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The Isomorphism Theorem of $*$-Bisimple Type A ω^2-Semigroups

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Abstract In this paper we study the isomorphisms of two $*$-bisimple type A ω^2-semigroups such that $\mathcal{D}^* = \widetilde{D}$ and obtain a criterion for isomorphisms of two such semigroups.

Keywords Type A semigroup; $*$-bisimple ω^2-semigroup; generalized Bruck-Reilly $*$-extension; isomorphism.

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1. Preliminaries

Earlier investigations in [1] studied $*$-bisimple type A ω^2-semigroups whose equivalence D^* and $\widetilde{\mathcal{D}}$ coincide, characterizing them as the generalized Bruck-Reilly $*$-extensions of cancellative monoids. The results of [1] generalize those of regular bisimple ω^2-semigroups. In this paper, we give necessary and sufficient conditions for two $*$-bisimple type A ω^2-semigroups with $\mathcal{D}^* = \widetilde{D}$ to be isomorphic. We complete this section with a summary of notions of type A semigroups, the details of which can be found in [1], [2] and [3].

For any semigroup S we shall denote by E_S the set of idempotents of S. Let S be a semigroup whose set E_S is non-empty. We define a partial ordering $\geq$ on E_S by the rule that $e \geq f$ if and only if $ef = f = fe$. Let N^0 denote the set of all non-negative integers and N denote the set of all positive integers. We define a partially order on $N^0 \times N^0$ in the following manner: if $(m,n),(p,q) \in N^0 \times N^0$,

$$(m,n) \leq (p,q) \text{ if and only if } m > p \text{ or, } m = p \text{ and } n \geq q.$$

The set $N^0 \times N^0$ with the above partially order is called an ω^2-chain, and denoted by C_{ω^2}. Any partially ordered set order isomorphic to C_{ω^2} is also called an ω^2-chain. We say that a semigroup S is an ω^2-semigroup if and only if E_S is order isomorphic to C_{ω^2}. Thus, if S is an ω^2-semigroup, then we can write

$$E_S = \{e_{m,n} : m, n \in N^0\},$$

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where $e_{m,n} \le e_{p,q}$ if and only if $(m,n) \le (p,q)$.

Let S be a semigroup. Let $a, b \in S$ such that for all $x, y \in S^1$, $ax = ay$ if and only if $bx = by$. Then a, b are said to be $\mathcal{L}^*$-equivalent and written $a\mathcal{L}^*b$. Dually, $a\mathcal{R}^*b$ if for all $x, y \in S^1$, $xa = ya$ if and only if $xb = yb$. If S has an idempotent e, then the following characterisation is known.

Lemma 1.1 ([3]) *Let S be a semigroup and e an idempotent in S. Then the following statements are equivalent:*

(i) $e\mathcal{L}^*a$;

(ii) $ae = a$ *and for all* $x, y \in S^1$, $ax = ay$ *implies* $ex = ey$.

By duality, a similar statement holds for $\mathcal{R}^*$. A semigroup in which each $\mathcal{L}^*$- and each $\mathcal{R}^*$-class contains an idempotent is called an abundant semigroup [2]. The join of the equivalence relations $\mathcal{L}^*$ and $\mathcal{R}^*$ is denoted by $\mathcal{D}^*$ and their intersection by $\mathcal{H}^*$. Thus $a\mathcal{H}^*b$ if and only if $a\mathcal{L}^*b$ and $a\mathcal{R}^*b$. In general $\mathcal{L}^* \circ \mathcal{R}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$ and neither equals $\mathcal{D}^*$. Basically, $a\mathcal{D}^*b$ if and only if there exist $x_1, x_2, \ldots, x_{2n-1}$ in S such that $a\mathcal{L}^*x_1\mathcal{R}^*x_2\mathcal{L}^* \cdots \mathcal{L}^*x_{2n-1}\mathcal{R}^*b$. Let H^* be an $\mathcal{H}^*$-class in a semigroup S with $e \in H^*$, where e is an idempotent in S. Then H^* is a cancellative monoid. Denote by $\mathcal{R}, \mathcal{L}$ the left and right Green's relations respectively, on S. Evidently $\mathcal{L} \subseteq \mathcal{L}^*, \mathcal{R} \subseteq \mathcal{R}^*$, and so, $\mathcal{D} \subseteq \mathcal{D}^*, \mathcal{H} \subseteq \mathcal{H}^*$. If S is a regular semigroup, then $\mathcal{L}^* = \mathcal{L}, \mathcal{R}^* = \mathcal{R}$. An $\mathcal{L}^*$-class containing an element $a \in S$ will be denoted by L_a^*. Similarly R_a^* is an $\mathcal{R}^*$-class with an element $a \in S$. To avoid ambiguity we at times denote a relation $\mathcal{K}$ on S by $\mathcal{K}(S)$. Let S be a semigroup with a semilattice E of idempotents. Then S is called a right adequate semigroup if each $\mathcal{L}^*$-class of S contains an idempotent. Dually, we have the notion of a left adequate semigroup. If S is a right (left) adequate semigroup, then each $\mathcal{L}^*$-($\mathcal{R}^*$-) class of S contains a unique idempotent. For an element a of such a semigroup, the idempotent in the $\mathcal{L}^*$-($\mathcal{R}^*$-)class containing a will be denoted by $a^*(a^+)$. A semigroup which is both left and right adequate will be called an adequate semigroup. A right (left) adequate semigroup S is called a right (left) type A semigroup if $ea = a(ea)^*(ae = (ae)^+a)$ for all elements a in S and all idempotents e in S. An adequate semigroup S is type A if it is both right and left type A. Let S be a type A semigroup with a semilattice of idempotents E. Then S is called type A ω^2-semigroup if E is an ω^2-chain. Thus in a type A ω^2-semigroup $E_S = \{e_{m,n} : m, n \in N\}$ and $e_{m,n} \ge e_{p,q}$ if and only if $(m,n) \ge (p,q)$. In such a semigroup S, we will denote by $L_{m,n}^*$ (resp. $R_{m,n}^*$) the $\mathcal{L}^*$-class (resp. $\mathcal{R}^*$-class) containing idempotent $e_{m,n}$. That is

