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Rings in which Every Element Is A Left Zero-Divisor

Yanli REN[1], **Yao WANG**[2,*]
1. *School of Mathematics and Information Technology, Nanjing Xiaozhuang University, Jiangsu* 211171*, P. R. China;*
2. *School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Jiangsu* 210044*, P. R. China*

Abstract We introduce the concepts of left (right) zero-divisor rings, a class of rings without identity. We call a ring R left (right) zero-divisor if $r_R(a) \neq 0$ $(l_R(a) \neq 0)$ for every $a \in R$, and call R strong left (right) zero-divisor if $r_R(R) \neq 0$ $(l_R(R) \neq 0)$. Camillo and Nielson called a ring right finite annihilated (RFA) if every finite subset has non-zero right annihilator. We present in this paper some basic examples of left zero-divisor rings, and investigate the extensions of strong left zero-divisor rings and RFA rings, giving their equivalent characterizations.

Keywords zero-divisor; left zero-divisor ring; strong left zero-divisor ring; RFA ring; extensions of rings.

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1. Some examples of left zero-divisor rings

Throughout this paper rings are general associative rings (with or without identity), $\mathbb{Z}$ denotes the ring of integers and $\mathbb{N}$ denotes the set of positive integers. Given a ring R, the right (left) annihilator of a subset X of R is defined by $r_R(X) = \{a \in R \mid Xa = 0\}$ $(l_R(X) = \{a \in R \mid aX = 0\})$, the polynomial ring over R in one indeterminate x is denoted by $R[x]$.

Definition 1.1 *A ring R is called left (right) zero-divisor if $r_R(a) \neq 0$ $(l_R(a) \neq 0)$ for every $a \in R$, and a ring R is called zero-divisor if it is both left and right zero-divisor.*

Obviously, any non-zero nil ring is zero-divisor; and rings with identity are never left (right) zero-divisor. If R is reversible (a ring R is called *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$.), then R is left zero-divisor if and only if R is right zero-divisor. In general, a left (right) zero-divisor ring need not be a nil ring and the zero-divisor property for a ring is not left-right symmetric.

Proposition 1.2 *If one of $\{R_i\}_{i\in W}$ is left zero-divisor, so is $R = \bigoplus_{i\in W} R_i$ $(R = \prod_{i\in W} R_i)$.*

Note that $R = \bigoplus_{i\in W} R_i$ $(R = \prod_{i\in W} R_i)$ is left zero-divisor does not imply that every R_i $(i \in W)$ is left zero-divisor.

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* Corresponding author
E-mail address: wangyao@nuist.edu.cn (Yao WANG)

For any ring R, we define $QM_2(R) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a+b=c+d, a,b,c,d \in R\right\}$, then $QM_2(R)$ is a subring of $M_2(R)$. Moreover, given an (R,R)-bimodule M, the trivial extension of R by M (see [4]) is the ring $T(R,M) = R\bigoplus M$ with the usual addition and the following multiplication:

$$(r_1,m_1)(r_2,m_2) = (r_1r_2,\ r_1m_2+m_1r_2).$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Theorem 1.3 *The following statements are equivalent for a ring R:*

(1) R is left zero-divisor.

(2) For any $n \in \mathbb{N}$, the ring $T_n(R)$ of $n \times n$ upper triangular matrices over R is left zero-divisor.

(3) $QM_2(R)$ is left zero-divisor.

(4) For any $n \in \mathbb{N}$, $S_n(R) = \left\{\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-3} \\ \cdots & & \cdots & & \cdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \mid a_i \in R, i=0,1,\dots,n-1\right\}$ is left zero-divisor.

(5) For any $n \in \mathbb{N}$, $R[x]/(x^n)$ is left zero-divisor, where (x^n) is the ideal generated by x^n.

(6) $T(R,R)$ is left zero-divisor.

Proof (1) $\Rightarrow$ (2). Assume that R is left zero-divisor and $A = (a_{ij}) \in T_n(R)$, where $a_{ij} = 0$ if $i > j$. Then there exists $0 \neq t_{ii} \in R$ such that $a_{ii}t_{ii} = 0$ for any $i, 1 \leq i \leq n$. Taking $D = (d_{ij})$, where $d_{11} = t_{11} \neq 0, d_{ij} = 0, 1 < i,j \leq n$, we get $0 \neq D \in T_n(R)$ such that $AD = 0$. Hence $T_n(R)$ is left zero-divisor.

