

Nonexistence of Iterative Roots on PM Functions

Lin LI¹, Xiaohua LIU^{2,*}, Liu LIU³

1. Department of Mathematics, Physics and Information Engineering, Jiaxing University, Zhejiang 314001, P. R. China;
2. Department of Mathematics and Information Science, Leshan Normal College, Sichuan 614004, P. R. China;
3. Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100190, P. R. China

Abstract Many results are given for iterative roots of PM functions, a class of non-monotonic continuous functions, when the characteristic interval exists. In this paper we discuss iterative roots in the opposite case and partly answer the Open Problem 1 proposed in [Ann. Polon. Math., 1997, 65(2): 119–128].

Keywords iterative root; PM function; fort.

MR(2010) Subject Classification 39B12; 37E05

1. Introduction

Given a non-empty set E and an integer $n > 0$, the iterative root of order n of a function $F : E \rightarrow E$, is a mapping $f : E \rightarrow E$ such that

$$f^n = F, \tag{1.1}$$

where f^n denotes the n th iterate of f , i.e., $f^n = f \circ f^{n-1}$ and $f^0 = \text{id}$. The problem of iterative roots is not only an important subject in the theory of functional equations but also an interesting spot to those mathematicians working in the field of dynamical systems. There have been plentiful results on iterative roots of monotone continuous functions [3, 4], but there are few results [1, 2, 5, 10, 11] on non-monotonic iterative roots. Usually, we call a point $x_0 \in I := [a, b]$ a fort of the continuous function $F : I \rightarrow I$ if F is not strictly monotone in every neighborhood of x_0 . As in [10, 11], a continuous function $F : I \rightarrow I$ is referred to as a strictly piecewise monotone function (or PM function simply) if the number $N(F)$ of forts of F is finite. Let $PM(I, I)$ consist of all PM functions from I into itself. We easily see the ascending relation

$$0 = N(F^0) \leq N(F) \leq N(F^2) \leq \dots \leq N(F^n) \leq \dots$$

in the class $PM(I, I)$. Based on the ascending relation, an important number $H(F)$ is considered naturally, which denotes the smallest positive integer k such that $N(F^k) = N(F^{k+1})$ as defined

Received September 25, 2011; Accepted September 3, 2012

Supported by the National Natural Foundation of China (Grant No. 11201184) and the Scientific Research Fund of Sichuan Provincial Education Department (Grant No. 11ZA160).

* Corresponding author

E-mail address: lsm901@163.com (Xiaohua LIU)

in [10, 11]. As known in [11], those PM functions with $H(F) \leq 1$ have “characteristic intervals” and we can find iterative roots of some of those PM functions by extension from characteristic intervals to the entire interval I ; to those PM functions with $H(F) > 1$ a result of non-existence is given as follows: Functions $F \in PM(I, I)$ with $H(F) > 1$ have no continuous iterative roots of order $n > N(F)$ (see Theorem 1 in [11]). Thus, two open problems were raised in [10, 11]:

(P1) Does $F \in PM(I, I)$ with $H(F) > 1$ have an iterative root of order n for $n \leq N(F)$?

(P2) Does $F \in PM(I, I)$ with $H(F) \leq 1$ have an iterative root of order n for $n \leq N(F) + 1$ if $F(x') = a'$ (or b') at $x' \in I \setminus [a', b']$, where $[a', b']$ is the characteristic interval of F ?

Some efforts have been made to the two problems. Problem (P2) was answered partly under the hypothesis that $\min F = a'$ and $\max F = b'$ in [5]. For Problem (P1), Sun TX discussed iterative solutions of all N and anti-N type functions on the interval $I = [0, 1]$ in [7, 8], i.e., in the case that $N(F) = n = 2$, while Zhang GY investigated the iterative roots for a class of piecewise expansion [9]. Recently, the authors generally considered the n -th iterative roots of function F in $PM(I, I)$ with $N(F) = n \geq 3$ (see [6]). In [6], two cases of iterative roots are determined if they exist, and one of them was completely solved, while another remains an open problem.

Concerning (P1) we consider the following class of PM functions defined by

$$\Phi(I) = \{F \in PM(I, I), H(F) > 1, N(F) = n > 1, F(a) = a, F(b) = b\}.$$

In this paper, unlike the partial results for general F in $PM(I, I)$ with $N(F) = n \geq 3$ in [6], the iterative root of PM function in above case is solved completely, so as to extend the known results in [10, 11].

2. Main result

We first give an auxiliary lemma.

Lemma 1 Suppose that $F \in \Phi(I)$ has a continuous iterative root f of order n . Then $N(f^i) = i$ for $i = 1, 2, \dots, n$.

