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Some Characterizations of Chains of Archimedean Ordered Semigroups

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Abstract In this paper, the concept of semiprimary fuzzy ideals of an ordered semigroup is introduced. Some characterizations for an ordered semigroup S to be a semilattice of archimedean ordered subsemigroups are given by some binary relations on S and the fuzzy radical of fuzzy ideals of S. Furthermore, some characterizations for an ordered semigroup S to be a chain of archimedean ordered subsemigroups are also given by means of fuzzy subsets of S. In particular, by using the fuzzy prime radical theorem of ordered semigroups, we prove that an ordered semigroup S is a chain of archimedean ordered subsemigroups if and only if S is a semilattice of archimedean ordered subsemigroups and all weakly completely prime fuzzy ideals of S form a chain.

Keywords ordered fuzzy point; archimedean ordered semigroup; completely semiprime fuzzy ideal; weakly completely prime fuzzy ideal.

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1. Introduction

Let S be a nonempty set. A fuzzy subset of S is, by definition, an arbitrary mapping $f : S \longrightarrow [0, 1]$, where $[0, 1]$ is the usual interval of real numbers. The important concept of a fuzzy set put forth by Zadeh in 1965 [1] has opened up keen insights and applications in a wide range of scientific fields. Following the terminology given by Zadeh, if S is an ordered semigroup, fuzzy sets in ordered semigroups have been first considered by Kehayopulu and Tsingelis in [2], they then defined "fuzzy" analogues for several notations that have been proved to be useful in the theory of ordered semigroups. A theory of fuzzy sets on ordered semigroups has been recently developed [3–9]. The concept of ordered fuzzy points of an ordered semigroup has been first introduced by Xie and Tang [7], and prime fuzzy ideals of an ordered semigroup were studied in [8].

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As we know, the concepts of archimedean (r-archimedean, t-archimedean) semigroups have been introduced and some necessary and sufficient conditions in order that a semigroup S (without order) is a semilattice (or band) of archimedean (or r-archimedean or t-archimedean) subsemigroups of S have been obtained in [10–12]. When we pass from semigroups to ordered semigroups, the concepts of archimedean (r-archimedean, t-archimedean) ordered semigroups have been introduced in [13]. Since then, Xie [14] gave some characterizations that an ordered semigroup S is a band of weakly r-archimedean ordered subsemigroups of S by those binary relations on S and generalized some important results in [15]. In [16], the authors characterized mainly ordered semigroups in which the radical of every ideal (right ideal, bi-ideal) is an ordered subsemigroup (resp., ideal, right ideal, left ideal, bi-ideal, interior ideal) by using some binary relations on S, and they have proved that an ordered semigroup S is a semilattice of archimedean ordered subsemigroups if and only if the radical subset of every ideal of S is an ideal of S.

In this paper, we attempt to study the semilattices and chains of archimedean ordered semigroups in detail by means of fuzzy subsets, and extend some similar results of semigroups or ordered semigroups to "fuzzy" ordered semigroups. Firstly, the concept of semiprimary fuzzy ideals of an ordered semigroup S is introduced. Some characterizations for an ordered semigroup S to be a semilattice of archimedean ordered subsemigroups are given by some binary relations on S and the fuzzy radical of fuzzy ideals of S. Furthermore, some characterizations for an ordered semigroup S to be a chain of archimedean (r-archimedean, t-archimedean) ordered subsemigroups are also given by means of fuzzy subsets of S. In particular, by using the fuzzy prime radical theorem [7], we prove that an ordered semigroup S is a chain of archimedean ordered subsemigroups if and only if S is a semilattice of archimedean ordered subsemigroups and all weakly completely prime fuzzy ideals of S form a chain. As an application of the results of this paper, the corresponding results of semigroup (without order) are also obtained by moderate modifications.