(2) $\Rightarrow$ (3). We construct a map $f: QM_2(R) \to T_2(R)$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+b & b \\ 0 & d-b \end{pmatrix}$, for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in QM_2(R)$. It is easy to verify that f is an injective and a ring homomorphism. For any $\begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \in T_2(R)$, since

$$f\left(\begin{pmatrix} x-z & z \\ x-y-z & y+z \end{pmatrix}\right) = \begin{pmatrix} x & z \\ 0 & y \end{pmatrix},$$

f is a ring isomorphism. This completes the proof by (2).

(3) $\Rightarrow$ (1). Let $r \in R$. Then $A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \in QM_2(R)$. Since $QM_2(R)$ is left zero-

divisor, there exists $0 \neq T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in QM_2(R)$ such that $AT = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix} = 0$, it follows that $ra = rb = rc = rd = 0$. Notice that $T \neq 0$, there must be $0 \neq s \in R$ such that $rs = 0$, as desired.

(1) $\Rightarrow$ (4). Let $A = (a_{ij}) \in S_n(R)$, where $a_{ii} = a_0, 1 \leq i \leq n$. Since R is left zero-divisor, there exists $0 \neq t_0 \in R$ such that $a_0 t_0 = 0$. Taking $0 \neq T = (t_{ij}) \in S_n(R)$, where $t_{1n} = t_0$ and $t_{ij} = 0, 1 < i \leq n, 1 \leq j < n$, we get $AT = 0$. Thus, $S_n(R)$ is left zero-divisor.

(4) $\Rightarrow$ (5). Note that $R[x]/(x^n) \cong S_n(R)$, we obtain the result by (4).

(5) $\Rightarrow$ (6). This is obvious since $T(R, R) \cong R[x]/(x^2)$.

(6) $\Rightarrow$ (1). Let $a \in R$. Then $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in T(R, R)$. Since $T(R, R)$ is left zero-divisor, there exists $0 \neq T = \begin{pmatrix} t & m \\ 0 & t \end{pmatrix} \in T(R, R)$ such that $AT = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} t & m \\ 0 & t \end{pmatrix} = \begin{pmatrix} at & am \\ 0 & at \end{pmatrix} = 0$, it follows that $at = 0$ and $am = 0$. Notice that $T \neq 0$, we have $t \neq 0$ or $m \neq 0$. Consequently in any case there is $0 \neq s \in R$ such that $as = 0$, as asserted. $\square$

Let $R[x; x^{-1}]$ be the ring of Laurent polynomials in one variable x with coefficients in a ring R, i.e., $R[x; x^{-1}]$ consists of all formal sums $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possible negative) integers.

Proposition 1.4 *Let R be a ring. Then $R[x]$ is left zero-divisor if and only if so is $R[x; x^{-1}]$.*

Proof Suppose that $R[x]$ is left zero-divisor. Let $f(x) \in R[x; x^{-1}]$. Then there exists an $n \in N$ such that $f_1(x) = f(x)x^n \in R[x]$. Hence there exists $0 \neq g(x) \in R[x]$ such that $f_1(x)g(x) = f(x)g(x)x^n = 0$, it follows that $f(x)g(x) = 0$ and $R[x; x^{-1}]$ is left zero-divisor.

Conversely, assume that $R[x; x^{-1}]$ is left zero-divisor, and let $f(x) \in R[x]$. Then there exists $0 \neq g(x) \in R[x; x^{-1}]$ such that $f(x)g(x) = 0$ since $R[x] \subseteq R[x; x^{-1}]$. As $g(x) = x^{-m}g_1(x)$ for some $m \in \mathbb{N}$ and $0 \neq g_1(x) \in R[x]$, $f(x)g(x) = x^{-m}f(x)g_1(x) = 0$, we obtain that $f(x)g_1(x) = 0$. $\square$

Proposition 1.5 *Let R and S be rings and $V = {}_R V_S$ be an (R, S)-bimodule. If R is left zero-divisor, so is $A = \begin{pmatrix} R & V \\ 0 & S \end{pmatrix}$.*