Proof Assume that there is a continuous function $f : I \rightarrow I$ satisfying (1.1). Since $H(F) > 1$, we have $N(F^2) > N(F)$, i.e., $N(f^{2n}) > N(f^n)$, which implies that $H(f) > n$ and

$$1 \leq N(f) < N(f^2) < \dots < N(f^n) = N(F) = n.$$

Thus $N(f^i) = i$ for $i = 1, 2, \dots, n$. \square

Let $S(F) = \{x_1, x_2, \dots, x_n\}$, where $x_1 < x_2 < \dots < x_n$. Since $F(a) = a$ and $F(b) = b$, which implies that $N(F)$ is even and F only has iterative roots of even order n because $n = N(F)$. Furthermore, suppose that F has a continuous iterative root f of order n satisfying $f(a) = x_1, f(x_1) = a$ or $f(b) = x_n, f(x_n) = b$. Note that $N(f^i) = i$ and then $f(x_{i+1}) = x_i$ or $f(x_i) = x_{i+1}$ for $i = 1, 2, \dots, n - 1$. It follows by iterating, that $F(x_{2j-1}) = x_1$ and $F(x_{2j}) = a$ in the case $f(a) = x_1, f(x_1) = a$ or $F(x_{2j-1}) = b$ and $F(x_{2j}) = x_n$ because $f(b) = x_n, f(x_n) = b$, for $j = 1, 2, \dots, \frac{n}{2}$. Hence, we define

$$\Lambda_1(I) = \{F \in \Phi(I) : F(x_{2j-1}) = x_1, F(x_{2j}) = a\},$$

$$\Lambda_2(I) = \{F \in \Phi(I) : F(x_{2j-1}) = b, F(x_{2j}) = x_n\},$$

for $j = 1, 2, \dots, \frac{n}{2}$ and thus $\Lambda_1(I) \cap \Lambda_2(I) = \emptyset$. Moreover, we only need to consider the set of $\Lambda_1(I)$, while the discussion for $\Lambda_2(I)$ is similar. In fact, for $F \in \Lambda_1(I)$, we can find corresponding $G \in \Lambda_2(I)$ such that $F = h^{-1} \circ G \circ h$ by an involution $h(x) = a + b - x$. For the iterative roots of $\Lambda_1(I)$ and $\Lambda_2(I)$, we have the following results in [6]:

Theorem 1 (Theorem 4.1 in [6]) *Let $F \in \Lambda_1(I)$. Then F has continuous iterative roots of any even order $n > 1$ if and only if $F|_{[a, x_1]}$ has a continuous strictly decreasing iterative root of order n .*

Corollary 1 *Let $F \in \Lambda_2(I)$. Then F has continuous iterative roots of any even order $n > 1$ if and only if $F|_{[x_1, b]}$ has a continuous strictly decreasing iterative root of order n .*

Example 1 Consider $F_1 : [0, 1] \rightarrow [0, 1]$ (see Figure 1), defined by

$$F_1(x) = \begin{cases} x, & \text{as } x \in [0, \frac{1}{2}], \\ -2x + \frac{3}{2}, & \text{as } x \in (\frac{1}{2}, \frac{3}{4}], \\ 4x - 3, & \text{as } x \in (\frac{3}{4}, \frac{7}{8}], \\ -8x + \frac{15}{2}, & \text{as } x \in (\frac{7}{8}, \frac{15}{16}], \\ 16x - 15, & \text{as } x \in (\frac{15}{16}, 1]. \end{cases}$$

Obviously, $F \in \Lambda_1(I)$ with $N(F_1) = 4$. Therefore, by Theorem 1, F has a 4-th continuous iterative root $f : [0, 1] \rightarrow [0, 1]$ as follows.

$$f(x) = \begin{cases} -x + \frac{1}{2}, & \text{as } x \in [0, \frac{1}{2}], \\ 2x - 1, & \text{as } x \in (\frac{1}{2}, 1]. \end{cases}$$

Example 2 Consider $F_2 : [0, 1] \rightarrow [0, 1]$ (see Figure 2), defined by

$$F_2(x) = \begin{cases} 4x, & \text{as } x \in [0, \frac{1}{4}], \\ -2x + \frac{3}{2}, & \text{as } x \in (\frac{1}{4}, \frac{1}{2}], \\ x, & \text{as } x \in (\frac{1}{2}, 1]. \end{cases}$$

Then $F \in \Lambda_2(I)$ satisfies that $N(F_2) = 2$. Therefore, by Theorem 1, F has a 2-nd continuous iterative root $f : [0, 1] \rightarrow [0, 1]$ as follows.

$$f(x) = \begin{cases} 2x, & \text{as } x \in [0, \frac{1}{2}], \\ -x + \frac{3}{2}, & \text{as } x \in (\frac{1}{2}, 1]. \end{cases}$$

Although we have given the necessary and sufficient conditions for iterative roots in $\Lambda_1(I) \cup \Lambda_2(I)$. In what follows, we continue to study the left part, i.e., $\Phi(I) \setminus (\Lambda_1(I) \cup \Lambda_2(I))$.

