

Class-Preserving Coleman Automorphisms of Finite Groups Whose Second Maximal Subgroups Are TI-Subgroups

Zhengxing LI*, Jinke HAI

College of Mathematics, Qingdao University, Shandong 266071, P. R. China

Abstract Recall that a subgroup H of a finite group G is called a TI-subgroup if $H \cap H^g = 1$ or H for each $g \in G$. Suppose that G is a finite group whose second maximal subgroups are TI-subgroups. It is shown that every class-preserving Coleman automorphism of G is an inner automorphism. As an immediate consequence of this result, we obtain that the normalizer property holds for G .

Keywords normalizer property; Coleman automorphism; class-preserving automorphism.

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1. Introduction

Let G be a finite group and σ be an automorphism of G . If σ maps every element $g \in G$ to some conjugate of g , then σ is called a class-preserving automorphism of G ; if the restriction of σ to every Sylow subgroup of G equals the restriction of some inner automorphism of G , then σ is called a Coleman automorphism of G ; if σ is both a class-preserving automorphism and a Coleman automorphism, then σ is called a class-preserving Coleman automorphism. All class-preserving automorphisms of G form a subgroup of $\text{Aut}(G)$, denoted by $\text{Aut}_c(G)$. All Coleman automorphisms of G also form a subgroup of $\text{Aut}(G)$, denoted by $\text{Aut}_{\text{Col}}(G)$. Obviously, $\text{Inn}(G) \leq \text{Aut}_c(G)$ and $\text{Inn}(G) \leq \text{Aut}_{\text{Col}}(G)$. Write $\text{Out}_c(G) := \text{Aut}_c(G)/\text{Inn}(G)$ and $\text{Out}_{\text{Col}}(G) := \text{Aut}_{\text{Col}}(G)/\text{Inn}(G)$, respectively.

The interest in class-preserving Coleman automorphisms of finite groups arose from the study of the normalizer problem for integral group rings. Denote by $\mathbb{Z}G$ the integral group ring of G over \mathbb{Z} , the ring of all rational integers. Let $U(\mathbb{Z}G)$ be the group of units of $\mathbb{Z}G$. Denote by $Z(U(\mathbb{Z}G))$ the center of $U(\mathbb{Z}G)$ and by $N_{U(\mathbb{Z}G)}(G)$ the normalizer of G in $U(\mathbb{Z}G)$. Obviously, $N_{U(\mathbb{Z}G)}(G) \geq G \cdot Z(U(\mathbb{Z}G))$. A question arising naturally is whether $N_{U(\mathbb{Z}G)}(G) = G \cdot Z(U(\mathbb{Z}G))$ for any finite group G . If the equality holds for G , then we will say that G has the normalizer property. This normalizer problem was first formulated by Sehgal [1]. Many positive results on this problem can be found in [2–6]. Let $u \in N_{U(\mathbb{Z}G)}(G)$. Then u induces an automorphism of G

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* Corresponding author

E-mail address: lzxlws@163.com (Zhengxing LI); haijinke2002@yahoo.com.cn (Jinke HAI)

by sending g to $u^{-1}gu$ for each $g \in G$. All such automorphisms of G form a subgroup of $\text{Aut}(G)$, denoted by $\text{Aut}_{\mathbb{Z}}(G)$. Clearly, $\text{Inn}(G) \leq \text{Aut}_{\mathbb{Z}}(G)$. Write $\text{Out}_{\mathbb{Z}}(G) := \text{Aut}_{\mathbb{Z}}(G)/\text{Inn}(G)$. It is well known that the normalizer property holds for G if and only if $\text{Out}_{\mathbb{Z}}(G) = 1$. In addition, it is known that $\text{Out}_{\mathbb{Z}}(G) \leq \text{Out}_c(G) \cap \text{Out}_{\text{Col}}(G)$. Thus if one can show that $\text{Out}_c(G) \cap \text{Out}_{\text{Col}}(G) = 1$, then $\text{Out}_{\mathbb{Z}}(G) = 1$ and thus the normalizer property holds for G . This is the starting point of studying class-preserving Coleman automorphisms of finite groups.

The aim of the present paper is to study class-preserving Coleman automorphisms of finite groups whose second maximal subgroups are TI-subgroups. Recall that a subgroup H of a finite group G is said to be second maximal if there exists a maximal subgroup M of G such that $H \leq M$ and H is maximal in M . Our main result is as follows.

Main Theorem *Let G be a finite group all of whose second maximal subgroups are TI-subgroups. Then class-preserving Coleman automorphisms of G are inner automorphisms, i.e., $\text{Out}_c(G) \cap \text{Out}_{\text{Col}}(G) = 1$.*

2. Notation and preliminaries

In this section, we present some results which will be used in the sequel section. First, we fix some notation. Through this paper, groups under consideration are finite. Let σ be an automorphism of a group G , $H \leq G$ and N be a normal subgroup of G . We shall write $\sigma|_H$ for the restriction of σ to H . If σ fixes N , then σ induces an automorphism of the quotient group G/N , denoted by $\sigma|_{G/N}$. Other notation used in this paper is mostly standard, refer to [7].