2. Notations and preliminaries

Throughout this paper, we denote by Z^+ the set of all positive integers. Recall that an ordered semigroup $(S, \cdot, \leq)$ is a semigroup $(S, \cdot)$ with an order relation " $\leq$ " such that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for any $x \in S$. For convenience we use the notation $S^1 := S \cup \{1\}$, where $1a = a1 := a$ for all $a \in S$ and $1 \cdot 1 = 1$. A function f from S to the real closed interval [0,1] is a fuzzy subset of S. The ordered semigroup S itself is a fuzzy subset of S such that $S(x) \equiv 1$ for all $x \in S$. Let f and g be two fuzzy subsets of S. Then the inclusion relation $f \subseteq g$ is defined by $f(x) \leq g(x)$ for all $x \in S$, and $f \cap g$, $f \cup g$ are defined by

$$(f \cap g)(x) = \min\{f(x), g(x)\} = f(x) \wedge g(x),$$

$$(f \cup g)(x) = \max\{f(x), g(x)\} = f(x) \vee g(x)$$

for all $x \in S$, respectively. The set of all fuzzy subsets of S is denoted by $F(S)$. One can easily show that $(F(S), \subseteq, \cap, \cup)$ forms a complete lattice.

Let $(S, \cdot, \leq)$ be an ordered semigroup. For $x \in S$, we define $A_x := \{(y, z) \in S \times S | x \leq yz\}$. For $\forall f, g \in F(S)$, the product $f \circ g$ is defined by

$$(\forall x \in S)\ (f \circ g)(x) = \begin{cases} \bigvee\limits_{(y,z) \in A_x} [\min\{f(y), g(z)\}], & \text{if } A_x \neq \emptyset, \\ 0, & \text{if } A_x = \emptyset. \end{cases}$$

It is well known [3, Theorem], that this operation "$\circ$" is associative.

We denote by a_λ an ordered fuzzy point of an ordered semigroup S, where

$$a_\lambda(x) = \begin{cases} \lambda, & \text{if } x \in (a], \\ 0, & \text{if } x \notin (a]. \end{cases}$$

It is easy to see that an ordered fuzzy point of an ordered semigroup S is a fuzzy subset of S. For any fuzzy subset f of S, we also denote $a_\lambda \subseteq f$ by $a_\lambda \in f$ in sequel [7].

Definition 2.1 ([8, Definition 3]) *Let f be a fuzzy subset of S. We define $(f]$ by the rule that*

$$(f](x) = \bigvee_{y \geq x} f(y)$$

for all $x \in S$. A fuzzy subset of S is called strongly convex if $f = (f]$.

Lemma 2.2 ([8, Theorem 2]) *Let f be a strongly convex fuzzy subset of an ordered semigroup S. Then $f = \bigcup_{y_s \in f} y_s$.*

We denote by f_A the characteristic mapping of A, that is the mapping of S into $[0, 1]$ defined by

$$f_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Let S be an ordered semigroup. For $H \subseteq S$, we define

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

For $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$.

For two subsets A, B of S, we have: (1) $A \subseteq (A]$; (2) If $A \subseteq B$, then $(A] \subseteq (B]$; (3) $(A](B] \subseteq (AB]$; (4) $((A]] = (A]$; (5) $((A](B]] = (AB]$ (see [17]). A nonempty subset A of S is called a left (resp., right) ideal of S if (1) $SA \subseteq A$ (resp., $AS \subseteq A$) and (2) If $a \in A$ and $S \ni b \leq a$, then $b \in A$. If A is both a left and right ideal of S, then it is called a (two-sided) ideal of S (see [17]). We denote by $L(a)$ (resp., $R(a), I(a)$) the left (resp., right, two-sided) ideal of S generated by a ($a \in S$). Then we have $L(a) = (a \cup Sa], R(a) = (a \cup aS]$ and $I(a) = (a \cup Sa \cup aS \cup SaS]$ (see [17]).

Lemma 2.3 ([17]) *Let S be an ordered semigroup. Then the following statements hold:*

(1) For every left (resp., right, two-sided) ideal T of S, $(T] = T$.

(2) If A, B are ideals of S, then $(AB], A \cap B$ are ideals of S.

(3) $(Sa], (aS]$ and $(SaS]$ are a left ideal, a right ideal and an ideal of S, respectively, $\forall a \in S$.