Theorem 2 *Let $F \in \Phi(I) \setminus (\Lambda_1(I) \cup \Lambda_2(I))$. Then F has no continuous iterative roots of any order n .*

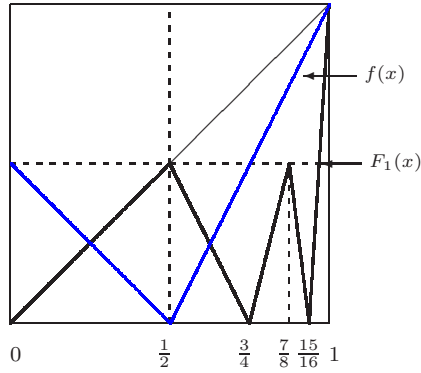


Figure 1 F_1 with $N(F_1) = 4$

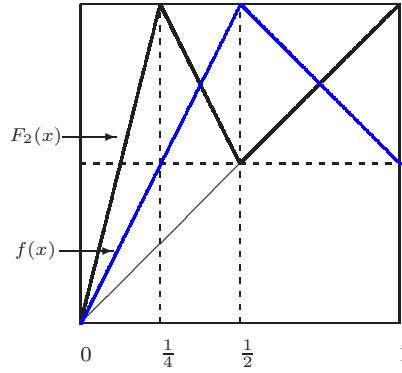


Figure 2 F_2 with $N(F_2) = 2$

Proof Suppose that $F \in \Phi(I) \setminus (\Lambda_1(I) \cup \Lambda_2(I))$ has a continuous iterative root f of order n . Obviously, n is even. Since $F(I) = I$, we have $f(I) = I$. Moreover, by Lemma 1, we have $N(f) = 1$. Let $S(f^i)$ be the set of all forts of f^i for $i \in \{1, 2, \dots, n\}$. Assume that x_1 is the unique fort of f , i.e., $S(f) = \{x_1\}$. Then there exist two possibilities: (i) $f(x_1) = a$; (ii) $f(x_1) = b$. For case (i), it can be divided into the following three subcases: (i-1) $f(a) = f(b) = b$, (i-2) $f(a) \in (a, b)$ and $f(b) = b$ and (i-3) $f(a) = b$ and $f(b) \in (a, b)$.

In subcase (i-1), by the intermediate value theorem, there must exist $x_2 \in (a, x_1)$ and $x'_2 \in (x_1, b)$ such that $f(x_2) = x_1$ and $f(x'_2) = x_1$. Consequently, $S(f^2) = \{x_1, x_2, x'_2\}$ and then $N(f^2) = 3 > 2$, which is a contradiction to Lemma 1.

In subcase (i-2). If $f(a) > x_1$, similarly to the proof on subcase (i-1), we can also infer that $N(f^2) = 3 > 2$, a contradiction to Lemma 1. If $f(a) = x_1$, then $F \in \Lambda_1(I)$, which is not in our consideration. If $f(a) < x_1$, then there exists in I a unique point, denoted by x_2 , satisfying $f(x_2) = x_1$. Clearly, $x_2 > x_1$ and $S(f^2) = \{x_1, x_2\}$. By induction, for $i = 2, \dots, n$, there exists a unique point x_i satisfying $f(x_i) = x_{i-1}$ and then $S(f^i) = \{x_1, x_2, \dots, x_i\}$ with $x_1 < x_2 < \dots < x_i$. Since $f(x_2) = x_1$, $f(x_1) = a$ and $F(a) = a$, we have

$$f(f(F(x_2))) = F(f^2(x_2)) = F(f(x_1)) = F(a) = a, \tag{2.2}$$

which implies that $f(F(x_2)) = x_1$. So $F(x_2) = x_2$ and $F(x_1) = F(f(x_2)) = f(F(x_2)) = x_1$. Thus $F(x_1) < F(x_2)$. However, it is also a contradiction. In fact, since $F(a) = a$ and $S(F) = S(f^n) = \{x_1, x_2, \dots, x_n\}$ with $x_1 < x_2 < \dots < x_n$, x_1 should be a maximum point of F and x_2 , adjacent with x_1 , be a minimum point of F , that is, $F(x_1) > F(x_2)$.

