)^{\frac{1}{p-1}},\ \delta^{\frac{1}{p-1}}\sum_{s=i}^{T}(s-i+1)^{\frac{1}{p-1}}\Big) = \delta^{\frac{1}{p-1}}b_1, \quad k=1,2.$$

Consequently (H$_5$) holds, since

$$\frac{1}{\phi^{-1}(1+\frac{h_1(r,r)}{g_1(r,r)})}\int_0^r \frac{\mathrm{d}u}{\phi^{-1}(g_1(u,0))} = \frac{p-1}{p-1+\alpha}\Big(\frac{r^{p-1+\alpha}}{1+(\sqrt{2}r)^{\alpha}+(\sqrt{2}r)^{\alpha+\beta}}\Big)^{\frac{1}{p-1}} > b_{10}$$

and

$$\frac{1}{\phi^{-1}(1+\frac{h_2(r,r)}{g_2(r,r)})}\int_0^r \frac{\mathrm{d}u}{\phi^{-1}(g_2(0,v))} = \frac{p-1}{p-1+\alpha}\Big(\frac{r^{p-1+\alpha}}{1+(\sqrt{2}r)^{\alpha}+(\sqrt{2}r)^{\alpha+\beta}}\Big)^{\frac{1}{p-1}} > b_{20}.$$

Finally notice that (since $\beta > p-1$)

$$\lim_{R\to\infty}\frac{R}{\phi^{-1}\Big(\bar{g}_1(R,R)(1+\frac{\bar{h}_1(\frac{R}{T+1},\,0)}{\bar{g}_1(\frac{R}{T+1},\,0)})\Big)} = \lim_{R\to\infty}\frac{R}{\Big((\sqrt{2}R)^{-\alpha}+(\sqrt{2}(T+1))^{-\alpha}(1+(T+1)^{-\beta}R^{\beta})\Big)} = 0$$

and

$$\lim_{R\to\infty}\frac{R}{\phi^{-1}\Big(\bar{g}_2(R,R)(1+\frac{\bar{h}_2(0,\frac{R}{T+1})}{\bar{g}_2(0,\frac{R}{T+1})})\Big)} = \lim_{R\to\infty}\frac{R}{\Big((\sqrt{2}R)^{-\alpha}+(\sqrt{2}(T+1))^{-\alpha}(1+(T+1)^{-\beta}R^{\beta})\Big)} = 0.$$

So there exists $R > 1$ with (H_7) holding. The result now follows from Theorem 3.3.

References

[1] P. R. AGARWAL, D. O'REGAN. *Singular discrete boundary value problems*. Appl. Math. Lett., 1999, **12**(4): 127–131.

[2] P. R. AGARWAL, D. O'REGAN. *Boundary value problems for discrete equations*. Appl. Math. Lett., 1997, **10**(4): 83–89.

[3] P. R. AGARWAL, D. O'REGAN. *Singular discrete (n,p) boundary value problems*. Appl. Math. Lett., 1999, **12**: 113–119.

[4] P. R. AGARWAL, D. O'REGAN. *Nonpositive discrete boundary value problems*. Nonlinear Anal., 2000, **39**: 207–215.

[5] P. R. AGARWAL, D. O'REGAN. *Existence theorem for single and multiple solutions to singular positone boundary value problems*. J. Differential Equations, 2001, **175**: 393–414.

[6] P. R. AGARWAL, D. O'REGAN. *Twin solutions to singular Dirichlet problems*. J. Math. Anal. Appl., 1999, **240**(2): 433–445.

[7] P. R. AGARWAL, D. O'REGAN. *Twin solutions to singular boundary value problems*. Proc. Amer. Math. Soc., 2000, **128**(7): 2085–2094.

[8] P. R. AGARWAL, D. O'REGAN. *Multiplicity results for singular conjugate, focal, and (n,p) problems*. J. Differential Equations, 2001, **170**(1): 142–156.

[9] K. DEIMLING. *Nonlinear Functional Analysis*. Springer-Verlag, Berlin, 1985.

[10] J. HENDERSON. *Singular boundary value problems for difference equations*. Dynam. Systems Appl., 1992, **1**(3): 271–282.

[11] J. HENDERSON. *Singular Boundary Value Problems for Higher Order Difference Equations*. de Gruyter, Berlin, 1996.

[12] Daqing JIANG. *Multiple positive solutions to singular boundary value problems for superlinear higher-order ODEs*. Comput. Math. Appl., 2000, **40**(2-3): 249–259.

[13] Daqing JIANG, Lili ZHANG, D. O'REGAN, et al. *Existence theory for single and multiple solutions to singular positone discrete Dirichlet boundary value problems to the one-dimension p-Laplacian*. Arch. Math. (Brno), 2004, **40**(4): 367–381.

[14] D. O'REGAN. *Existence Theory for Nonlinear Ordinary Differential Equations*. Kluwer Academic Publishers Group, Dordrecht, 1997.

[15] Weimin HU, Daqing JIANG, Gexin LUO. *Existence theory for single and multiple solutions to singular discrete boundary value problems of second-order differential systems*. Dynam. Systems Appl., 2008, **17**(1): 221–234.

[16] Xingqiu ZHANG. *Existence of positive solutions for systems of nonlinear second-order differential equations on the half line in a Banach space*. J. Math. Res. Exposition, 2011, **31**(4): 587–604.