Definition 2.4 ([13]) *An ordered semigroup S is called archimedean if for any $a, b \in S$, there* *exists $m \in Z^+$ such that $b^m \in (S^1aS^1]$. Equivalently, for any $a, b \in S$, there exists $m \in Z^+$ such that $b^m \leq xay$ for some $x, y \in S^1$. An ordered subsemigroup T of S is called archimedean if the ordered semigroup $(T, \cdot, \leq)$ is archimedean.*

In this paper, the following binary relation on S defined in [13] will be used frequently:

$$a\eta b \Leftrightarrow (\exists m \in Z^+)\ (\exists x, y \in S^1)\ b^m \leq xay.$$

Definition 2.5 *A subset A of an ordered semigroup S is called semiprimary if*

$$(\forall a, b \in S)\ (\exists n \in Z^+)\ ab \in A \Rightarrow a^n \in A \text{ or } b^n \in A.$$

Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called a fuzzy left (resp., right) ideal of S if (1) $f(xy) \geq f(y)$ (resp., $f(xy) \geq f(x)$) for all $x, y \in S$ and (2) $x \leq y$ implies $f(x) \geq f(y)$. Equivalent definition: (1) $S \circ f \subseteq f$ (resp., $f \circ S \subseteq f$) and (2) $x \leq y$ implies $f(x) \geq f(y)$ (see [9]). A fuzzy ideal of S is a fuzzy subset of S which is both a fuzzy left and a fuzzy right ideal of S. Let f be a fuzzy ideal of S. Then f is called weakly completely prime if for any two ordered fuzzy points $a_\lambda, b_\lambda \in S$ $(\forall \lambda \in (0,1])$, $a_\lambda \circ b_\lambda \in f$ implies $a_\lambda \in f$ or $b_\lambda \in f$; f is called completely semiprime if for any ordered fuzzy point $a_\lambda \in S$ $(\forall \lambda \in (0,1])$, $a_\lambda^2 \in f$ implies $a_\lambda \in f$ (see [7]).

Lemma 2.6 ([2]) *Let S be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then A is a left (resp., right) ideal of S if and only if the characteristic mapping f_A of A is a fuzzy left (resp., right) ideal of S.*

Lemma 2.7 ([7]) *Let a_λ, b_μ be ordered fuzzy points of S, $\lambda \neq 0, \mu \neq 0$ and f a strongly convex fuzzy subset of S. Then the following statements are true:*

(1) $(\forall x \in S)(S \circ a_\lambda \circ S)(x) = \begin{cases} \lambda, & \text{if } x \in (SaS], \\ 0, & \text{if } x \notin (SaS], \end{cases}$ *and $S \circ a_\lambda \circ S$ is a fuzzy ideal of S.*

(2) $(\forall x \in S)(a_\lambda \circ S)(x) = \begin{cases} \lambda, & \text{if } x \in (aS], \\ 0, & \text{if } x \notin (aS], \end{cases}$ *and $a_\lambda \circ S$ is a fuzzy right ideal of S.*

(3) $a_\lambda \circ b_\mu = (ab)_{\lambda \wedge \mu}$.

(4) For any $\lambda, \mu > 0$, $b_\mu \in S \circ a_\lambda \circ S$ if and only if $b \in (SaS]$, $\mu \leq \lambda$.

(5) If S is commutative, then, for every ordered fuzzy point a_λ of S, $S \circ a_\lambda = a_\lambda \circ S$.

(6) f is a fuzzy left (resp., right) ideal of S if and only if for any $a_\lambda \in S, b_\mu \in f$, implies $a_\lambda \circ b_\mu \in f$ (resp., $b_\mu \circ a_\lambda \in f$).

Definition 2.8 ([7]) *Let X be a set. If f is a fuzzy subset of X, then f is said to have sup property if every subset of $\{f(x) \mid x \in X\} = \mathrm{Im}(X)$ has a maximal element. Equivalently, if for every subset Y of X, there exists $y_0 \in Y$ such that $f(y_0) = \sup\{f(y) | y \in Y\}$.*

Let S be an ordered semigroup and f a fuzzy subset of S satisfying the sup property. The fuzzy radical of f, denoted by $\sqrt{f}$, is defined by

$$(\forall x \in S)\ \sqrt{f}(x) := \sup_{n \in Z^+} \{f(x^n)\}.$$

Lemma 2.9 ([7]) *Let f and g be any two fuzzy subsets of an ordered semigroup S. Then the following statements are true:*

(1) $f \subseteq \sqrt{f}$.

