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On Nonlocal Elliptic Systems of $p(x)$-Kirchhoff-Type under Neumann Boundary Condition

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Abstract This paper is concerned with the existence of solutions to a class of $p(x)$-Kirchhoff-type systems under Neumann boundary condition. By Ekeland Variational Principle and the theory of the variable exponent Sobolev spaces, we establish conditions ensuring the existence of solutions for the problem. Since the Poincaré's inequality does not hold in the space $W^{1,p(x)}(\Omega)$, we shall prove the Poincaré-Wirtinger's inequality in a subspace of $W^{1,p(x)}(\Omega)$.

Keywords variational method; elliptic systems; nonlocal; Neumann boundary.

MR(2010) Subject Classification 35D30; 35J60; 35J70

1. Introduction

In this paper we study the following nonlocal elliptic systems of gradient type with non-standard growth conditions

$$\begin{cases} -M_1(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x)\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \frac{\partial F}{\partial u}F(u,v)+\rho_1(x) & \text{in } \Omega,\\ -M_2(\int_\Omega \frac{1}{q(x)}|\nabla v|^{q(x)}\,\mathrm{d}x)\mathrm{div}(|\nabla v|^{q(x)-2}\nabla v) = \frac{\partial F}{\partial v}F(u,v)+\rho_2(x) & \text{in } \Omega,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, ν is the unit exterior vector on $\partial\Omega$, $p(x), q(x) \in C(\overline{\Omega})$ with $1 < p^- := \min_{\overline{\Omega}} p(x) \le p^+ := \max_{\overline{\Omega}} p(x) < +\infty$ and $1 < q^- := \min_{\overline{\Omega}} q(x) \le q^+ := \max_{\overline{\Omega}} q(x) < +\infty$, $M_1(t)$, $M_2(t)$ are continuous functions. We confine ourselves to the case where $M_1 = M_2$ for simplicity. Notice that the results of this paper remain valid for $M_1 \ne M_2$ by adding some slight changes in the hypothesis (H_2). The function $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is assumed to be of class C^1 in $u, v \in \mathbb{R}$.

The operator $-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is said to be the $p(x)$-Laplacian, and becomes p-Laplacian when $p(x) \equiv p$ (a constant). The $p(x)$-Laplacian possesses more complicated nonlinearities than the p-Laplacian; for example, it is inhomogeneous. The study of various mathematical problems with variable exponent growth condition has been received considerable attention in recent years. These problems are interesting in applications and raise many difficult mathematical problems. One of the most studied models leading to the problem of this type is the model of motion

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of electrorheological fluids, which are characterized by their ability to drastically change the mechanical properties under the influence of an exterior electromagnetic field [1, 2]. Problems with variable exponent growth conditions also appear in the mathematical modeling of stationary thermo-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium [3, 4]. Another field of application of equations with variable exponent growth conditions is image processing [5]. The variable nonlinearity is used to outline the borders of the true image and to eliminate possible noise. We refer the readers to [6–10] for an overview and references on this subject, and to [11–17] for the study of the $p(x)$-Laplacian equations and the corresponding variational problems.

The problem (1.1) is related to the stationary version of a model introduced by Kirchhoff [18]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2\mathrm{d}x\Big)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

where ρ,ρ_0,h,E,L are constants, which extends the classical D'Alembert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. A distinguishing feature of equation (1.2) is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2\mathrm{d}x$ which depends on the average $\frac{1}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2\mathrm{d}x$, and hence the equation is no longer a pointwise identity. Some early classical studies of Kirchhoff equations can be found in Bernstein [19] and Pohožaev [20]. The equation

$$\begin{cases} -(a+b\int_\Omega|\nabla u|^2\mathrm{d}x)\Delta u=f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega\end{cases} \tag{1.3}$$

is related to the stationary analogue of the equation (1.2). Eq. (1.3) received much attention only after Lions [21] proposed an abstract framework to the problem. Some important and interesting results can be found, for example, in [22–24]. More recently Alves et al. [25] and Ma and Rivera [26] obtained positive solutions of such problems by variational methods. The study of Kirchhoff type equations has already been extended to the case involving the p-Laplacian [27–29] and $p(x)$-Laplacian [30, 31].