$$R_{m,n}^* = \{a \in S : a\mathcal{R}^*e_{m,n}\},$$

$$L_{p,q}^* = \{a \in S : a\mathcal{L}^*e_{p,q}\}.$$

Let $H_{(m,n),(q,p)}^*$ denote the $R_{m,n}^* \cap L_{p,q}^*$. That is

$$H_{(m,n),(q,p)}^* = \{a \in S : a\mathcal{R}^*e_{m,n},\ a\mathcal{L}^*e_{p,q}\}.$$

If $H_{(m,n),(q,p)}^* \neq \emptyset$, evidently $H_{(m,n),(q,p)}^*$ is an $\mathcal{H}^*$-class of S. Also, observe that $a^+ = e_{m,n}$ and $a^* = e_{p,q}$. If S is a type A ω^2-semigroup, then S is $*$-bisimple if and only if it has a single

$\mathcal{D}^*$-class. Let S be a type A semigroup, and let $a, b \in S$. The relation $\widetilde{\mathcal{D}}$ is defined on S by

$$a\widetilde{\mathcal{D}}b \text{ if and only if } a^*\mathcal{D}b^* \text{ and } a^+\mathcal{D}b^+ \text{ for } a^*, b^*, a^+, b^+ \in E(S).$$

$\widetilde{\mathcal{D}}$ is an equivalence relation and satisfies the inclusion $\mathcal{D} \subseteq \widetilde{\mathcal{D}} \subseteq \mathcal{D}^*$ on a type A semigroup S (see [4]).

Lemma 1.2 ([5]) *Let S be an adequate semigroup. The following conditions are equivalent:*

(i) $\mathcal{D}^* = \widetilde{\mathcal{D}}$;

(ii) Every nonempty $\mathcal{H}^*$-class contains a regular element.

Furthermore, if (i) or (ii) holds, then $\mathcal{D}^* = \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{R}^* \circ \mathcal{L}^*$.

In [1], Shang and Wang introduced the generalized Bruck-Reilly $*$-extension. Consider a monoid T with H_e^* and H_e as the $\mathcal{H}^*$- and $\mathcal{H}$-class which contains the identity e of T, respectively. Let β, γ be two homomorphisms from T into H_e^*. Let u be an element in H_e and τ_u be the inner automorphism of H_e^* defined by $x \to uxu^{-1}$ such that

$$\gamma\tau_u = \beta\gamma. \tag{1.1}$$

We can make $S = N^0 \times N^0 \times T \times N^0 \times N^0$ into a semigroup by defining

$$(m, n, a, q, p)(m', n', a', q', p')$$
$$= \begin{cases} (m, n - q + \max(q, n'), a\beta^{\max(q,n')-q}a'\beta^{\max(q,n')-n'}, q' - n' + \max(q, n'), p') & \text{if } p = m' \\ (m, n, a(u^{-n'}a'\gamma u^{q'})\gamma^{p-m'-1}\beta^q, q, p' - m' + p) & \text{if } p > m' \\ (m - p + m', n', (u^{-n}a\gamma u^q)\gamma^{m'-p-1}\beta^{n'}a', q', p') & \text{if } p < m' \end{cases}$$

where β^0, γ^0 are interpreted as the identity map of T and u^0 is interpreted as the identity e of T. Then S is a semigroup with identity $(0, 0, e, 0, 0)$. The semigroup $S = N^0 \times N^0 \times T \times N^0 \times N^0$ constructed above will be called the generalized Bruck-Reilly $*$-extension of T determined by β, γ, u and will be denoted by $S = GBR^*(T; \beta, \gamma; u)$. Let $(m, n, a, q, p) \in S$. Then (m, n, a, q, p) is an idempotent if and only if $m = p$, $n = q$ and a is an idempotent.

Lemma 1.3 ([1]) *Let $S = GBR^*(T; \beta, \gamma; u)$ be a generalized Bruck-Reilly $*$-extension of a monoid T determined by β, γ, u. Suppose that (m, n, a, q, p) and (m', n', a', q', p') are elements in S. Then*

(i) $(m, n, a, q, p)\mathcal{L}^*(S)(m', n', a', q', p')$ if and only if $q = q'$, $p = p'$ and $a\mathcal{L}^*(T)a'$.

(ii) $(m, n, a, q, p)\mathcal{R}^*(S)(m', n', a', q', p')$ if and only if $m = m'$, $n = n'$ and $a\mathcal{R}^*(T)a'$.