Proof Take any $\begin{pmatrix} r & v \\ 0 & s \end{pmatrix} \in A$. For $r \in R$, there exists $0 \neq t \in R$ such that $rt = 0$ since R is left zero-divisor. Thus, we get $0 \neq \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \in A$ such that $\begin{pmatrix} r & v \\ 0 & s \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} = 0$, which implies that A is left zero-divisor. $\square$

Proposition 1.6 *If a ring R is left zero-divisor, so is the ring*

$$V(R)=\left\{\begin{pmatrix} a & d & 0 & 0 & 0 & 0\\ 0 & b & 0 & 0 & 0 & 0\\ 0 & 0 & c & e & 0 & 0\\ 0 & 0 & 0 & a & 0 & 0\\ 0 & 0 & 0 & 0 & b & f\\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix} \mid a,b,c,d,e,f\in R\right\}.$$

Proof Fix $A\in V(R)$. Since R is left zero-divisor, there exists $0\neq a'\in R$ such that $aa'=0$. Taking $0\neq T=(t_{ij})\in V(R)$, where $t_{12}=a'$ and 0 elsewhere, we obtain that $AT=0$. □

Let R be a commutative ring, M an R-module and σ an endomorphism of R. Recall that the *Nagata extension* of R by M and σ (see [4]), denoted by $N(R,M,\sigma)$, is the ring $R\bigoplus M$ with the usual addition and the multiplication $(r_1,m_1)(r_2,m_2)=(r_1r_2,\ \sigma(r_1)m_2+r_2m_1)$, where $r_i\in R$ and $m_i\in M, i=1,2$.

Proposition 1.7 *Let R be a commutative left zero-divisor ring. Then the Nagata extension $N(R,R,\sigma)$ of R by R and σ is left zero-divisor.*

Proof For any $(r,m)\in N(R,R,\sigma)$, we have $0\neq t\in R$ such that $\sigma(r)t=0$ since R is left zero-divisor and $\sigma(R)\subseteq R$. Putting $0\neq(0,t)\in N(R,R,\sigma)$, we get that $(r,m)(0,t)=(r0,\sigma(r)t+0m)=(0,0)$. Therefore $N(R,R,\sigma)$ is left zero-divisor. □

It is interesting to know if the polynomial ring of a ring share the same property with the ring. If $R[x]$ is left zero-divisor, then R is again left zero-divisor. We raise the following question: if R is left zero-divisor, is the polynomial ring $R[x]$ necessarily left zero-divisor?

We do not know whether R is left zero-divisor when both R/I and I are left zero-divisor for an ideal I of R. In view of this question, the following proposition may be of some interest. According to Lambek [5], a ring R is called *symmetric* if $abc=0\Leftrightarrow acb=0$ for all $a,b,c\in R$, i.e., if $bc\in r_R(a)\Leftrightarrow cb\in r_R(a)$. We call a ring R *left symmetric* if $rst=0$ implies $srt=0$ for all $r,s,t\in R$. For example, let $R=2\mathbb{Z}$. Then $T(R,R)\cong\left\{\begin{pmatrix} r & s\\ 0 & r\end{pmatrix}\mid r,s\in R\right\}$ is left symmetric. Note that this definition is equivalent to that of symmetric rings for rings with identity, but in general they are different. For instance, $R=\begin{pmatrix} 0 & \mathbb{Z}\\ 0 & \mathbb{Z}\end{pmatrix}$ is symmetric but not left symmetric.

Proposition 1.8 *Let R be left symmetric and I a non-trivial ideal of R which is a right annihilator in R. If R/I is left zero-divisor, then R is left zero-divisor.*

Proof Since I is non-trivial, we assume that $I=r_R(S)$ where $0\neq S\subseteq R$. For any $a\in R$, there exists $\overline{0}\neq\overline{t}\in R/I$ such that $\overline{a}\overline{t}=\overline{0}$, i.e., $at\in I=r_R(S)$ since R/I is left zero-divisor. It follows that $Sat=0$. Consequently $aSt=0$ since R is left symmetric. Note that $t\notin I$, we have $St\neq 0$. This implies that there exists $s_0\in S$ such that $s_0t\neq 0$ and $a(s_0t)=0$. Thus $r_R(a)\neq 0$, as required. □

It is natural to conjecture that the homomorphic image R/I of R and eR, eRe may also be left (right) zero-divisor for a left (right) zero-divisor ring R, $I \lhd R$ and $e = e^2 \in R$. We have, however, a negative answer to these situations by the following example.