(2) If $f \subseteq g$, then $\sqrt{f} \subseteq \sqrt{g}$.

(3) If f is a strongly convex fuzzy subset of S, then $\sqrt{f}$ is also a strongly convex fuzzy subset of S.

(4) If f is a weakly completely prime fuzzy ideal of S, then $\sqrt{f} = f$.

(5) If f is a strongly convex fuzzy subset of S, then

$$(\forall a_\lambda \in S)\ a_\lambda \in \sqrt{f} \Leftrightarrow a_\lambda^n \in f \text{ for some } n \in Z^+.$$

Lemma 2.10 ([7, Corollary 8.10]) *Let S be an ordered semigroup (not necessarily commutative). Then every completely semiprime fuzzy ideal of S is the intersection of all weakly completely prime fuzzy ideals of S containing it.*

The reader is referred to [15, 18] for notation and terminology not defined in this paper.

3. Semiprimary fuzzy ideals of ordered semigroups

Definition 3.1 *A fuzzy subset f of an ordered semigroup S is called semiprimary if*

$$(\forall x, y \in S)\ (\exists n \in Z^+)\ f(xy) \leq f(x^n) \vee f(y^n).$$

Theorem 3.2 *Let S be an ordered semigroup and f a fuzzy ideal of S. Then f is semiprimary if and only if for any ordered fuzzy points $a_\lambda, b_\lambda \in S$ $(\lambda \neq 0)$, $a_\lambda \circ b_\lambda \in f$ implies $a_\lambda^n \in f$ or $b_\lambda^n \in f$ for some $n \in Z^+$.*

Proof Suppose that f is a semiprimary fuzzy ideal of an ordered senmigroup S. Let $a_\lambda, b_\lambda (\lambda \neq 0)$ be any ordered fuzzy points of S such that $a_\lambda \circ b_\lambda \in f$, i.e., $(ab)_\lambda \in f$, which implies that $f(ab) \geq \lambda$. By hypothesis, we have

$$(\exists n \in Z^+)\ f(a^n) \vee f(b^n) \geq f(ab) \geq \lambda.$$

Then we obtain that $f(a^n) \geq \lambda$ or $f(b^n) \geq \lambda$. Therefore, $a_\lambda^n \in f$ or $b_\lambda^n \in f$.

Conversely, for any $a, b \in S$, let $\lambda = f(ab)$. Then $a_\lambda \circ b_\lambda = (ab)_\lambda \in f$. Since f is a semiprimary fuzzy ideal of S, by hypothesis we have $a_\lambda^n \in f$ or $b_\lambda^n \in f$ for some $n \in Z^+$. Then $f(a^n) \geq \lambda$ or $f(b^n) \geq \lambda$, which implies that $f(a^n) \vee f(b^n) \geq f(ab)$ for some $n \in Z^+$.

Definition 3.3 *An ordered semigroup S is called a semiprimary ordered semigroup if all its fuzzy ideals are semiprimary.*

Proposition 3.4 *An ordered semigroup S is semiprimary if and only if for any $a, b \in S$, we have*

$$(ab, a) \in \eta \text{ or } (ab, b) \in \eta.$$

Proof Suppose first that S is semiprimary. Let $a, b \in S$. Since $f = S \circ a_\lambda \circ b_\lambda \circ S$ is a fuzzy ideal of S and $a_\lambda^2 \circ b_\lambda^2 \in f$, we have, by hypothesis, that

$$a_\lambda^{2n} \in f = S \circ (ab)_\lambda \circ S \text{ or } b_\lambda^{2n} \in f = S \circ (ab)_\lambda \circ S$$

for some $n \in Z^+$. By Lemma 2.7, we have $a^{2n} \in (SabS]$, or $b^{2n} \in (SabS]$, that is, $(ab, a) \in \eta$ or $(ab, b) \in \eta$.