Lemma 1.4 ([1]) *Let $S = GBR^*(T; \beta, \gamma; u)$. Then an element (m, n, a, q, p) in S has an inverse $(x, y, b, z, w) \in S$ if and only if b is the inverse of a in T, $x = p, y = q, z = n$ and $w = m$.*

Lemma 1.5 ([1]) *Let M be a cancellative monoid with identity e and $S = GBR^*(M; \beta, \gamma; u)$ the generalized Bruck-Reilly $*$-extension of M determined by β, γ, u, where $\beta : M \to H_e^*, \gamma : M \to H_e^*$, $u \in H_e$ and H_e^*, H_e are the $\mathcal{H}^*$-class and $\mathcal{H}$-class of M containing the identity e of M, respectively. Then $S = GBR^*(M; \beta, \gamma; u)$ is a $*$-bisimple type A ω^2-semigroup such that $\mathcal{D}^*(S) = \widetilde{D}(S)$. Conversely, every $*$-bisimple type A ω^2-semigroup such that $\mathcal{D}^* = \widetilde{D}$ is* *isomorphic to some* $GBR^*(M;\beta,\gamma;u)$.

2. The isomorphism theorem

In this section, we give necessary and sufficient conditions for two $*$-bisimple type A ω^2-semigroups with $\mathcal{D}^*=\widetilde{D}$ to be isomorphic. Let S, and S' be $*$-bisimple type A ω^2-semigroups satisfying $\mathcal{D}^*=\widetilde{D}$. Suppose that $\sigma: S\to S'$ is a mapping between them. We prove the following theorem.

Theorem 2.1 *Let* $S=GBR^*(M;\beta,\gamma;u)$ *and* $S'=GBR^*(M';\beta',\gamma';u')$ *be* $*$*-bisimple type* A ω^2*-semigroups such that* $\mathcal{D}^*=\widetilde{D}$. *Then* $S\cong S'$ *if and only if there exist isomorphism* α *of* M *onto* M' *and two inner automorphisms* τ_v *and* τ_w *of* M' *such that*

$$\beta\alpha=\alpha\beta'\tau_w,\quad \gamma\alpha=\alpha\gamma'\tau_v,\quad u\alpha=(w\gamma'u')\tau_v \tag{2.1}$$

where v *and* w *are two units of* M'.

Proof Let σ be an isomorphism of S onto S'. Then σ must induce a one-to-one order preserving mapping of E_S onto $E_{S'}$. Thus $(m,n,e,n,m)\sigma=(m,n,e',n,m)$, for all m and n in N^0, where we have denoted the identities of both M and M' by e and e', respectively.

Let $s=(m,n,x,q,p)\in S$ and $s\sigma=(m,n,x,q,p)\sigma=(i,j,y,l,k)=r$. Then

$$s^*\sigma=(s\sigma)^*=r^*,\quad s^+\sigma=(s\sigma)^+=r^+,$$

which implies that $(p,q,e,q,p)\sigma=(k,l,e',l,k)$, and $(m,n,e,n,m)\sigma=(i,j,e',j,i)$; consequently $p=k, q=l, m=i$ and $n=j$.

Now $H^*_{(0,0),(0,0)}(S)\sigma=H^*_{(0,0),(0,0)}(S')$ so that $M\cong M'$ where $M=H^*_{(0,0),(0,0)}(S)$ and $M'=H^*_{(0,0),(0,0)}(S')$. Denote the isomorphism between M and M' by α. By definition then

$$(0,0,x,0,0)\sigma=(0,0,x\alpha,0,0).$$

As $(0,0,e,1,0)$ is a regular element in S, so is its image $(0,0,e,1,0)\sigma$ in S'. Suppose that $(0,0,e,1,0)\sigma=(0,0,w,1,0)$. Then w is evidently a unit in M' and $(0,1,e,0,0)\sigma=(0,1,w^{-1},0,0)$. Thus for all $x\in M$,

$$(0,1,x\beta,1,0)\sigma=(0,1,e,1,0)\sigma(0,0,x,0,0)\sigma=(0,1,e',1,0)(0,0,x\alpha,0,0)=(0,1,x\alpha\beta',1,0).$$

Also

$$\begin{aligned}(0,1,x\beta,1,0)\sigma&=(0,1,e,0,0)\sigma(0,0,x\beta,0,0)\sigma(0,0,e,1,0)\sigma\\&=(0,1,w^{-1},0,0)(0,0,x\beta\alpha,0,0)(0,0,w,1,0)\\&=(0,1,w^{-1}x\beta\alpha,0,0)(0,0,w,1,0)\\&=(0,1,w^{-1}x\beta\alpha w,1,0).\end{aligned}$$