Example 1.9 The ring $R = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$ is left zero-divisor. We have $I = \begin{pmatrix} 0 & 0 \\ \mathbb{Z} & 0 \end{pmatrix} \lhd R, e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = e^2 \in R$ and $ReR = R$, but $R/I \cong \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & 0 \end{pmatrix} = eR = eRe$ is not left zero-divisor.

From the above example it also follows that the left zero-divisor property of rings is not a radical property in the sense of Amitsur and Kurosh.

2. Strong left zero-divisor rings and RFA rings

Observe that for some rings, they not only satisfy $r_R(a) \neq 0$ for any $a \in R$ but also have $r_R(R) \neq 0$. In this section, we will focus on these rings.

Definition 2.1 *A ring R is called strong left (right) zero-divisor if $r_R(R) \neq 0$ ($l_R(R) \neq 0$).*

Any strong left (right) zero-divisor ring is left (right) zero-divisor, but the converse does not hold.

Example 2.2 Let $R = \sum_{i=2}^{\infty} \mathbb{Z}_2 x_i$ be a countably infinite dimensional vector space over the field $\mathbb{Z}_2 = \{0, 1\}$, with basis $T = \{x_2, x_3, \dots, x_n, \dots\}$. Multiplication of the base vectors is defined as

$$x_i x_j = \begin{cases} 0, & \text{if } (i,j) \neq 1, \\ x_{ij}, & \text{if } (i,j) = 1, \end{cases}$$

where (i, j) is the maximal prime divisor of i and j. Thought of as a ring, R is the set of all finite sums $\sum a_i x_i$, where a_i are elements in the field Z_2. Addition is defined articulately as $a_i x_i + a_j x_j$ just written together, if $i \neq j$; and if $i = j$, then $a_i x_i + a'_i x_i = (a_i + a'_i) x_i$. Multiplication is distributive and defined as above. The ring R is then commutative. Moreover, for any $a = a_{i_1} x_{i_1} + a_{i_2} x_{i_2} + \cdots + a_{i_n} x_{i_n} \in R$, we have $a^2 = 0$, and hence R is zero-divisor.

For any $a = x_{i_1} + x_{i_2} + \cdots + x_{i_n} \in R$ and any positive integer $n \geq 2$, since $(n, n+1) = 1$, we get that

$$x_{i+1} a = x_{i_1\ (i_1+1)} + \cdots, \quad x_{[i_1(i_1+1)+1]} a = x_{j_1} + \cdots,$$

where $j_1 = i_1(i_1 + 1)[i_1\ (i_1 + 1) + 1], \dots$. Thus, if $a \in r_R(T)$, then necessarily $a = 0$, whence $r_R(R) \subseteq r_R(T) = 0$. So R is not strong left zero-divisor.

For a ring R with a ring endomorphism $\alpha : R \to R$, a skew polynomial ring $R[x; \alpha]$ of R is the ring obtained by giving the polynomial ring over R with the new multiplication $xr = \alpha(r)x$ for all $r \in R$.

Theorem 2.3 *Let R be a ring and $\alpha : R \to R$ an epimorphism. Then R is strong left zero-divisor if and only if so is $R[x; \alpha]$.*

Proof If $Rb=0$, then $R\alpha(b)=\alpha(R)\alpha(b)\subseteq\alpha(Rb)=0$, hence $\alpha(r_R(R))\subseteq r_R(R)$. Now assume that R is strong left zero-divisor, then $T=r_R(R)\neq 0$. For every $f(x)=\sum_{i=o}^{n}a_ix^i\in R[x;\alpha]$, taking any $0\neq t(x)=\sum_{j=0}^{m}b_jx^j\in T[x;\alpha]\subseteq R[x;\alpha]$, we obtain

$$f(x)t(x)=\sum_{k=0}^{m+n}\sum_{i+j=k}a_i\alpha^i(b_j)x^k=0.$$

Hence $r_{R[x;\alpha]}(R[x;\alpha])\neq 0$.

Conversely, assume that $R[x;\alpha]$ is strong left zero-divisor. For any $0\neq f(x)=\sum_{i=0}^{n}a_ix^i\in r_{R[x;\alpha]}(R[x;\alpha])$, there exists at least one $a_{i_k}\neq 0, 0\leq i_k\leq n, a_{i_k}\in R$. Note that $R\subseteq R[x;\alpha]$ and $Rf(x)=0$. It follows that $Ra_{ik}=0$ and $r_R(R)\neq 0$. □

Theorem 2.3 answers partially the question raised in the above section.