Lemma 2.1 ([2, Theorem 2]) *If G is a non-nilpotent group all of whose second maximal subgroups are TI-subgroups, then one of the following holds:*

- (a) $G \cong S_4$, the symmetric group of degree 4;
- (b) $G = \text{PSL}(2, 5)$;
- (c) $G = PQ$ is a minimal non-abelian group, where P is a cyclic Sylow p -subgroup of G and Q is a normal Sylow q -subgroup of order q of G with p and q distinct primes.
- (d) G is a non-abelian group of order pq^2 or pqr , where p , q and r are distinct primes.
- (e) $G = KH$ is a Frobenius group with kernel K elementary abelian and with complement H , each maximal subgroup of H acts irreducibly on K , and H is either cyclic or the direct product of a cyclic group of odd order with the quaternion group Q_8 of order 8.

Lemma 2.2 ([3, Theorem 14]) *For any finite simple group G , there is a prime p dividing $|G|$ such that p -central automorphisms of G are inner automorphisms.*

Lemma 2.3 ([4, Proposition 2.7]) *Let G be a finite group having an abelian normal subgroup A with cyclic quotient G/A . Then class-preserving automorphisms of G are inner automorphisms.*

Lemma 2.4 ([5, Theorem C]) *Let G be a finite group whose Sylow subgroups of odd order are all cyclic, and whose Sylow 2-subgroups are either cyclic, dihedral, or generalized quaternion. Then class-preserving automorphisms of G are inner automorphisms.*

Lemma 2.5 ([5, Corollary 2]) *Let G be an A -group (i.e., solvable groups whose Sylow subgroups are abelian) with elementary abelian Sylow subgroups. Then Class-preserving automorphisms of G are inner automorphisms.*

Lemma 2.6 ([3, Corollary 3]) *Let N be a normal subgroup of a finite group G and let p be a prime which does not divide the order of G/N . Then the following hold:*

- (1) *If σ is a Coleman automorphism of G of p -power order, then σ induces a Coleman automorphism of N ;*
- (2) *If $\text{Out}_{\text{Col}}(N)$ is a p' -group, then so is $\text{Out}_{\text{Col}}(G)$.*

Lemma 2.7 ([6, Proposition 3.1]) *Suppose that G is a finite solvable group. If G has a cyclic Sylow p -subgroup, then $\text{Out}_c(G) \cap \text{Out}_{\text{Col}}(G)$ is a p' -group.*

Lemma 2.8 ([7, Theorem]) *If the Sylow 2-subgroups of a finite group G are dihedral, or generalized quaternion, then $\text{Out}_c(G) \cap \text{Out}_{\text{Col}}(G)$ is a $2'$ -group.*

Lemma 2.9 ([3, Proposition 1]) *The prime divisors of $|\text{Aut}_c(G)|$ and $|\text{Aut}_{\text{Col}}(G)|$ lie in $\pi(G)$, the set of prime divisors of $|G|$.*

3. Proof of main theorem

In this section, we are in position to prove Main Theorem. For reader's convenience, we rewrite Main Theorem here as

Theorem 3.1 *Let G be a finite group all of whose second maximal subgroups are TI-subgroups. Then class-preserving Coleman automorphisms of G are inner automorphisms, i.e., $\text{Out}_c(G) \cap \text{Out}_{\text{Col}}(G) = 1$.*

Proof Let G be a finite group all of whose second maximal subgroups are TI-subgroups. We first consider the case where G is a nilpotent group. Without loss of generality, we may assume that $G = P_1 \times P_2 \times \cdots \times P_r$, where P_i 's are the Sylow p_i -subgroups of G . Let $\sigma \in \text{Aut}_c(G) \cap \text{Aut}_{\text{Col}}(G)$ be an arbitrary class-preserving Coleman automorphism of G . We have to show that σ is an inner automorphism of G . Since σ is a Coleman automorphism of G , there exists $x_i \in G$ such that $\sigma|_{P_i} = \text{conj}(x_i)|_{P_i}$. Since G is nilpotent, we may pick $x_i \in P_i$ for each P_i with $1 \leq i \leq r$. Write $x := x_1 x_2 \cdots x_r$. Then for any $g = g_1 g_2 \cdots g_r \in G$ with $g_i \in P_i$, we have $\sigma(g) = \sigma(g_1) \sigma(g_2) \cdots \sigma(g_r) = g_1^{x_1} g_2^{x_2} \cdots g_r^{x_r} = g^x$, which implies that $\sigma = \text{conj}(x)$. Hence $\text{Aut}_c(G) \cap \text{Aut}_{\text{Col}}(G) \subseteq \text{Inn}(G)$. It follows that $\text{Out}_c(G) \cap \text{Out}_{\text{Col}}(G) = 1$.

Hereafter we assume that G is a non-nilpotent group. In view of Lemma 2.1, we may complete the proof by considering the following cases, respectively.

Case 1 $G \cong S_4$, the symmetric group of degree 4.

Since G is a complete group in this case, it follows that $\text{Aut}(G) = \text{Inn}(G)$ and thus $\text{Out}(G) = 1$. In particular, $\text{Out}_c(G) \cap \text{Out}_{\text{Col}}(G) = 1$.