Therefore, for all $x\in M$, $x\alpha\beta'=w^{-1}x\beta\alpha w$ and hence

$$x\beta\alpha=wx\alpha\beta'w^{-1}=x\alpha\beta'\tau_w,$$

where $\tau_w : M' \to M'$ is the automorphism defined by $y\tau_w = wyw^{-1}$. Thus we obtain

$$\beta\alpha = \alpha\beta'\tau_w.$$

Similarly, suppose that $(0,0,e,0,1)\sigma = (0,0,v,0,1)$ for some $v \in M'$. Then v is evidently a unit in M' and $(1,0,e,0,0)\sigma = (1,0,v^{-1},0,0)$. Thus, for all x in M, we have

$$\begin{aligned}(1,0,x\alpha\gamma',0,1) &= (1,0,e',0,1)(0,0,x\alpha,0,0)\\ &= (1,0,e,0,1)\sigma(0,0,x,0,0)\sigma = (1,0,x\gamma,0,1)\sigma\\ &= (1,0,e,0,0)\sigma(0,0,x\gamma,0,0)\sigma(0,0,e,0,1)\sigma\\ &= (1,0,v^{-1},0,0)(0,0,x\gamma\alpha,0,0)(0,0,v,0,1)\\ &= (1,0,v^{-1}x\gamma\alpha,0,0)(0,0,v,0,1)\\ &= (1,0,v^{-1}x\gamma\alpha v,0,1).\end{aligned}$$

Hence, for all elements x of M, we have $x\alpha\gamma' = v^{-1}x\gamma\alpha v$; that is $vx\alpha\gamma' v^{-1} = x\gamma\alpha$. Thus we obtain

$$\gamma\alpha = \alpha\gamma'\tau_v.$$

Now,

$$\begin{aligned}(0,0,u\alpha,0,0) &= (0,0,u,0,0)\sigma = (0,0,e,0,1)\sigma(0,0,e,1,0)\sigma(1,0,e,0,0)\sigma\\ &= (0,0,v,0,1)(0,0,w,1,0)(1,0,v^{-1},0,0)\\ &= (0,0,vw\gamma'u',0,1)(1,0,v^{-1},0,0)\\ &= (0,0,vw\gamma'u'v^{-1},0,0)\end{aligned}$$

and so $u\alpha = vw\gamma'u'v^{-1}$. Thus $u\alpha = (w\gamma'u')\tau_v$.

Conversely, suppose that there exists an isomorphism α of M onto M' such that $\beta\alpha = \alpha\beta'\tau_w$, $\gamma\alpha = \alpha\gamma'\tau_v$ and $u\alpha = (w\gamma'u')\tau_v$ for some unit w of M' and some unit v of M'.

For each positive integer m, let $w_m = w(w\beta')\cdots(w\beta'^{m-1})$ and $v_m = v(v\gamma')\cdots(v\gamma'^{m-1})$. Then $w_m^{-1} = (w^{-1}\beta'^{m-1})\cdots(w^{-1}\beta')w^{-1}$ and $v_m^{-1} = (v^{-1}\gamma'^{m-1})\cdots(v^{-1}\gamma')v^{-1}$. Let $w_0 = v_0 = e'$. Note that $w_1 = w$ and $v_1 = v$. We claim that

$$\beta^i\alpha = \alpha\beta'^i\tau_{w_i} \tag{2.2}$$

where $i \in N^0$. Clearly this is true for $i = 0$. Suppose that it is true for $i = m$, and let $x \in M'$. Then

$$x\tau_w\beta'^m\tau_{w_m} = (wxw^{-1})\beta'^m\tau_{w_m} = (w\beta'^m x\beta'^m w^{-1}\beta'^m)\tau_{w_m} = x\beta'^m\tau_{w_{m+1}}$$

and so $\tau_w\beta'^m\tau_{w_m} = \beta'^m\tau_{w_{m+1}}$. Thus, by virtue of (2.1), we have

$$\beta^{m+1}\alpha = \beta\alpha\beta'^m\tau_{w_m} = \alpha\beta'\tau_w\beta'^m\tau_{w_m} = \alpha\beta'^{m+1}\tau_{w_{m+1}}$$

and so (2.2) holds by induction. Similarly, we have that

$$\gamma^i\alpha = \alpha\gamma'^i\tau_{v_i} \tag{2.3}$$

where $i \in N^0$.

We note that

$$u^r\alpha = (w_r\gamma' u'^r)\tau_v \tag{2.4}$$

where $r \in N^0$. Clearly this is true for $r = 0$. Assume that it holds for $r = n$. Then, by (1.1), we have $\gamma'\tau_{u'} = \beta'\gamma'$ and so

$$u'^n w\gamma' = u'^{n-1} w\beta'\gamma' u' = u'^{n-2} w\beta'^2\gamma' u'^2 = \cdots = w\beta'^n\gamma' u'^n$$

and

$$\begin{aligned} u^{n+1}\alpha &= u^n\alpha u\alpha = (w_n\gamma' u'^n)\tau_v(w\gamma' u')\tau_v = (w_n\gamma' u'^n w\gamma' u')\tau_v \\ &= (w_n\gamma' w\beta'^n\gamma' u'^n u')\tau_v = (w_{n+1}\gamma' u'^{n+1})\tau_v. \end{aligned}$$

Hence the result holds for $r = n+1$ and so, by induction, it holds for all positive non-negative r.

Write $h = (0,0,w,1,0)$. As w is a unit in M', then $(0,0,w,1,0),(0,1,w^{-1},0,0)$ must be mutually inverse regular elements of S' and so $h^{-1} = (0,1,w^{-1},0,0)$. Let $k = (0,0,v,0,1)$. Similarly, $(0,0,v,0,1),(1,0,v^{-1},0,0)$ must be mutually inverse regular elements of S' and so $k^{-1} = (1,0,v^{-1},0,0)$. Write $h^0 = (0,0,e',0,0)$ and $k^0 = (0,0,e',0,0)$. A straightforward calculation shows that

$$\begin{aligned} h^n &= (0,0,w_n,n,0), \quad h^{-n} = (0,n,w_n^{-1},0,0), \\ k^n &= (0,0,v_n,0,n), \quad k^{-n} = (n,0,v_n^{-1},0,0) \end{aligned} \tag{2.5}$$

for all non-negative integers n.