Recall that for an infinite set of commuting indeterminates $\{x_\lambda\}$ over R, Gilmer-Grams [3] defined rings

$$R[\{x_\lambda\}]=\bigcup\{R[F]\mid F \text{ is a finite subset of } \{x_\lambda\}\} \text{ and}$$

$$R[[\{x_\lambda\}]]=\bigcup\{R[[F]]\mid F \text{ is a finite subset of } \{x_\lambda\}\}.$$

Theorem 2.4 *Let R be a ring. Then the following statements are equivalent:*

(1) R is strong left zero-divisor.

(2) $T_n(R)$ is strong left zero-divisor for any $n\in\mathbb{N}$.

(3) $QM_2(R)$ is strong left zero-divisor.

(4) $S_n(R)$ is strong left zero-divisor for any $n\in\mathbb{N}$.

(5) $R[x]/(x^n)$ is strong left zero-divisor for any $n\in\mathbb{N}$.

(6) $T(R,R)$ is strong left zero-divisor.

(7) $R[x;x^{-1}]$ is strong left zero-divisor.

(8) $R[\{x_\lambda\}]$ is strong left zero-divisor.

(9) $R[[\{x_\lambda\}]]$ is strong left zero-divisor.

Proof Note that if R is strong left zero-divisor, then $S=r_R(R)\neq 0$ and there exists $0\neq t(\{x_\lambda\})\in S[\{x_\lambda\}]\subseteq S[[\{x_\lambda\}]]$ such that $R[\{x_\lambda\}]t(\{x_\lambda\})=0$ ($R[[\{x_\lambda\}]]t(\{x_\lambda\})=0$), hence $R[\{x_\lambda\}]$ ($R[[\{x_\lambda\}]]$) is strong left zero-divisor. It follows that (1) $\Leftrightarrow$ (8) and (1) $\Leftrightarrow$ (9).

Making a little modification in the proof of Theorem 1.3, we can prove that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6).

By Theorem 2.3, we know that R is strong left zero-divisor if and only if so is $R[x]$. By analogy with the proof of Proposition 1.4, it is easy to prove that (1) $\Leftrightarrow$ (7). □

Theorem 2.5 *A ring R is strong left zero-divisor if and only if so is $M_n(R)$, the ring of $n\times n$ matrices over R, for any positive integer n.*

Proof Assume that R is strong left zero-divisor and $A=(a_{ij})\in M_n(R)$. Then $r_R(R)\neq 0$. For any $0\neq r\in r_R(R)$, we have $a_{ij}r=0,\forall 1\leq i,j\leq n$. Putting $T=(t_{ij})\in M_n(R)$, where $t_{ii}=r$ and $t_{ij}=0$ if $i\neq j, 1\leq i,j\leq n$, we get that $T\neq 0$ and $AT=0$. Hence $r_{M_n(R)}(M_n(R))\neq 0$ because A is arbitrary.

Conversely, assume that $M_n(R)$ is strong left zero-divisor and $r \in R$. Take any $0 \neq A = (a_{ij}) \in r_{M_n(R)}(M_n(R)) \neq 0$, and suppose that some $a_{kl} \neq 0, 1 \leq k, l \leq n$. If we put $T = (t_{ij})$ as above, then from $TA = 0$ one can get that $ra_{kl} = 0$. This implies that $a_{kl} \in r_R(R) \neq 0$. $\square$

Given a monoid G and a ring R, we use $R[G]$ to denote the monoid ring of G over R.

Theorem 2.6 *A ring R is strong left zero-divisor if and only if so is $R[G]$ for any monoid G.*

Proof Assume that $r_R(R) \neq 0$ and $\sum r_i g_i \in R[G]$. For any $0 \neq a \in r_R(R)$, we have $(\sum r_i g_i)(ae) = \sum (r_i a) g_i = 0$, where e is the identity of G. Thus $0 \neq ae \in r_{R[G]}(R[G])$.