Case 2 $G = \text{PSL}(2, 5)$.

Since G is a simple group in this case, it follows from Lemma 2.2 that there exists a prime p dividing $|G|$ such that p -central automorphisms of G are inner automorphisms. Let $\sigma \in \text{Aut}_c(G) \cap \text{Aut}_{\text{Col}}(G)$ be an arbitrary class-preserving Coleman automorphism of G . We have to show that σ is an inner automorphism of G . Let P be a Sylow p -subgroup of G with p being the prime mentioned above. Since σ is definitely a Coleman automorphism of G , there exists some $x \in G$ such that $\sigma|_P = \text{conj}(x)|_P$, or equivalently, $\text{conj}(x^{-1})\sigma|_P = \text{id}|_P$. This is to say that $\text{conj}(x^{-1})\sigma$ is a p -central automorphism of G . Hence $\text{conj}(x^{-1})\sigma \in \text{Inn}(G)$, which yields $\sigma \in \text{Inn}(G)$. Since σ is arbitrary, we have $\text{Out}_c(G) \cap \text{Out}_{\text{Col}}(G) = 1$.

Case 3 $G = PQ$ is a minimal non-abelian group, where P is a cyclic Sylow p -subgroup of G and Q is a normal Sylow q -subgroup of order q of G with p and q distinct primes.

Note that $G/Q \cong P$ in this case. Since both P and Q are cyclic, it follows from Lemma 2.3 that class-preserving automorphisms of G are inner automorphisms, that is, $\text{Out}_c(G) = 1$. In particular, $\text{Out}_c(G) \cap \text{Out}_{\text{Col}}(G) = 1$.

Case 4 G is a non-abelian group of order pq^2 or pqr , where p, q and r are distinct primes.

If $|G| = pqr$ with p, q and r distinct primes, then it is clear that the Sylow subgroups of G are all cyclic. It follows from Lemma 2.4 that class-preserving automorphisms of G are inner automorphisms. That is, $\text{Out}_c(G) = 1$. In particular, $\text{Out}_c(G) \cap \text{Out}_{\text{Col}}(G) = 1$.

Now we assume that $|G| = pq^2$ with p and q distinct primes. By the well-known Burnside's theorem, G is a solvable group. Further, since any q -group of order q^2 must be an abelian group, it follows that G has an abelian Sylow q -subgroup. In addition, it is clear that G has a cyclic Sylow p -subgroup by Sylow's theorem. Hence G must be an A-group. Let Q be a Sylow q -subgroup of G . Then Q is either a cyclic group or an elementary abelian q -group since $|Q| = q^2$. If Q is cyclic, then by Lemma 2.4 all class-preserving automorphisms of G are inner automorphisms. In particular, $\text{Out}_c(G) \cap \text{Out}_{\text{Col}}(G) = 1$. If Q is an elementary abelian q -group, then by Lemma 2.5 all class-preserving automorphisms of G are inner automorphisms, i.e., $\text{Out}_c(G) = 1$. In particular, $\text{Out}_c(G) \cap \text{Out}_{\text{Col}}(G) = 1$.

Case 5 $G = KH$ is a Frobenius group with kernel K elementary abelian and with complement H , each maximal subgroup of H acts irreducibly on K , and H is either cyclic or the direct product of a cyclic group of odd order with the quaternion group Q_8 of order 8.

Assume that H is a cyclic group. Then by Lemma 2.3 every class-preserving automorphism of G is an inner automorphism, i.e., $\text{Out}_c(G) = 1$. It follows that $\text{Out}_c(G) \cap \text{Out}_{\text{Col}}(G) = 1$. From now on assume that H is the direct product of a cyclic group of odd order with the quaternion group Q_8 of order 8. Since G is a Frobenius group, it follows that $|K|$ and $|H|$ are relatively prime. Thus for any prime p dividing $|G|$, either $p \in \pi(K)$ or $p \in \pi(H)$. If $p \in \pi(K)$, then $p = q$ and thus the Sylow p -subgroup K of G is normal in G . Then, by Lemma 2.6, $\text{Out}_c(G) \cap \text{Out}_{\text{Col}}(G)$ is a p' -group. If $p \in \pi(H)$ and $p \neq 2$, then it is easy to see that the Sylow p -subgroup of G is cyclic. By Lemma 2.7, $\text{Out}_c(G) \cap \text{Out}_{\text{Col}}(G)$ is a p' -group. If $p \in \pi(H)$ and

$p = 2$, then the Sylow 2-subgroup of G is a quaternion 2-group Q_8 of order 8. Thus, by Lemma 2.8, $\text{Out}_c(G) \cap \text{Out}_{\text{Col}}(G)$ is a $2'$ -group. As $p \in \pi(G)$ is arbitrary, it follows from Lemma 2.9 that $\text{Out}_c(G) \cap \text{Out}_{\text{Col}}(G) = 1$. This completes the proof of Main Theorem. \square

Corollary 3.2 *Let G be a finite group all of whose second maximal subgroups are TI-subgroups. Then the normalizer property holds for G .*

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