Let σ be a mapping from S into S', defined by

$$(m,n,x,q,p)\sigma = k^{-m}h^{-n}(0,0,x\alpha,0,0)h^q k^p.$$

Thus, applying (2.5), we see that

$$(m,n,x,q,p)\sigma = (m,n,(v_m^{-1}\beta'^n w_n^{-1})x\alpha(w_q v_p\beta'^q),q,p)$$

and so σ is a bijection.

Let $(m,n,x,q,p),(m',n',x',q',p') \in S$. We shall show that

$$(m,n,x,q,p)\sigma(m',n',x',q',p')\sigma = [(m,n,x,q,p)(m',n',x',q',p')]\sigma. \tag{2.6}$$

It is convenient to consider separately the following five cases.

Case I $p = m'$ and $q = n'$. In this case, we have

$$\begin{aligned} &(m,n,x,q,p)\sigma(m',n',x',q',p')\sigma \\ &\quad = k^{-m}h^{-n}(0,0,x\alpha,0,0)h^q k^p k^{-m'}h^{-n'}(0,0,x'\alpha,0,0)h^{q'}k^{p'} \\ &\quad = k^{-m}h^{-n}(0,0,(xx')\alpha,0,0)h^{q'}k^{p'} = (m,n,xx',q',p')\sigma \\ &\quad = [(m,n,x,q,p)(m',n',x',q',p')]\sigma. \end{aligned}$$

Case II $p = m'$ and $q > n'$. In this case, by (2.2), we have

$$w_{q-n'}x'\alpha\beta'^{q-n'} = x'\beta^{q-n'}\alpha w_{q-n'}.$$

Thus, using (2.5), we obtain

$$\begin{aligned}
&(m,n,x,q,p)\sigma(m',n',x',q',p')\sigma \\
&\quad = k^{-m}h^{-n}(0,0,x\alpha,0,0)h^{q}k^{p}k^{-m'}h^{-n'}(0,0,x'\alpha,0,0)h^{q'}k^{p'} \\
&\quad = k^{-m}h^{-n}(0,0,x\alpha,0,0)(0,0,w_{q-n'},q-n',0)(0,0,x'\alpha,0,0)h^{q'}k^{p'} \\
&\quad = k^{-m}h^{-n}(0,0,x\alpha w_{q-n'},q-n',0)(0,0,x'\alpha,0,0)h^{q'}k^{p'} \\
&\quad = k^{-m}h^{-n}(0,0,x\alpha w_{q-n'}x'\alpha\beta'^{q-n'},q-n',0)h^{q'}k^{p'} \\
&\quad = k^{-m}h^{-n}(0,0,x\alpha x'\beta^{q-n'}\alpha w_{q-n'},q-n',0)h^{q'}k^{p'} \\
&\quad = k^{-m}h^{-n}(0,0,(xx'\beta^{q-n'})\alpha,0,0)(0,0,w_{q-n'},q-n',0)h^{q'}k^{p'} \\
&\quad = k^{-m}h^{-n}(0,0,(xx'\beta^{q-n'})\alpha,0,0)h^{q'-n'+q}k^{p'} \\
&\quad = (m,n,xx'\beta^{q-n'},q'-n'+q,p')\sigma \\
&\quad = [(m,n,x,q,p)(m',n',x',q',p')]\sigma.
\end{aligned}$$

Case III $p=m'$ and $q<n'$. In this case, by (2.2), we have

$$w_{n'-q}^{-1}x\beta^{n'-q}\alpha = x\alpha\beta'^{n'-q}w_{n'-q}^{-1}.$$

Thus, using (2.5), we obtain

$$\begin{aligned}
&(m,n,x,q,p)\sigma(m',n',x',q',p')\sigma \\
&\quad = k^{-m}h^{-n}(0,0,x\alpha,0,0)h^{q}k^{p}k^{-m'}h^{-n'}(0,0,x'\alpha,0,0)h^{q'}k^{p'} \\
&\quad = k^{-m}h^{-n}(0,0,x\alpha,0,0)(0,n'-q,w_{n'-q}^{-1},0,0)(0,0,x'\alpha,0,0)h^{q'}k^{p'} \\
&\quad = k^{-m}h^{-n}(0,n'-q,x\alpha\beta'^{n'-q}w_{n'-q}^{-1},0,0)(0,0,x'\alpha,0,0)h^{q'}k^{p'} \\
&\quad = k^{-m}h^{-n}(0,n'-q,x\alpha\beta'^{n'-q}w_{n'-q}^{-1}x'\alpha,0,0)h^{q'}k^{p'} \\
&\quad = k^{-m}h^{-n}(0,n'-q,w_{n'-q}^{-1}x\beta^{n'-q}\alpha x'\alpha,0,0)h^{q'}k^{p'} \\
&\quad = k^{-m}h^{-n}(0,n'-q,w_{n'-q}^{-1},0,0)(0,0,(x\beta^{n'-q}x')\alpha,0,0)h^{q'}k^{p'} \\
&\quad = k^{-m}h^{-(n+n'-q)}(0,0,(x\beta^{n'-q}x')\alpha,0,0)h^{q'}k^{p'} \\
&\quad = (m,n+n'-q,x\beta^{n'-q}x',q',p')\sigma \\
&\quad = [(m,n,x,q,p)(m',n',x',q',p')]\sigma.
\end{aligned}$$

