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Travelling-Wave Solutions and Interfaces for Non-Newtonian Diffusion Equations with Strong Absorption

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Abstract In this paper, we first find finite travelling-wave solutions, and then investigate the short time development of interfaces for non-Newtonian diffusion equations with strong absorption. We show that the initial behavior of the interface depends on the concentration of the mass of $u(x,0)$ near $x=0$. More precisely, we find a critical value of the concentration, which separates the heating front of interfaces from the cooling front of them.

Keywords travelling-wave solutions; interfaces; non-Newtonian diffusion equations; strong absorption.

MR(2010) Subject Classification 35K50; 35K55; 35K65

1. Introduction

In this paper, we study the following non-Newtonian diffusion equations with strong absorption

$$u_t = (|u_x|^{p-2}u_x)_x - \lambda u^q, \quad x \in \mathbb{R}, \ t > 0, \tag{1.1}$$

subject to initial value conditions $u(x,0) = u_0(x)$, where $p > 2$, $0 < q < 1$, $\lambda > 0$ and u_0 is a nonnegative and continuous function and has compact support. The particular feature of the equation (1.1) is their gradient-dependent diffusivity with absorption. Such equations are widely used models for various physical, chemical, and biological problems involving diffusion with absorption. In the non-Newtonian fluids theory, in particular, the parameter p is a characteristic quantity of the medium. According to the behavior of the absorption near $u=0$, we say that u^q is strong absorption if $0 < q < 1$ and u^q is weak absorption if $q > 1$.

We are first interested in a particular class of solutions to (1.1), the finite travelling waves. By a travelling-wave solution with velocity $0 \neq k \in \mathbb{R}$ we mean a solution $u(x,t)$ of (1.1) in $Q = \{(x,t) : -\infty < x < +\infty, t > 0\}$ of the form $u(x,t) = \phi(kt - x)$, where $\phi(\eta) \geq 0, \phi \not\equiv 0$

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and $\phi \to 0$ as $\eta \to -\infty$. In the case $\phi(\eta) = 0$ for $\eta \leq \eta_0$ and some $\eta_0 \in \mathbb{R}$ we say that u is a finite travelling wave. A finite travelling wave with positive (resp., negative) velocity k is called a heating wave (resp., a cooling wave) in thermal propagation. For classical heat equation $u_t = \Delta u - \lambda u^q$ not in the nonlinear equation $u_t = \Delta u^m - \lambda u^q$, the travelling waves have been widely studied since [19] was published [20, 22, 32].

Many authors studied the interfaces of nonlinear diffusion equations in absorbing medium

$$\begin{aligned} &u_t = (u^m)_{xx} - u^q, \quad x \in \mathbb{R},\ t > 0, \\ &u(x, 0) = u_0(x), \qquad x \in \mathbb{R}, \end{aligned} \tag{1.2}$$

where $m > 1$, $q > 0$ and $u_0(x)$ is nonnegative and has compact support. Since the diffusion rate mu^{m-1} vanishes at points where $u = 0$, the initial support propagates at finite speed; that is, there appear interface curves between the region where $u > 0$ and the region where $u = 0$. It is shown in the following papers that supp$u(x, t)$ exhibits three properties:

(i) Positivity. supp$u(x, t)$ expands as t increases and $\lim_{t\to\infty}$supp$u(x, t) = \mathbb{R}$ for $q \geq m$ (see [6, 8, 21, 26, 29]);

(ii) Localization. supp$u(x, t)$ expands as t increases and is uniformly bounded with respect to t for $1 \leq q < m$. There exist constants L_1, L_2 such that supp$u(x, t) \subset [L_1, L_2]$ for all $t > 0$ (see [8, 18, 24, 26, 28, 29]);

(iii) Total extinction. supp$u(x, t)$ is compact for $0 < q < 1$. Thus supp$u(x, t)$ expands and/or shrinks and $u(x, t)$ becomes extinct in a finite time: $u(x, t) \equiv 0$ for $t > T^*$, and $u(x, t) \not\equiv 0$ for $t < T^*$, where $T^* > 0$ is some constant and is called the extinction time of $u(x, t)$ (see [24, 26–28]).

Chen et al. [9] studied the support dynamics of the problem (1.2) with strong absorption. They showed that, given a nonnegative, continuous and compactly supported initial function u_0, the number of connected components of the positivity set $\Omega(t) = \{x \in R : u(x, t) > 0\}$ is always finite (even if the initial data have infinitely many peaks). If an initial function as above has only one peak (is "bell-shaped"), it was shown in [9] that the interfaces $\zeta^+(t)$ and $\zeta^-(t)$ are continuous, $\Omega(t) = \{x \in R : \zeta^-(t) < u(x, t) < \zeta^+(t)\}$, where

$$\zeta^+(t) = \sup\{x : u(x, t) > 0\}, \quad \zeta^-(t) = \inf\{x : u(x, t) > 0\}$$

denote the interfaces of solution u at time t and are called the right and left interfaces.