Conversely, let $a, b \in S$ and f be any a fuzzy ideal of S. By assumption, there exist $x, y, u, v \in S^1$ such that $a^m \leq xaby$ or $b^n \leq uabv$ for some $m, n \in Z^+$. Put $t = mn$. Then $a^t \leq xab(ya^n)$ or $b^t \leq (b^m u)abv$. Since f is a fuzzy ideal of S, we have

$$f(a^t) \geq f(xab(ya^n)) \geq f(xab) \geq f(ab)$$

or

$$f(b^t) \geq f((b^m u)abv) \geq f(abv) \geq f(ab),$$

which thus implies that $f(a^t) \vee f(b^t) \geq f(ab)$ for some $t \in Z^+$.

Proposition 3.5 *Let S be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then A is a semiprimary ideal of S if and only if the characteristic mapping f_A of A is a semiprimary fuzzy ideal of S.*

Proof $\Rightarrow$. Let $x, y \in S$. Then $f_A(xy) \leq f_A(x^n) \vee f_A(y^n)$ for some $n \in Z^+$. Indeed if $xy \notin A$, then it is clear that $f_A(xy) \leq f_A(x^n) \vee f_A(y^n)$. If $xy \in A$, then, since A is a semiprimary ideal of S, we have $x^n \in A$ or $y^n \in A$ for some $n \in Z^+$, which implies that $f_A(x^n) = 1$ or $f_A(y^n) = 1$. Thus we have $f_A(xy) = 1 = f_A(x^n) \vee f_A(y^n)$.

$\Leftarrow$. For any $x, y \in S$, let $xy \in A$. Then $f_A(xy) = 1$. Since f_A is a semiprimary fuzzy ideal of S, we have

$$(\exists n \in Z^+)\ f_A(x^n) \vee f_A(y^n) \geq f_A(xy) = 1,$$

which implies that $f_A(x^n) \vee f_A(y^n) = 1$, i.e., $f_A(x^n) = 1$ or $f_A(y^n) = 1$. Therefore, $x^n \in A$ or $y^n \in A$.

The rest of the proof is a consequence of Lemma 2.6.

Definition 3.6 ([7]) *Let f be any fuzzy subset of an ordered semigroup S. Then the set*

$$f_t := \{x \in S |\ f(x) \geq t\},\ \text{where } t \in [0, 1]$$

is called a level subset of f.

Lemma 3.7 ([7]) *Let S be an ordered semigroup and f a fuzzy subset of S. Then f is a fuzzy ideal of S if and only if the level subset f_t $(t \in (0, 1])$ of f is an ideal of S for $f_t \neq \emptyset$.*

Proposition 3.8 *Let S be an ordered semigroup and f a fuzzy subset of S. Then f is a semiprimary fuzzy ideal of S if and only if the level subset f_t $(t \in (0, 1])$ of f is a semiprimary ideal of S for $f_t \neq \emptyset$.*

Proof $\Rightarrow$. Let $x, y \in S$ such that $xy \in f_t$. Then $f(xy) \geq t$. Since f is a semiprimary fuzzy ideal of S, we have

$$(\exists n \in Z^+)\ f(x^n) \vee f(y^n) \geq f(xy) \geq t,$$

which implies that $f(x^n) \geq t$ or $f(y^n) \geq t$. Therefore, $x^n \in f_t$ or $y^n \in f_t$.

$\Leftarrow$. For any $x, y \in S$, let $t = f(xy)$. Then $xy \in f_t$. By hypothesis, there exists $n \in Z^+$ such that $x^n \in f_t$ or $y^n \in f_t$, which implies that $f(x^n) \geq t$ or $f(y^n) \geq t$. It thus follows that $f(x^n) \vee f(y^n) \geq f(xy)$.

The rest of the proof is a consequence of Lemma 3.7.