In subcase (i-3), we first claim that $f(b) \neq x_1$. Otherwise, there exist $x_3 \in (a, x_1)$ and $x'_3 \in (x_1, b)$ such that $f(x_3) = x_2$ and $f(x'_3) = x_2$, which is impossible. If $f(b) > x_1$, the discussion is similar to subcase (i-2) when $f(a) > x_1$. If $f(b) < x_1$, there also exists $x_2 \in I$ satisfying $f(x_2) = x_1$. By induction, for $i = 2, 3, \dots, n$, we have $f(x_i) = x_{i-1}$ with $x_1 > x_3 > \dots > x_4 > x_2$. Furthermore, it follows from (2.2) that $F(x_1) = x_1$ and $F(x_3) = x_3$, which is a contradiction. Actually, since $F(b) = b$ and $x_1 > x_3 > \dots > x_4 > x_2$, x_1 should be a minimum point of F and

x_3 be a maximum point of F , i.e., $F(x_1) < F(x_3)$.

Case (ii) can be proved similarly and the proof is completed. \square

As shown in Figures 3 and 4, by Theorem 2, F has no continuous iterative roots of order 4 since $F \in \Phi(I) \setminus (\Lambda_1(I) \cup \Lambda_2(I))$.

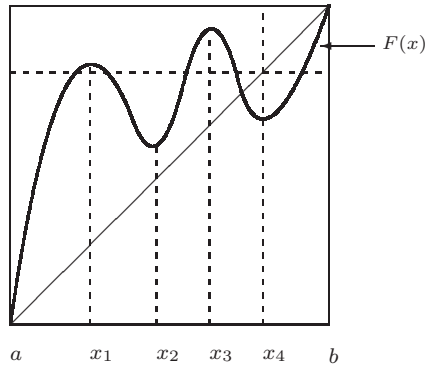


Figure 3 $F \notin (\Lambda_1(I) \cup \Lambda_2(I))$

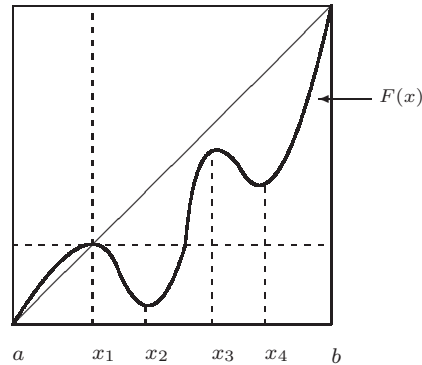


Figure 4 $F \notin (\Lambda_1(I) \cup \Lambda_2(I))$

Acknowledgment We would like to thank the referees for careful reading and many helpful suggestions.

References

- [1] K. BARON, W. JARCZYK. *Recent results on functional equations in a single variable, perspectives and open problems*. Aequationes Math., 2001, **61**(1-2): 1–48.
- [2] A. BLOKH, E. COVEN, M. MISIUREWICZ, et al. *Roots of continuous piecewise monotone maps of an interval*. Acta Math. Univ. Comenian. (N.S.), 1991, **60**(1): 3–10.
- [3] M. KUCZMA. *Functional Equations in a Single Variable*. Państwowe Wydawnictwo Naukowe, Warsaw, 1968.
- [4] M. KUCZMA, B. CHOCZEWSKI, R. GER. *Iterative Functional Equations*. Cambridge University Press, Cambridge, 1990.
- [5] Lin LI, Dilian YANG, Weinian ZHANG. *A note on iterative roots of PM functions*. J. Math. Anal. Appl., 2008, **341**(2): 1482–1486.
- [6] Liu LIU, W. JARCZYK, Lin LI, et al. *Iterative roots of piecewise monotonic functions of nonmonotonicity height not less than 2*. Nonlinear Anal., 2012, **75**(1): 286–303.
- [7] Taixiang SUN, Hongjian XI. *Iterative solutions of N-type of functions on intervals*. J. Math. Study, 1996, **29**(2): 40–45. (in Chinese)
- [8] Taixiang SUN. *Iterative roots of anti-N-type functions on intervals*. J. Math. Study, 2000, **33**(3): 274–284. (in Chinese)
- [9] Guangyuan ZHANG. *Conjugation and iterative roots of a class of self-mappings of a line segment (I)*. Chinese Ann. Math. Ser. A, 1992, **13**(1): 33–40 (in Chinese).
- [10] Jingzhong ZHANG, Lu YANG. *Discussion on iterative roots of piecewise monotone functions*. Acta Math. Sinica, 1983, **26**(4): 398–412. (in Chinese)
- [11] Weinian ZHANG. *PM functions, their characteristic intervals and iterative roots*. Ann. Polon. Math., 1997, **65**(2): 119–128.