It is worth mentioning that, the support of the corresponding solution of (1.2) is bounded for each time $t > 0$ if $0 < q < 1$ and $u_0(x) = O(1 + |x|)^{-\gamma}$ as $|x| \to \infty$, where $\gamma > 0$ is some constant. For this phenomenon, we say that the support of solution $u(x, t)$ has the property of instantaneous shrinking. namely, $|\zeta^{\pm}(t)| < \infty$ for any $t > 0$ despite the fact $|\zeta^{\pm}(0)| = \infty$. The effect of instantaneous shrinking for (1.2) was discovered in [1, 10], the authors studied the effect of the spatially dependent coefficient on the effect of instantaneous shrinking for the semilinear equation $u_t = u_{xx} - c(x)u^q$ with power-law initial functions. For the Cauchy problem for the equations $u_t = (u^m)_{xx} + (u^q)_x$, this effect was established in [14]. Instantaneous shrinking property of solutions has been also investigated by other authors (see for example [7, 38, 41–43] and references therein).

In this paper we are also interested in the short time development of interfaces of (1.1) with strong absorption. We show the law governing the motion of the free boundaries in a small time. For convenience, we only consider the right interface $\zeta^+(t)$, and the left interface can be handled in a similar way. Since (1.1) is invariant under the transformations $x \to -x, x \to x + c, c \in \mathbb{R}$, without loss of generality we will investigate the case when $\zeta^+(0) = 0$. More precisely, we are interested in the short time behavior of the right interface $\zeta^+(t)$ with $\zeta^+(0) = 0$. Since $p > 2$, the solutions to equation (1.1) have the property of finite propagation locally in time. This means that behaviour of u_0 as $x \to -\infty$ has no influence on our results. Accordingly, we may suppose that $u_0(x) = 0$ as $x < -K$, here the constant K is large enough, which is suitable for existence, uniqueness and comparison results [36]. For more problems concerned with the interfaces of solution to degenerate parabolic equations, we refer the reader to [2–5, 12, 13, 15–17, 31, 33–35, 37, 39, 40] and references therein. We also mention the recent work [11] of Ferreira et al., where the authors investigated the interfaces of inhomogeneous porous medium equations with convection.

Now we state the main results of this paper.

Theorem 1.1 *The equation (1.1) admits a finite travelling-wave solution $u(x,t) = \phi((kt - x))$ with $\phi(0) = 0$ if $k \neq 0$. Moreover,*

$$\begin{array}{lll} \textit{(i)}\ \lim_{\eta\to 0}\eta^{-\frac{p}{p-1-q}}\phi(\eta) = [\lambda\frac{p-1-q}{(p-1)(q+1)}(\frac{p-1-q}{p})^{p-1}]^{\frac{1}{p-1-q}} & \text{if } (p-1)q<1; & \\ \textit{(ii)}\ \lim_{\eta\to +\infty}\eta^{-\frac{p}{p-1-q}}\phi(\eta) = [\lambda\frac{p-1-q}{(p-1)(q+1)}(\frac{p-1-q}{p})^{p-1}]^{\frac{1}{p-1-q}} & \text{if } (p-1)q>1; & \\ \textit{(iii)}\ \lim_{\eta\to +\infty}\eta^{-\frac{p-1}{p-2}}\phi(\eta) = (\frac{p-2}{p-1})^{\frac{p-1}{p-2}}k^{\frac{1}{p-2}} & \text{if } k>0,\ (p-1)q<1; & (1.3)\\ \textit{(iv)}\ \lim_{\eta\to 0}\eta^{-\frac{p-1}{p-2}}\phi(\eta) = (\frac{p-2}{p-1})^{\frac{p-1}{p-2}}k^{\frac{1}{p-2}} & \text{if } k>0,\ (p-1)q>1; & \\ \textit{(v)}\ \lim_{\eta\to +\infty}\eta^{-\frac{1}{1-q}}\phi(\eta) = [(1-q)(-\frac{\lambda}{k})]^{\frac{1}{1-q}} & \text{if } k<0,\ (p-1)q<1; & \\ \textit{(vi)}\ \lim_{\eta\to 0}\eta^{-\frac{1}{1-q}}\phi(\eta) = [(1-q)(-\frac{\lambda}{k})]^{\frac{1}{1-q}} & \text{if } k<0,\ (p-1)q>1. & \end{array}$$

In many applications it is important to know whether, for given initial data, the interface $\zeta^+(t)$ is a heating or cooling front (support of the solution $u(x,t)$ expands or contracts with time). Next results shows that the initial behavior of the interface depends on the concentration of the mass of $u(x,0)$ near $x = 0$. We shall compare, locally, $u(x,0)$ with the auxiliary function $u_s(x)$ given by $u_s(x) = C_0(-x)_+^{\frac{p}{p-1-q}}$ where

$$C_0 = [\lambda\frac{p-1-q}{(p-1)(q+1)}(\frac{p-1-q}{p})^{p-1}]^{\frac{1}{p-1-q}}. \tag{1.4}$$

It is easy to see that u_s is the nonnegative stationary solution to the equation (1.1) in $\mathbb{R}$ vanishing on $[0, +\infty)$.