Case IV $p>m'$. Utilizing (2.2), (2.3) and (2.4), it is easy to see that $(u^{-n'}x'\gamma u^{q'})\alpha = (u^{n'}\alpha)^{-1}x'\gamma\alpha u^{q'}\alpha = vu'^{-n'}(w_{n'}^{-1}x'\alpha w_{q'})\gamma' u'^{q'}v^{-1}$ and so

$$\begin{aligned}
&(u^{-n'}x'\gamma u^{q'})\alpha\gamma'^{p-m'-1}\beta'^{q}\tau_{w_q(v_{p-m'-1}\beta'^{q})} \\
&\quad = [vu'^{-n'}(w_{n'}^{-1}x'\alpha w_{q'})\gamma' u'^{q'}v^{-1}]\gamma'^{p-m'-1}\beta'^{q}\tau_{w_q(v_{p-m'-1}\beta'^{q})} \\
&\quad = \{v\gamma'^{p-m'-1}\beta'^{q}[u'^{-n'}(w_{n'}^{-1}x'\alpha w_{q'})\gamma' u'^{q'}]\gamma'^{p-m'-1}\beta'^{q}v^{-1}\gamma'^{p-m'-1}\beta'^{q}\}\tau_{w_q(v_{p-m'-1}\beta'^{q})} \\
&\quad = [u'^{-n'}(w_{n'}^{-1}x'\alpha w_{q'})\gamma' u'^{q'}]\gamma'^{p-m'-1}\beta'^{q}\tau_{v\gamma'^{p-m'-1}\beta'^{q}}\tau_{w_q(v_{p-m'-1}\beta'^{q})} \\
&\quad = [u'^{-n'}(w_{n'}^{-1}x'\alpha w_{q'})\gamma' u'^{q'}]\gamma'^{p-m'-1}\beta'^{q}\tau_{w_q(v_{p-m'}\beta'^{q})}.
\end{aligned}$$

By virtue of (2.2), (2.3) and (2.4) we have

$$
\begin{aligned}
&(u^{-n'}x'\gamma u^{q'})\gamma^{p-m'-1}\beta^q\alpha w_q(v_{p-m'}\beta'^q)\\
&\quad= (u^{-n'}x'\gamma u^{q'})\gamma^{p-m'-1}\alpha\beta'^q\tau_{w_q}w_q(v_{p-m'}\beta'^q)\\
&\quad= (u^{-n'}x'\gamma u^{q'})\alpha\gamma'^{p-m'-1}\tau_{v_{p-m'-1}}\beta'^q\tau_{w_q}w_q(v_{p-m'}\beta'^q)\\
&\quad= (u^{-n'}x'\gamma u^{q'})\alpha\gamma'^{p-m'-1}\beta'^q\tau_{w_q(v_{p-m'-1}\beta'^q)}w_q(v_{p-m'}\beta'^q)\\
&\quad= [u'^{-n'}(w_{n'}^{-1}x'\alpha w_{q'})\gamma' u'^{q'}]\gamma'^{p-m'-1}\beta'^q\tau_{w_q(v_{p-m'}\beta'^q)}w_q(v_{p-m'}\beta'^q)\\
&\quad= w_q(v_{p-m'}\beta'^q)(u'^{-n'}w_{n'}^{-1}\gamma' u'^0)\gamma'^{p-m'-1}\beta'^q\\
&\qquad x'\alpha\gamma'\gamma'^{p-m'-1}\beta'^q(u'^0 w_{q'}\gamma' u'^{q'})\gamma'^{p-m'-1}\beta'^q
\end{aligned}
$$

and so

$$
\begin{aligned}
&x\alpha w_q(v_{p-m'}\beta'^q)(u'^{-n'}w_{n'}^{-1}\gamma')\gamma'^{p-m'-1}\beta'^q x'\alpha\gamma'^{p-m'}\beta'^q(w_{q'}\gamma' u'^{q'})\gamma'^{p-m'-1}\beta'^q\\
&\quad= (x(u^{-n'}x'\gamma u^{q'})\gamma^{p-m'-1}\beta^q)\alpha w_q(v_{p-m'}\beta'^q).
\end{aligned}
$$