Conversely, assume that $R[G]$ is strong left zero-divisor and $a \in R$. If $0 \neq \sum r_i g_i \in r_{R[G]}(R[G])$, then from $0 = (ae)(\sum r_i g_i) = \sum (a_i r_i) g_i$ we get that $ar_i = 0$ for any i. This shows that $r_i \in r_R(R)$ for any i, and $r_R(R) \neq 0$. $\square$

Let G denote a group with identity e, and $R = \bigoplus_{g \in G} R_g$ be a G-graded ring. Beattie [1] defined the generalized smash product $R\#G^*$ of R and G to be the free left R-module $\bigoplus_{g \in G} RP_g$ with multiplication defined for elements aP_g and bP_h by $(aP_g)(bP_h) = ab_{gh^{-1}}P_h$, and extended to general elements of $R\#G^*$ by linearity.

Theorem 2.7 *Let $R = \bigoplus_{g \in G} R_g$ be a G-graded ring. Then R is strong left zero-divisor if and only if so is $R\#G^*$.*

Proof Assume that $r_R(R) \neq 0$ and $\sum a_i P_{g_i} \in R\#G^*$. Take any $0 \neq r \in r_R(R)$. Since $r_R(R)$ is a graded ideal of R, $(\sum a_i P_{g_i}) r P_e = \sum a_i r_{g_i} P_e = 0$. This implies that $0 \neq rP_e \in r_{R\#G^*}(R\#G^*)$.

Conversely, assume that $R\#G^*$ is strong left zero-divisor. Taking

$$0 \neq \sum_i a^{(i)} P_{g_i} \in r_{R\#G^*}(R\#G^*),$$

we get that $0 = r_g P_h(\sum_i a^{(i)} P_{g_i}) = \sum_i r_g a^{(i)}_{hg_i^{-1}} P_{g_i}$ for any $g, h \in G$ and $r_g \in R_g$. Thus for every $g_i \in G, r_g a^{(i)}_{hg_i^{-1}} = 0$. If $a^{(i_0)} \neq 0$, then there exists an $h_0 \in G$ such that $a^{(i_0)}_{h_0 g_{i_0}^{-1}} \neq 0$, and hence $a^{(i_0)}_{h_0 g_{i_0}^{-1}} \in r_R(r_g) \neq 0$. Since $g \in G$ and $r_g \in R_g$ are arbitrary, we have $a^{(i_0)}_{h_0 g_{i_0}^{-1}} \in r_R(R)$. $\square$

Camillo-Nielson [2] introduced the concept of right finite annihilated rings (in short, RFA rings) to describe exactly when a direct product or direct sum of rings is right McCoy. A ring R is called RFA if every finite subset of R has a nonzero right annihilator.

Clearly, strong left zero-divisor rings are RFA rings, but the converse does not hold.

Example 2.8 Let $R = \mathbb{Z}[x_1, x_2, x_3, \ldots]/(x_1^2, x_2^3, x_3^4, \ldots)$, and $A = \langle \overline{x_1}, \overline{x_2}, \overline{x_3}, \ldots \rangle$ be the ideal of R generated by $\overline{x_1}, \overline{x_2}, \overline{x_3}, \ldots$. Then A is nil, left zero-divisor and RFA. But A is neither nilpotent nor strong left zero-divisor.

For RFA rings, we have the following

Proposition 2.9 *Let R be a ring and $S = \{(a_n)_{n=1}^{\infty} \in \prod R \mid a_n$ is a eventually constant$\}$, a subring of the countable direct product $\prod_{n=1}^{\infty} R$. Then ring R is RFA if and only if so is S.*

Proof It is a trivial verification.

Theorem 2.10 *Let R be a ring. Then the following statements are equivalent:*

(1) R is RFA.

(2) $T_n(R)$ is RFA for any $n \in \mathbb{N}$.

(3) $QM_2(R)$ is RFA.

(4) $S_n(R)$ is RFA for any $n \in \mathbb{N}$.

(5) $R[x]/(x^n)$ is RFA for any $n \in \mathbb{N}$.

(6) $T(R,R)$ is RFA.

(7) $R[x;x^{-1}]$ is RFA.

(8) $R[\{x_\lambda\}]$ is RFA.

Proof (1) $\Rightarrow$ (2). Assume that $F = \{A_k = (a_{ij}^k) \in T_n(R), k = 1,2,\dots,m\}$ is a finite subset of $T_n(R)$. Then $E = \{a_{ij}^k | 1 \leq i,j \leq n, k = 1,2,\dots,m\}$ is a finite subset of R, there exists $0 \neq t \in R$ such that $a_{ij}^k t = 0$ for every a_{ij}^k $(1 \leq i,j \leq n, 1 \leq k \leq m)$ since R is RFA. Putting $D = (d_{ij}) \in T_n(R)$ with $d_{11} = t$ and zeros elsewhere, we have that $A_k D = 0$ for $1 \leq k \leq m$.