Proposition 3.9 *Let S be a commutative ordered semigroup and f a fuzzy ideal of S. If for any fuzzy ideals f_1 and f_2 of S, the conditions $f_1 \circ f_2 \subseteq f$ and $f_1(x) > f(x^m)$ for some $x \in S$ and for all $m \in Z^+$ together imply that, given $y \in S$, there exists $n \in Z^+$ such that $f_2(y) \leq f(y^n)$, then f is a semiprimary fuzzy ideal of S.*

Proof Let $a, b \in S$ such that $ab \in f_t$ $(t \in (0, 1])$ and $a^m \notin f_t$, $\forall m \in Z^+$. By Lemma 3.7, f_t is an ideal of S. Define fuzzy subsets f_1 and $f_2 :\ S \to [0, 1]$ by

$$f_1(x) := \begin{cases} t, & \text{if } x \in I(a), \\ 0, & \text{if } x \notin I(a), \end{cases} \quad f_2(x) := \begin{cases} t, & \text{if } x \in I(b), \\ 0, & \text{if } x \notin I(b). \end{cases}$$

Then $f_1 \circ f_2 \subseteq f$, i.e., $(\forall x \in S)\ (f_1 \circ f_2)(x) \leq f(x)$. Indeed: If $x \notin (I(a)I(b)]$, then $(f_1 \circ f_2)(x) = 0 \leq f(x)$. If $x \in (I(a)I(b)]$, then, since S is commutative, we have

$$x \in (ab \cup abS] \subseteq (f_t \cup f_t S] \subseteq (f_t] = f_t,$$

that is, $f(x) \geq t$. Clearly, $(f_1 \circ f_2)(x) \leq t$. Therefore, $f_1 \circ f_2 \subseteq f$. Moreover, $f_1(a) = t > f(a^m)$, $\forall m \in Z^+$. By hypothesis, $t = f_2(b) \leq f(b^n)$ for some $n \in Z^+$. Hence, $b^n \in f_t$ and f_t is a semiprimary ideal of S. It thus follows, by Proposition 3.8, that f is a semiprimary fuzzy ideal of S.

4. Chains of archimedean ordered semigroups by terms of fuzzy subsets

Unless stated otherwise f in sequel will denote a fuzzy subset of S satisfying the sup property. In this section, we shall give some characterizations for an ordered semigroup S to be a chain of archimedean ordered subsemigroups by means of fuzzy subsets of S, and investigate the relations between semiprimary ordered semigroups and chains of archimedean ordered semigroups.

Lemma 4.1 ([16]) *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S is a semilattice of archimedean ordered subsemigroups.*

(2) $(\forall a, b \in S)\ (\exists m \in Z^+)\ (ab)^m \in (S^1 a^2 S^1]$, *i.e.,* $(a^2, ab) \in \eta$.

(3) *The radical of every ideal of S is an ideal of S.*

Theorem 4.2 *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S is a semilattice of archimedean ordered subsemigroups.*

(2) $(\forall a, b \in S)\ (\exists m \in Z^+)\ (ab)^m \in (S^1 a^2 S^1]$, *i.e.,* $(a^2, ab) \in \eta$.

(3) *The fuzzy radical of every fuzzy ideal of S is a fuzzy ideal of S.*

Proof (1) $\Rightarrow$ (3). Suppose that S is a semilattice Y of archimedean ordered subsemigroups S_α $(\alpha \in Y)$. Let f be a fuzzy ideal of S. Assume that $a_\lambda \in S, b_\mu \in \sqrt{f}$. Then, by Lemma 2.9, $b_\mu^k \in f$ for some $k \in Z^+$. Now let $a \in S_\alpha, b \in S_\beta$. Then we have that $ab, ab^k \in S_{\alpha\beta}$. Since $S_{\alpha\beta}$ is archimedean, and so there exists $m \in Z^+$ such that $(ab)^m \in (S^1ab^kS^1]$. It thus follows, by Lemma 2.7, that $(a_\lambda \circ b_\mu)^m \in S^1 \circ a_\lambda \circ b_\mu^k \circ S^1 \subseteq S^1 \circ f \circ S^1 \subseteq f$. Therefore, $a_\lambda \circ b_\mu \in \sqrt{f}$. In the same way we may show that $b_\mu \circ a_\lambda \in \sqrt{f}$, as required.