Thus, by (2.5), we obtain

$$
\begin{aligned}
&(m,n,x,q,p)\sigma(m',n',x',q',p')\sigma\\
&\quad= k^{-m}h^{-n}(0,0,x\alpha,0,0)h^q k^{p-m'}h^{-n'}(0,0,x'\alpha,0,0)h^{q'}k^{p'}\\
&\quad= k^{-m}h^{-n}(0,0,x\alpha,0,0)(0,0,w_q,q,0)(0,0,v_{p-m'},0,p-m')(0,n',w_{n'}^{-1},0,0)\\
&\qquad (0,0,x'\alpha,0,0)h^{q'}k^{p'}\\
&\quad= k^{-m}h^{-n}(0,0,x\alpha w_q,q,0)(0,0,v_{p-m'},0,p-m')(0,n',w_{n'}^{-1},0,0)(0,0,x'\alpha,0,0)h^{q'}k^{p'}\\
&\quad= k^{-m}h^{-n}(0,0,x\alpha w_q(v_{p-m'})\beta'^q,q,p-m')(0,n',w_{n'}^{-1},0,0)(0,0,x'\alpha,0,0)h^{q'}k^{p'}\\
&\quad= k^{-m}h^{-n}(0,0,x\alpha w_q(v_{p-m'})\beta'^q(u'^{-n'}w_{n'}^{-1}\gamma' u'^0)\gamma'^{p-m'-1}\beta'^q,q,p-m')\\
&\qquad (0,0,x'\alpha,0,0)h^{q'}k^{p'}\\
&\quad= k^{-m}h^{-n}(0,0,x\alpha w_q(v_{p-m'})\beta'^q(u'^{-n'}w_{n'}^{-1}\gamma' u'^0)\gamma'^{p-m'-1}\beta'^q x'\alpha\gamma'\gamma'^{p-m'-1}\beta'^q,q,p-m')\\
&\qquad (0,0,w_{q'},q',0)k^{p'}\\
&\quad= k^{-m}h^{-n}(0,0,x\alpha w_q(v_{p-m'}\beta'^q)(u'^{-n'}w_{n'}^{-1}\gamma')\gamma'^{p-m'-1}\beta'^q x'\alpha\gamma'^{p-m'}\beta'^q\\
&\qquad (w_{q'}\gamma' u'^{q'})\gamma'^{p-m'-1}\beta'^q,q,p-m')k^{p'}\\
&\quad= k^{-m}h^{-n}(0,0,(x(u^{-n'}x'\gamma u^{q'})\gamma^{p-m'-1}\beta^q)\alpha w_q(v_{p-m'}\beta'^q),q,p-m')k^{p'}\\
&\quad= k^{-m}h^{-n}(0,0,(x(u^{-n'}x'\gamma u^{q'})\gamma^{p-m'-1}\beta^q)\alpha w_q,q,0)(0,0,v_{p-m'},0,p-m')k^{p'}\\
&\quad= k^{-m}h^{-n}(0,0,(x(u^{-n'}x'\gamma u^{q'})\gamma^{p-m'-1}\beta^q)\alpha,0,0)(0,0,w_q,q,0)(0,0,v_{p-m'},0,p-m')k^{p'}\\
&\quad= k^{-m}h^{-n}(0,0,(x(u^{-n'}x'\gamma u^{q'})\gamma^{p-m'-1}\beta^q)\alpha,0,0)h^q k^{p-m'}k^{p'}\\
&\quad= (m,n,x(u^{-n'}x'\gamma u^{q'})\gamma^{p-m'-1}\beta^q,q,p+p'-m')\sigma\\
&\quad= [(m,n,x,q,p)(m',n',x',q',p')]\sigma.
\end{aligned}
$$

Case V $p < m'$. Utilizing (2.2), (2.3) and (2.4), it is easy to see that

$$
(u^{-n}x\gamma u^q)\alpha = (u^n\alpha)^{-1}x\gamma\alpha u^q\alpha = vu'^{-n}(w_n^{-1}x\alpha w_q)\gamma' u'^q v^{-1}
$$

and so

$$\begin{aligned}
&(u^{-n}x\gamma u^q)\alpha\gamma'^{m'-p-1}\beta'^{n'}\tau_{w_{n'}(v_{m'-p-1}\beta'^{n'})}\\
&\quad=[vu'^{-n}(w_n^{-1}x\alpha w_q)\gamma'u'^qv^{-1}]\gamma'^{m'-p-1}\beta'^{n'}\tau_{w_{n'}(v_{m'-p-1}\beta'^{n'})}\\
&\quad=\{v\gamma'^{m'-p-1}\beta'^{n'}[u'^{-n}(w_n^{-1}x\alpha w_q)\gamma'u'^q]\gamma'^{m'-p-1}\beta'^{n'}v^{-1}\gamma'^{m'-p-1}\beta'^{n'}\}\tau_{w_{n'}(v_{m'-p-1}\beta'^{n'})}\\
&\quad=[u'^{-n}(w_n^{-1}x\alpha w_q)\gamma'u'^q]\gamma'^{m'-p-1}\beta'^{n'}\tau_{v\gamma'^{m'-p-1}\beta'^{n'}}\tau_{w_{n'}(v_{m'-p-1}\beta'^{n'})}\\
&\quad=[u'^{-n}(w_n^{-1}x\alpha w_q)\gamma'u'^q]\gamma'^{m'-p-1}\beta'^{n'}\tau_{w_{n'}(v_{m'-p}\beta'^{n'})}.
\end{aligned}$$