In [32], the authors considered a nonlocal elliptic system of the p-Kirchhoff type. By Ekeland Variational Principle [33], they established the existence of weak solutions of the problem. Motivated by above, we consider the nonlocal elliptic system (1.1). We establish conditions ensuring the existence of solutions for system (1.1).

The rest of this paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces. In Sections 3, we give our main results and their proofs.

2. Preliminaries

In order to discuss problem (1.1), we need some theories on $W^{1,p(x)}(\Omega)$ which we call variable exponent Sobolev space. Firstly we state some basic properties of spaces $W^{1,p(x)}(\Omega)$

which will be used later (for details, see [16]). Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on Ω.

Write

$$C_+(\overline{\Omega}) = \left\{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\right\}$$

and

$$L^{p(x)}(\Omega) = \left\{u \in \mathbf{S}(\Omega) : \int_\Omega |u(x)|^{p(x)} \,\mathrm{d}x < +\infty\right\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \le 1\right\},$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{p(x)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Proposition 2.1 ([16]) *The spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2.2 ([16]) *Set $\rho(u) = \int_\Omega |u(x)|^{p(x)} \,\mathrm{d}x$. For any $u \in L^{p(x)}(\Omega)$, then*

1) *For $u \neq 0$, $|u|_{p(x)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$;*
2) *$|u|_{p(x)} < 1$ $(= 1; > 1) \Leftrightarrow \rho(u) < 1$ $(= 1; > 1)$;*
3) *If $|u|_{p(x)} > 1$, then $|u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$;*
4) *If $|u|_{p(x)} < 1$, then $|u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}$;*
5) *$\lim_{k\to+\infty} |u_k|_{p(x)} = 0 \iff \lim_{k\to+\infty} \rho(u_k) = 0$;*
6) *$\lim_{k\to+\infty} |u_k|_{p(x)} = +\infty \iff \lim_{k\to+\infty} \rho(u_k) = +\infty$.*

Proposition 2.3 ([13, 16]) *If $q \in C_+(\overline{\Omega})$ and $q(x) \le p^*(x)$ $(q(x) < p^*(x))$ for $x \in \overline{\Omega}$, then there is a continuous (compact) embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, where*

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

Proposition 2.4 ([14, 16]) *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have the following Hölder-type inequality*

$$\left|\int_\Omega uv\mathrm{d}x\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)}.$$

Let $W_c = \{1\}$, that is, the subspace of $W^{1,p(x)}(\Omega)$ spanned by the constant function 1, and $W_0 = \{z \in W^{1,p(x)}(\Omega) : \int_\Omega z\mathrm{d}x = 0\}$ which is called the space of functions of $W^{1,p(x)}(\Omega)$ with null mean in Ω. By Hahn-Banach Theorem, we can see that

$$W^{1,p(x)}(\Omega) = W_0 \oplus W_c,$$

i.e., every function $u \in W^{1,p(x)}(\Omega)$ is of the form

$$u = u_0 + \alpha,$$

where $\int_\Omega u_0 \mathrm{d}x = 0$ and α is a constant. Consequently, if $(u, v) \in W^{1,p(x)}(\Omega) \times W^{1,p(x)}(\Omega)$, then

$$(u, v) = (u_0 + \alpha, v_0 + \beta) = (u_0, v_0) + (\alpha, \beta),$$

where $\int_\Omega u_0 \mathrm{d}x = \int_\Omega v_0 \mathrm{d}x = 0$ and α, β are constants.

It is well known that the Poincaré's inequality does not hold in the space $W^{1,p(x)}(\Omega)$. However, it is true in W_0 as shown in the next lemma.