(3) $\Rightarrow$ (2). For any $a, b \in S, \lambda \in (0,1]$, let $f = S \circ a_\lambda^2 \circ S$. Then f is a fuzzy ideal of S and $a_\lambda \in \sqrt{f}$. By hypothesis, $\sqrt{f}$ is a fuzzy ideal of S, and so we have $a_\lambda \circ b_\lambda = (ab)_\lambda \in \sqrt{f}$, i.e., there exists $m \in Z^+$ such that $(ab)_\lambda^m \in f = S \circ a_\lambda^2 \circ S$. It thus follows, by Lemma 2.7, that $(ab)^m \in (Sa^2S]$, i.e., $(a^2, ab) \in \eta$.

(2) $\Rightarrow$ (1). This implication follows by Lemma 4.1.

Theorem 4.3 *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) S is a chain of archimedean ordered subsemigroups.

(2) S is semiprimary.

(3) $\sqrt{f}$ is a weakly completely prime fuzzy ideal of S, for every fuzzy ideal f of S.

(4) $\sqrt{f}$ is a weakly completely prime fuzzy subset of S, for every fuzzy ideal f of S.

(5) S is a semilattice of archimedean ordered subsemigroups and all weakly completely prime fuzzy ideals of S form a chain.

Proof (1) $\Rightarrow$ (2). Let S be a chain Y of archimedean ordered subsemigroups S_α $(\alpha \in Y)$. Let $x \in S_\alpha, y \in S_\beta$ $(\alpha, \beta \in Y)$. Since $\alpha \le \beta$ or $\beta \le \alpha$, and so we have $x, xy \in S_\alpha$ or $y, xy \in S_\beta$, then

$$x^m \in (S_\alpha^1 xy S_\alpha^1] \subseteq (S^1xyS^1] \text{ or } y^n \in (S_\beta^1 xy S_\beta^1] \subseteq (S^1xyS^1]$$

for some $m, n \in Z^+$. By Proposition 3.4, we obtain that S is semiprimary.

(2) $\Rightarrow$ (3). Suppose that S is a semiprimary ordered semigroup. Let $a, b \in S$. Then by Proposition 3.4 there exists $m \in Z^+$ such that $(ba)^m \in (S^1(ba)(ab)S^1]$ or $(ab)^m \in (S^1(ba)(ab)S^1]$, whence we have

$$(ab)^{m+1} \in (S^1a^2S^1]. \tag{$*$}$$

Thus it follows, by Theorem 4.2, that $\sqrt{f}$ is a fuzzy ideal of S, for every fuzzy ideal f of S. Now let $a_\lambda, b_\lambda (\lambda \ne 0)$ be any ordered fuzzy points of S such that $a_\lambda \circ b_\lambda \in \sqrt{f}$. Then $a_\lambda^2 \circ b_\lambda^2 \in S \circ a_\lambda \circ b_\lambda \circ S \subseteq S \circ \sqrt{f} \circ S \subseteq \sqrt{f}$. Since S is semiprimary and $\sqrt{f}$ is a fuzzy ideal of S, there exists $n \in Z^+$ such that $a_\lambda^{2n} \in \sqrt{f}$ or $b_\lambda^{2n} \in \sqrt{f}$. Therefore, $a_\lambda \in \sqrt{f}$ or $b_\lambda \in \sqrt{f}$, i.e., $\sqrt{f}$ is weakly completely prime.

(3) $\Rightarrow$ (4). Clearly.