By virtue of (2.2), (2.3) and (2.4) we have

$$\begin{aligned}
&v_{m'-p}^{-1}\beta'^{n'}w_{n'}^{-1}(u^{-n}x\gamma u^q)\gamma^{m'-p-1}\beta^{n'}\alpha\\
&\quad=v_{m'-p}^{-1}\beta'^{n'}w_{n'}^{-1}(u^{-n}x\gamma u^q)\gamma^{m'-p-1}\alpha\beta'^{n'}\tau_{w_{n'}}\\
&\quad=v_{m'-p}^{-1}\beta'^{n'}w_{n'}^{-1}(u^{-n}x\gamma u^q)\alpha\gamma'^{m'-p-1}\tau_{v_{m'-p-1}}\beta'^{n'}\tau_{w_{n'}}\\
&\quad=v_{m'-p}^{-1}\beta'^{n'}w_{n'}^{-1}(u^{-n}x\gamma u^q)\alpha\gamma'^{m'-p-1}\beta'^{n'}\tau_{w_{n'}(v_{m'-p-1}\beta'^{n'})}\\
&\quad=v_{m'-p}^{-1}\beta'^{n'}w_{n'}^{-1}[u'^{-n}(w_n^{-1}x\alpha w_q)\gamma'u'^q]\gamma'^{m'-p-1}\beta'^{n'}\tau_{w_{n'}(v_{m'-p}\beta'^{n'})}\\
&\quad=[u'^{-n}(w_n^{-1}x\alpha w_q)\gamma'u'^q]\gamma'^{m'-p-1}\beta'^{n'}v_{m'-p}^{-1}\beta'^{n'}w_{n'}^{-1}
\end{aligned}$$

and so

$$\{[u'^{-n}(w_n^{-1}x\alpha w_q)\gamma'u'^q]\gamma'^{m'-p-1}v_{m'-p}^{-1}\}\beta'^{n'}w_{n'}^{-1}x'\alpha=v_{m'-p}^{-1}\beta'^{n'}w_{n'}^{-1}((u^{-n}x\gamma u^q)\gamma^{m'-p-1}\beta^{n'}x')\alpha.$$

Thus, by (2.5), we obtain

$$\begin{aligned}
&(m,n,x,q,p)\sigma(m',n',x',q',p')\sigma\\
&\quad=k^{-m}h^{-n}(0,0,x\alpha,0,0)h^qk^{-(m'-p)}h^{-n'}(0,0,x'\alpha,0,0)h^{q'}k^{p'}\\
&\quad=k^{-m}(0,n,w_n^{-1},0,0)(0,0,x\alpha,0,0)(0,0,w_q,q,0)(m'-p,0,v_{m'-p}^{-1},0,0)(0,n',w_{n'}^{-1},0,0)\\
&\qquad(0,0,x'\alpha,0,0)h^{q'}k^{p'}\\
&\quad=k^{-m}(0,n,w_n^{-1}x\alpha,0,0)(0,0,w_q,q,0)(m'-p,0,v_{m'-p}^{-1},0,0)(0,n',w_{n'}^{-1},0,0)\\
&\qquad(0,0,x'\alpha,0,0)h^{q'}k^{p'}\\
&\quad=k^{-m}(0,n,w_n^{-1}x\alpha w_q,q,0)(m'-p,0,v_{m'-p}^{-1},0,0)(0,n',w_{n'}^{-1},0,0)\\
&\qquad(0,0,x'\alpha,0,0)h^{q'}k^{p'}\\
&\quad=k^{-m}(m'-p,0,[u'^{-n}(w_n^{-1}x\alpha w_q)\gamma'u'^q]\gamma'^{m'-p-1}\beta'^0v_{m'-p}^{-1},0,0)(0,n',w_{n'}^{-1},0,0)\\
&\qquad(0,0,x'\alpha,0,0)h^{q'}k^{p'}\\
&\quad=k^{-m}(m'-p,n',\{[u'^{-n}(w_n^{-1}x\alpha w_q)\gamma'u'^q]\gamma'^{m'-p-1}v_{m'-p}^{-1}\}\beta'^{n'}w_{n'}^{-1},0,0)\\
&\qquad(0,0,x'\alpha,0,0)h^{q'}k^{p'}\\
&\quad=k^{-m}(m'-p,n',\{[u'^{-n}(w_n^{-1}x\alpha w_q)\gamma'u'^q]\gamma'^{m'-p-1}v_{m'-p}^{-1}\}\beta'^{n'}w_{n'}^{-1}x'\alpha,0,0)h^{q'}k^{p'}\\
&\quad=k^{-m}(m'-p,n',v_{m'-p}^{-1}\beta'^{n'}w_{n'}^{-1}((u^{-n}x\gamma u^q)\gamma^{m'-p-1}\beta^{n'}x')\alpha,0,0)h^{q'}k^{p'}\\
&\quad=k^{-m}(m'-p,n',v_{m'-p}^{-1}\beta'^{n'}w_{n'}^{-1},0,0)(0,0,((u^{-n}x\gamma u^q)\gamma^{m'-p-1}\beta^{n'}x')\alpha,0,0)h^{q'}k^{p'}\\
&\quad=k^{-m}(m'-p,0,v_{m'-p}^{-1},0,0)(0,n',w_{n'}^{-1},0,0)(0,0,((u^{-n}x\gamma u^q)\gamma^{m'-p-1}\beta^{n'}x')\alpha,0,0)h^{q'}k^{p'}
\end{aligned}$$

$$= k^{-(m+m'-p)}h^{-n'}(0,0,((u^{-n}x\gamma u^{q})\gamma^{m'-p-1}\beta^{n'}x')\alpha,0,0)h^{q'}k^{p'}$$
$$= (m+m'-p, n', (u^{-n}x\gamma u^{q})\gamma^{m'-p-1}\beta^{n'}x', q', p')\sigma$$
$$= [(m,n,x,q,p)(m',n',x',q',p')]\sigma.$$

This completes the proof. □

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