Proposition 2.5 (Poincaré-Wirtinger's Inequality) *There is a positive constant C such that*

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)} \quad \textit{for all} \quad z \in W_0.$$

Proof Let $\varphi : W_0 \to \mathbb{R}$ be the functional given by $\varphi(u) = |\nabla u|_{p(x)}$ for all $u \in W_0$ and S be the manifold

$$S = \left\{u \in W_0 : |u|_{p(x)} = 1\right\}.$$

Since φ is bounded from below on S and lower semicontinuous, it follows that there is a minimizing sequence $(u_n) \subset S$, that is,

$$\varphi(u_n) \to \inf_S \varphi = \varphi_0 \geq 0.$$

Consequently, $|u_n|_{p(x)} = 1$ and there is a positive constant C_1 such that $|\nabla u_n|_{p(x)} \leq C_1$, for all $n \in \mathbb{N}$. From these facts we infer that the sequence (u_n) is bounded in $W^{1,p(x)}(\Omega)$. Then there is a subsequence still denoted by (u_n) which converges weakly in $W^{1,p(x)}(\Omega)$. Without loss of generality, we assume that $u_n \rightharpoonup u$ in $W^{1,p(x)}(\Omega)$. By virtue of compactness of the Sobolev embedding we have that $u_n \to u$ in $L^{r(x)}(\Omega)$, $r \in C_+(\overline{\Omega})$ and $r(x) < p^*(x)$ for all $x \in \overline{\Omega}$. In particular, $0 = \int_\Omega u_n \mathrm{d}x \to \int_\Omega u \mathrm{d}x = 0$ and $1 = |u_n|_{p(x)} \to |u|_{p(x)}$ and thus $u \in S$.

Let us show that $\varphi_0 > 0$. Suppose on the contrary that $\varphi_0 = 0$. In this case, up to subsequences, we have

$$\begin{aligned}
0 &= \lim_{n\to+\infty} |\nabla u_n|_{p(x)} = \lim_{n\to+\infty} (|\nabla u_n|_{p(x)} + |u_n|_{p(x)} - |u_n|_{p(x)}) \\
&= \lim_{n\to+\infty} (\|u_n\|_{p(x)} - |u_n|_{p(x)}) = \lim_{n\to+\infty} \|u_n\|_{p(x)} - \lim_{n\to+\infty} |u_n|_{p(x)} \geq \|u\|_{p(x)} - |u|_{p(x)} \\
&= |\nabla u|_{p(x)},
\end{aligned}$$

which yields $|\nabla u|_{p(x)} \leq 0$. Therefore, $u(x) = C_2$ a.e., in Ω, with C_2 a real constant. Since $u \in S \subset W_0$, one has

$$\int_\Omega u \mathrm{d}x = \int_\Omega C_2 \mathrm{d}x = 0$$

and we conclude that $C_2 = 0$, which is impossible because $|u|_{p(x)} = 1$. Consequently, $\varphi_0 > 0$. Thus,

$$\varphi_0 = \lim_{n\to+\infty} |\nabla u_n|_{p(x)} \geq |\nabla u|_{p(x)} \geq \varphi_0.$$

Hence, $\varphi(u) = |\nabla u|_{p(x)} = \varphi_0$, which shows that the infimum of φ is attained on S. Consequently,

$$\varphi_0 \leq |\nabla u|_{p(x)}$$

for all $u \in W_0$ with $|u|_{p(x)} = 1$. If $0 \neq u \in W_0$,

$$\varphi_0 \leq |\nabla \frac{u}{|u|_{p(x)}}|_{p(x)} = \frac{|\nabla u|_{p(x)}}{|u|_{p(x)}}.$$

It follows that

$$|u|_{p(x)} \leq \frac{1}{\varphi_0}|\nabla u|_{p(x)} \quad \text{for all} \quad u \in W_0,$$

which shows the Poincaré-Wirtinger's inequality in W_0.