Theorem 1.2 *Let u be any local non-negative solution to (1.1) with $\zeta^+(0) = 0$.*

(i) Suppose that there exist $x_0 \in (-\infty, 0)$ and $C \in (0, C_0)$ such that

$$u(x,0) \le C(-x)_+^{\frac{p}{p-1-q}} \quad \textit{for} \quad x \in [x_0, 0].$$

Then

$$\zeta^+(t) \le -Lt^{\frac{p-1-q}{p(1-q)}} \quad \textit{for any} \quad t \in [0, t_0],$$

for some $L > 0$ and $t_0 > 0$.

(ii) Suppose that there exist $x_0 \in (-\infty, 0)$ and $C \in (C_0, +\infty)$ such that

$$u(x,0) \geq C(-x)_+^{\frac{p}{p-1-q}} \quad \text{for} \quad x \in [x_0, 0].$$

Then

$$\zeta^+(t) \geq Lt^{\frac{p-1-q}{p(1-q)}} \quad \text{for any} \quad t \in [0, t_0],$$

for some $L > 0$ and $t_0 > 0$.

Finally, we give a brief outline of the rest of this paper. In Section 2, we consider the travelling-wave solutions of (1.1) and prove Theorem 1.1. The proof of Theorem 1.2 is the subject of Section 3.

2. Travelling-wave solutions

In this section, by using a phase-plane argument, we find finite travelling-wave solutions for the equation (1.1), and give the asymptotic behavior of these solutions by constructing various upper and lower solutions.

Inserting $u(x,t) = \phi((kt - x))$ into (1.1), we have

$$(|\phi'|^{p-2}\phi')' - k\phi' - \lambda\phi^q = 0 \quad \text{in} \quad \mathbb{R}.$$

The above equation is understood in weak sense, i.e., $|\phi'|^{p-2}\phi'$ and ϕ are continuous functions in $\mathbb{R}$ and the equation is satisfied in its standard integral version. Since the finite travelling-wave solution ϕ vanishes for $\eta \leq \eta_0$, by translation we may put $\eta_0 = 0$. Then the equation (1.1) admits a finite travelling-wave solution if we find a positive function ϕ in $\mathbb{R}^+$ such that

$$\begin{aligned} &(|\phi'|^{p-2}\phi')' - k\phi' - \lambda\phi^q = 0, \\ &\phi(0) = 0, \quad \phi'(0) = 0 \end{aligned} \tag{2.1}$$

with $k \neq 0$ since we can prolong solution ϕ by 0 on $(-\infty, 0)$. In the sequel we analyze the corresponding phase portrait of the problem (2.1). To this aim, we first give a monotonicity property of ϕ.

Lemma 2.1 *If ϕ is a positive solution to (2.1), then ϕ is increasing in $(0, +\infty)$.*

Proof For $k < 0$, we suppose that ϕ is not increasing in $(0, +\infty)$. There exists some η_0 such that ϕ is strictly increasing on $(0, \eta_0)$ and η_0 is local maximum, then $(|\phi'|^{p-2}\phi')'(\eta_0) \leq 0$. But by (2.1), we see that $(|\phi'|^{p-2}\phi')'(\eta_0) > 0$, which is a contradiction. Consequently $\phi'(\eta) > 0$ for any $\eta > 0$.

For $k > 0$, we define $\Phi(\eta) = \frac{p-1}{p}|\phi'(\eta)|^p - \frac{\lambda}{q+1}\phi^{q+1}(\eta)$. Then it follows from (2.1) that $\Phi'(\eta) = k(\phi')^2 \geq 0$. Assume that ϕ is not increasing in $(0, +\infty)$ and let η_0 be the first zero of ϕ'. Then $0 = \Phi(0) \leq \Phi(\eta_0) = -\frac{\lambda}{q+1}\phi^{q+1}(\eta_0) < 0$, which is a contradiction. Consequently $\phi'(\eta) > 0$ for any $\eta > 0$. $\Box$