(2) $\Rightarrow$ (3). Holds since $QM_2(R) \cong T_2(R)$.

(3) $\Rightarrow$ (1). For any finite subset F of R, $E = \{A_r = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} | r \in F\}$ is a finite subset of $QM_2(R)$. Then there exists $0 \neq T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in QM_2(R)$ such that $A_r T = 0$ for every $r \in F$, it follows that $ra = rb = rc = rd = 0$. Notice that $T \neq 0$, there is $0 \neq s \in R$ such that $Fs = 0$, as desired.

(1) $\Rightarrow$ (4). Let $F = \{A_k = (a_{ij}^k) \in S_n(R), k = 1,2,\dots,m\}$ be a finite subset of $S_n(R)$. Then $E = \{a_{ij}^k | 1 \leq i,j \leq n, k = 1,2,\dots,m\}$ is a finite subset of R, there exists $0 \neq t \in R$ such that $a_{ij}^k t = 0$ for every a_{ij}^k $(1 \leq i,j \leq n, 1 \leq k \leq m)$ since R is RFA. Taking $0 \neq T = (t_{ij}) \in S_n(R)$ with $t_{1n} = t$ and zeros elsewhere, we obtain that $A_k T = 0$ for $1 \leq k \leq m$.

(4) $\Rightarrow$ (5). Holds by $R[x]/(x^n) \cong S_n(R)$.

(5) $\Rightarrow$ (6). Follows from $T(R,R) \cong R[x]/(x^2)$.

(6) $\Rightarrow$ (1). Let F be a finite subset of R and $E = \{A_r = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} | r \in F\}$. Then there exists $0 \neq T = \begin{pmatrix} t & m \\ 0 & t \end{pmatrix} \in T(R,R)$ such that $A_r T = 0$ for any $r \in F$, it follows that $rt = 0$ and $rm = 0$. Notice that $T \neq 0$, we have that $t \neq 0$ or $m \neq 0$. Consequently in any case there is $0 \neq s \in R$ such that $Fs = 0$, as desired.

(1) $\Rightarrow$ (8). Let $E = \{f_i\{x_\lambda\} | i = 1,2,\dots,m\}$ be a finite subset of $R[\{x_\lambda\}]$. Then $E \subseteq R[F]$ for some finite subset F of $\{x_\lambda\}$, and the set H of coefficients of all $f_i\{x_\lambda\} \subseteq E$ is a finite subset of R. Hence there exists $0 \neq t \in R$ such that $Ht = 0$, it follows that $f_i\{x_\lambda\}t = 0$ for $1 \leq k \leq m$.

(8) $\Rightarrow$ (1). Let E be a finite subset of R. Then $E \subseteq R[\{x_\lambda\}]$, and there exists $0 \neq f\{x_\lambda\} \in R[\{x_\lambda\}]$ such that $Ef\{x_\lambda\} = 0$. Thus $Ea = 0$ for any nonzero coefficient a of $f\{x_\lambda\}$.

(1) $\Leftrightarrow$ (7). The proof is analogous to that of (1) $\Leftrightarrow$ (8). $\square$

Example 2.11 Consider the ring $R = \begin{pmatrix} 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \end{pmatrix}$. For any $A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & d \end{pmatrix} \in R$ and $T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R$, we have $AT = 0$, which implies that R is strong left zero-divisor.

We conclude this paper with the following chart:

$$\begin{tikzcd}
\text{nilpotent} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{locally nilpotent} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{nil} \arrow[d, Rightarrow] \\
\text{strong left zero-divisor} \arrow[r, Rightarrow] & \text{RFA} \arrow[r, Rightarrow] & \text{left zero-divisor}
\end{tikzcd}$$

No other implications hold (except by transitivity). Note that Example 2.11 shows that a strong left zero-divisor, left zero-divisor and RFA ring are not necessarily nilpotent, nil and locally nilpotent, respectively; and Example 20.2 in Szasz [6] also shows that a left zero-divisor ring is not necessarily an RFA ring.

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