For every (u, v) and (φ, ψ) in $W := W^{1,p(x)}(\Omega) \times W^{1,q(x)}(\Omega)$, let

$$\mathcal{F}(u, v) := \int_\Omega F(u, v)\mathrm{d}x.$$

Then $\mathcal{F}'(u, v)(\varphi, \psi) = D_1\mathcal{F}(u, v)(\varphi) + D_2\mathcal{F}(u, v)(\psi)$, where

$$D_1\mathcal{F}(u, v)(\varphi) = \int_\Omega \frac{\partial F}{\partial u}(u, v)\varphi \mathrm{d}x$$

and

$$D_2\mathcal{F}(u, v)(\psi) = \int_\Omega \frac{\partial F}{\partial v}(u, v)\psi \mathrm{d}x.$$

The Euler-Lagrange functional associated to (1.1) is given by

$$\begin{aligned} J(u, v) =&\widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x\Big) + \widehat{M}\Big(\int_\Omega \frac{1}{q(x)}|\nabla v|^{q(x)}\mathrm{d}x\Big) - \\ &\mathcal{F}(u, v) - \int_\Omega \rho_1(x)u\mathrm{d}x - \int_\Omega \rho_2(x)v\mathrm{d}x, \end{aligned}$$

where $\widehat{M}(t) := \int_0^t M(\tau)\mathrm{d}\tau$. It is easy to verify that $J \in C^1(W, \mathbb{R})$ and $(u, v) \in W$ is a weak solution of (1.1) if and only if (u, v) is a critical point of J. Moreover, we have

$$J'(u, v)(\varphi, \psi) = D_1J(u, v)(\varphi) + D_2J(u, v)(\psi),$$

where

$$D_1J(u, v)(\varphi) = M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x\Big)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi\mathrm{d}x - D_1\mathcal{F}(u, v)(\varphi) - \int_\Omega \rho_1\varphi\mathrm{d}x,$$

$$D_2J(u, v)(\varphi) = M\Big(\int_\Omega \frac{1}{q(x)}|\nabla v|^{q(x)}\mathrm{d}x\Big)\int_\Omega |\nabla v|^{q(x)-2}\nabla v\nabla\psi\mathrm{d}x - D_2\mathcal{F}(u, v)(\psi) - \int_\Omega \rho_2\psi\mathrm{d}x.$$

Let us choose on W the norm $\|\cdot\|$ defined by

$$\|(u, v)\| := \|u\|_{p(x)} + \|v\|_{q(x)}.$$

3. Existence of solution

In this section we shall discuss the existence of weak solution of (1.1). For simplicity, we use C, C_i, $i = 1, 2, \ldots$ to denote the general positive constant (the exact value may change from line to line).

Before stating our results, we introduce some natural hypotheses on the righthand side of (1.1) and the nonlocal coefficient $M(t)$.

(H_1) There is $k>0$ such that $F(u+k,v+k)=F(u,v)$, for all $(u,v)\in\mathbb{R}^2$.

(H_2) $\exists m_0>0$ such that $M(t)\geq m_0$.

(H_3) $\rho_1(x)\in L^{p'(x)}(\Omega)$ with $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$, $\rho_2(x)\in L^{q'(x)}(\Omega)$ with $\frac{1}{q(x)}+\frac{1}{q'(x)}=1$ and $\int_\Omega \rho_1(x)\mathrm{d}x=\int_\Omega\rho_2(x)\mathrm{d}x=0$.

Lemma 3.1 *If (H_1), (H_2) and (H_3) hold, the functional J is bounded from below.*

Proof Firstly, we shall prove that J is well defined. To do this, it is enough to show that $\mathcal{F}$, $\int_\Omega \rho_1 u\mathrm{d}x$ and $\int_\Omega\rho_2 v\mathrm{d}x$ are well defined. Since F is continuous on $[0,k]\times[0,k]$ and $F(u+k,v+k)=F(u,v)$ for all $(u,v)\in\mathbb{R}^2$, it follows that $|F(u,v)|\leq C_3$, for all $(u,v)\in\mathbb{R}^2$, and so

$$\mathcal{F}(u,v)\leq C_3|\Omega| \quad \text{for all} \quad (u,v)\in W.$$

On the other hand, from (H_3), we can easily see that $\int_\Omega\rho_1 u\mathrm{d}x\leq C_4$ for $u\in W^{1,p(x)}(\Omega)$ and $\int_\Omega \rho_2 v\mathrm{d}x\leq C_5$ for $v\in W^{1,q(x)}(\Omega)$.