(4) $\Rightarrow$ (2). Let $\sqrt{f}$ be a weakly completely prime fuzzy subset of S, for every fuzzy ideal f of S. Since $S \circ (ab)_\lambda \circ S$ $(\lambda \ne 0)$ is a fuzzy ideal of S, $\sqrt{S \circ (ab)_\lambda \circ S}$ is a weakly completely prime fuzzy subset of S, for every $a, b \in S$. Since $a_\lambda^2 \circ b_\lambda^2 \in S \circ a_\lambda \circ b_\lambda \circ S = S \circ (ab)_\lambda \circ S \subseteq \sqrt{S \circ (ab)_\lambda \circ S}$, we have that $a_\lambda^2 \in \sqrt{S \circ (ab)_\lambda \circ S}$ or $b_\lambda^2 \in \sqrt{S \circ (ab)_\lambda \circ S}$, i.e., $a_\lambda^{2m} \in S \circ (ab)_\lambda \circ S$ or $b_\lambda^{2n} \in S \circ (ab)_\lambda \circ S$ for some $m, n \in Z^+$. By Lemma 2.7, we have $a^{2m} \in (SabS]$ or $b^{2n} \in (SabS]$. It thus follows, by Proposition 3.4, that S is a semiprimary ordered semigroup.

(2) $\Rightarrow$ (1). Let S be a semiprimary ordered semigroup. Then by (2) $\Rightarrow$ (3) we have shown that the condition $(*)$ holds and by Theorem 4.2, S is a semilattice Y of archimedean ordered subsemigroups S_α $(\alpha \in Y)$. Let $\alpha, \beta \in Y$. Suppose that $a \in S_\alpha, b \in S_\beta$. Then, by Proposition 3.4, there exist $m, n \in Z^+$ such that

$$a^m \in (S^1abS^1] \text{ or } b^n \in (S^1abS^1],$$

which implies $\alpha \leq \beta$ or $\beta \leq \alpha$. Thus, Y is a chain.

(3) $\Rightarrow$ (5). By (3) and Theorem 4.2, S is a semilattice of archimedean ordered subsemigroups. Let P_1 and P_2 be weakly completely prime fuzzy ideals of S. Suppose that $P_1 \nsubseteq P_2$ and $P_2 \nsubseteq P_1$. Then there exist $a_\lambda \in P_1 \backslash P_2$ and $b_\lambda \in P_2 \backslash P_1$, whence $a_\lambda \circ b_\lambda \in P_1 \cap P_2 \subseteq \sqrt{P_1 \cap P_2}$, and by hypothesis, $a_\lambda \in \sqrt{P_1 \cap P_2}$ or $b_\lambda \in \sqrt{P_1 \cap P_2}$. By Lemma 2.9(4), $a_\lambda \in P_1 \cap P_2$ or $b_\lambda \in P_1 \cap P_2$, which is a contradiction. Thus, $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$, which means that all weakly compltely prime fuzzy ideals of S form a chain.

(5) $\Rightarrow$ (3). By Theorem 4.2 we have that $\sqrt{f}$ is a fuzzy ideal of S, for every fuzzy ideal f of S. Clearly, $\sqrt{f}$ is a completely semiprime fuzzy ideal of S. Then we have, by Lemma 2.10, that $\sqrt{f}$ is the intersection of all weakly completely prime fuzzy ideals of S containing it. Now we shall show that $\sqrt{f}$ is the intersection of all weakly completely prime fuzzy ideals of S containing f. Indeed: Let P be a weakly completely prime fuzzy ideal of S containing f. Then, by Lemma 2.9, $\sqrt{f} \subseteq \sqrt{P} = P$. Therefore, $\sqrt{f}$ is the intersection of all weakly completely prime fuzzy ideals of S containing f. Let $\sqrt{f} = \bigcap P_i$, where P_i are weakly completely prime fuzzy ideals of S containing f. Assume that $a_\lambda \notin \bigcap P_i, b_\lambda \notin \bigcap P_i$. Then there exist P_j, P_k such that $a_\lambda \notin P_j$ and $b_\lambda \notin P_k$. By hypothesis, $P_j \subseteq P_k$ or $P_k \subseteq P_j$. If $P_j \subseteq P_k$ (the case $P_k \subseteq P_j$ can be similarly treated), then $a_\lambda \notin P_j$ and $b_\lambda \notin P_j$, which implies, since P_j is weakly completely prime, that $a_\lambda \circ b_\lambda \notin P_j$. Hence, $a_\lambda \circ b_\lambda \notin \bigcap P_i$. We have thus shown that $\sqrt{f} = \bigcap P_i$ is weakly completely prime, as required.

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