Let us show that J is bounded from below. If $(u,v)\in W$, u and v may be written as

$$u=u_0+\alpha \quad \text{and} \quad v=v_0+\beta,$$

where $\alpha,\beta\in\mathbb{R}$ and $\int_\Omega u_0\mathrm{d}x=\int_\Omega v_0\mathrm{d}x=0$. Thus by Poincaré-Wirtinger's Inequality, we have

$$\begin{aligned}
J(u,v)\geq&\frac{m_0}{p^+}\int_\Omega|\nabla u|^{p(x)}\mathrm{d}x+\frac{m_0}{q^+}\int_\Omega|\nabla v|^{q(x)}\mathrm{d}x-\\
&\int_\Omega\rho_1\left(u_0+\alpha\right)\mathrm{d}x-\int_\Omega\rho_2\left(v_0+\beta\right)\mathrm{d}x-C_6\\
\geq&\frac{m_0}{p^+}\int_\Omega|\nabla u|^{p(x)}\mathrm{d}x+\frac{m_0}{q^+}\int_\Omega|\nabla v|^{q(x)}\mathrm{d}x-\int_\Omega\rho_1u_0\mathrm{d}x-\int_\Omega\rho_2v_0\mathrm{d}x-C_6\\
\geq&\frac{m_0}{p^+}\int_\Omega|\nabla u|^{p(x)}\mathrm{d}x+\frac{m_0}{q^+}\int_\Omega|\nabla v|^{q(x)}\mathrm{d}x-|\rho_1|_{p'}|u_0|_{p(x)}-|\rho_2|_{q'(x)}|v_0|_{q(x)}-C_6\\
\geq&\frac{m_0}{p^+}\int_\Omega|\nabla u|^{p(x)}\mathrm{d}x+\frac{m_0}{q^+}\int_\Omega|\nabla v|^{q(x)}\mathrm{d}x-C_7|\rho_1|_{p'(x)}|\nabla u_0|_{p(x)}-C_8|\rho_2|_{q'(x)}|\nabla v_0|_{q(x)}-C_6\\
\geq&\frac{m_0}{p^+}\min\{|\nabla u_0|_{p(x)}^{p^-},|\nabla u_0|_{p(x)}^{p^+}\}+\frac{m_0}{q^+}\min\{|\nabla v_0|_{q(x)}^{q^-},|\nabla v_0|_{q(x)}^{q^+}\}-\\
&C_7|\rho_1|_{p'(x)}|\nabla u_0|_{p(x)}-C_8|\rho_2|_{q'(x)}|\nabla v_0|_{q(x)}-C_6.
\end{aligned}$$

Because the function

$$(s,t)\mapsto\frac{m_0}{p^+}\min\{s^{p^-},s^{p^+}\}+\frac{m_0}{q^+}\min\{t^{q^-},t^{q^+}\}-C_7|\rho_1|_{p'(x)}s-C_8|\rho_2|_{q'(x)}t-C_6,\quad s,t\geq0$$

is bounded from below, we conclude that J is also bounded from below.

Theorem 3.1 *Under assumptions (H_1)–(H_3), problem (1.1) possesses a weak solution $(u,v)\in W$.*

Proof We shall find a critical point of the functional J. As J is a C^1 and bounded from below functional, it follows from the Ekeland Variational Principle that there exists $(u_n,v_n)\in W$ such that

$$J\left(u_n,v_n\right)\to\inf_W J \quad\text{and}\quad J'\left(u_n,v_n\right)\to0. \tag{1}$$

For each $n \in \mathbb{N}$, we have

$$u_n = u_n^0 + \alpha_n \quad \text{and} \quad v_n = v_n^0 + \beta_n,$$

where α_n and β_n are real constants and $\int_\Omega u_n^0 \mathrm{d}x = \int_\Omega v_n^0 \mathrm{d}x = 0$. From (3.1) we have $|J(u_n, v_n)| \leq C_9$, for some positive constant C_9 and for all $n \in \mathbb{N}$. We now use the Lemma 3.1 to obtain

$$\begin{aligned} C_{10} \leq & \frac{m_0}{p^+} \min\{\left|\nabla u_n^0\right|_{p(x)}^{p^-}, \left|\nabla u_n^0\right|_{p(x)}^{p^+}\} + \frac{m_0}{q^+} \min\{\left|\nabla v_n^0\right|_{q(x)}^{q^-}, \left|\nabla v_n^0\right|_{q(x)}^{q^+}\} - \\ & c\left|\rho_1\right|_{p(x)'} \left|\nabla u_n^0\right|_{p(x)} - c\left|\rho_2\right|_{q'(x)} \left|\nabla v_n^0\right|_{q(x)} \leq C_9 \end{aligned}$$

which implies that the sequences $\left|\nabla u_n^0\right|_{p(x)}$ and $\left|\nabla v_n^0\right|_{q(x)}$ are bounded. By virtue of the Poincaré-Wirtinger's inequality $\left|u_n^0\right|_{p(x)}$ and $\left|v_n^0\right|_{q(x)}$ are bounded too. Consequently, $\left(u_n^0\right)$ is bounded sequences in $W^{1,p(x)}(\Omega)$ and $\left(v_n^0\right)$ is bounded sequences in $W^{1,q(x)}(\Omega)$. It is obvious that there exists constant k large enough such that $\alpha_n, \beta_n \in [0, k]$ for all $n \in \mathbb{N}$. So (u_n) is bounded sequence in $W^{1,p(x)}(\Omega)$ and (v_n) is bounded sequence in $W^{1,q(x)}(\Omega)$. Hence, up to a subsequence, we have

$$(u_n, v_n) \rightharpoonup (u, v) \quad \text{in} \quad W,$$

$$\int_\Omega \rho_1 u_n \mathrm{d}x \to \int_\Omega \rho_1 u \mathrm{d}x, \quad \int_\Omega \rho_2 v_n \mathrm{d}x \to \int_\Omega \rho_2 v \mathrm{d}x$$

and

$$(u_n, v_n) \to (u, v) \quad \text{a.e. in } \Omega.$$

Due to the continuity of F, $F(u_n, v_n) \to F(u, v)$ a.e., in Ω and because $|F(u_n(x), v_n(x))| \leq C_3$ for all $n \in \mathbb{N}$ a.e., in Ω, we may use the Lebesgue dominated convergence theorem to conclude that

$$\int_\Omega F(u_n, v_n) \mathrm{d}x \to \int_\Omega F(u, v) \mathrm{d}x.$$

On the other hand, by Proposition 3.1 of [31], we have

$$\widehat{M}\Big(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \mathrm{d}x\Big) \leq \lim_{n \to +\infty} \widehat{M}\Big(\int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} \mathrm{d}x\Big)$$

and

$$\widehat{M}\Big(\int_\Omega \frac{1}{q(x)} |\nabla v|^{q(x)} \mathrm{d}x\Big) \leq \lim_{n \to +\infty} \widehat{M}\Big(\int_\Omega \frac{1}{q(x)} |\nabla v_n|^{q(x)} \mathrm{d}x\Big).$$

Consequently,

$$\inf_W J = \lim_{n \to +\infty} J(u_n, v_n) \geq J(u, v),$$

which implies that $J(u, v) = \inf_W J$. Since $(u, v) \in W$ and is a weak solution of problem (1.1), we conclude that such a function satisfies the Neumann boundary condition in the trace sense. This finishes the proof of the theorem. □

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