We shall show that there exists a unique $\phi(\eta) > 0$ in $\mathbb{R}^+$ satisfying (2.1). Let us introduce the variables:

$$X = \phi, \quad Y = (\phi')^{p-1}$$

and then our problem can be reformulated as finding the nontrivial trajectories of the differential system

$$X' = Y^{\frac{1}{p-1}}, \quad Y' = kY^{\frac{1}{p-1}} + \lambda X^q,$$

which start from $(0,0)$ at $\eta = 0$, exist for $0 < \eta < +\infty$, and are contained in the first quadrant $\Omega_1 = \{(X,Y) : X > 0, Y > 0\}$ for $\eta > 0$. We claim that there exists one and only one such trajectory $Y(X)$. To show this, we write the system of O.D.E for the trajectories:

$$\begin{aligned} &\frac{\mathrm{d}Y}{\mathrm{d}X} = f(X,Y) = k + \lambda X^q Y^{-\frac{1}{p-1}}, \\ &Y(0) = 0. \end{aligned} \tag{2.2}$$

We shall find the nontrivial trajectories $Y(X)$ to (2.2) by two steps. First we prove the global existence of the solution of the following approximation problem

$$\begin{aligned} &\frac{\mathrm{d}Y}{\mathrm{d}X} = k + \lambda X^q Y^{-\frac{1}{p-1}}, \\ &Y(0) = \epsilon, \quad \epsilon > 0. \end{aligned} \tag{2.3}$$

Since the function $f(X,Y)$ is locally Lipschitz continuous function in $\mathbb{R}^+ \times (\epsilon, +\infty)$, from the theory of the ordinary differential equations, we have the existence of unique local solution Y_ϵ. For $k > 0$, the function $Y_\epsilon(X)$ is strictly increasing and satisfies the following inequality

$$\frac{\mathrm{d}Y_\epsilon}{\mathrm{d}X} \leq k + \lambda X^q \epsilon^{-\frac{1}{p-1}}$$

and then Y_ϵ is global solution.

For $k < 0$, define the curve $\widetilde{C} : Y(X) = (-\frac{\lambda}{k}X^q)^{p-1}$, then we have $f(X,Y) = 0$ on $\widetilde{C}$ and the curve $\widetilde{C}$ divides the first quadrant Ω_1 into two regions: $R_l = \{(X,Y) : f(X,Y) < 0\}$ and $R_r = \{(X,Y) : f(X,Y) > 0\}$ (see Figure 1). Y_ϵ starts in region R_l, then Y_ϵ must cross $\widetilde{C}$ at some point with horizontal tangent and after Y_ϵ lies in the region R_r, where Y_ϵ is strictly increasing. Hence the minimum M_ϵ of Y_ϵ reaches on $\widetilde{C}$ and is strictly positive. So

$$\frac{\mathrm{d}Y_\epsilon}{\mathrm{d}X} \leq k + \lambda X^q M_\epsilon^{-\frac{1}{p-1}}$$

and then Y_ϵ is a global solution.

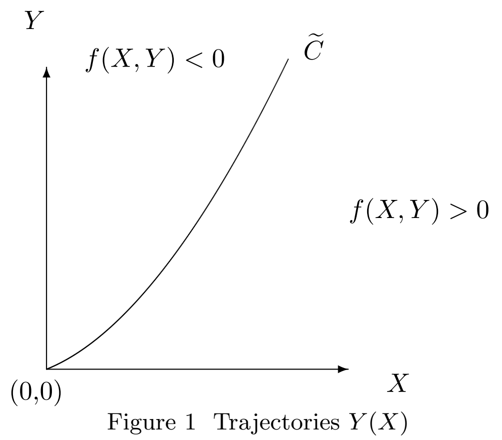

Figure 1 Trajectories $Y(X)$

Next we prove the global existence of the Cauchy problem

$$\begin{aligned} &\frac{\mathrm{d}Y}{\mathrm{d}X} = k + \lambda X^q Y^{-\frac{1}{p-1}}, \\ &Y(\epsilon) = 0. \end{aligned} \tag{2.4}$$

To this end, we consider the Cauchy problem

$$\begin{aligned} &\frac{\mathrm{d}v}{\mathrm{d}t} = g(t,v) = \frac{1}{f(v,t)} = \frac{t^{\frac{1}{p-1}}}{\lambda v^q + kt^{\frac{1}{p-1}}}, \\ &v(0) = \epsilon. \end{aligned} \tag{2.5}$$

It is easy to see that the problem (2.5) has a unique local solution v_ϵ. For $k > 0$, we have $0 \le \frac{\mathrm{d}v_\epsilon}{\mathrm{d}t} \le \frac{1}{k}$ and then v_ϵ is global. For $k < 0$, we denote by $\widetilde{C}$ the curve $f(v,t) = 0$. Then $\widetilde{C}$ divides the first quadrant Ω_1 into two regions: $R_l = \{(v,t) : f(v,t) > 0\}$ and $R_r = \{(v,t) : f(v,t) < 0\}$ (see Figure 2). v_ϵ starts in region R_l and $\frac{\mathrm{d}v_\epsilon}{\mathrm{d}t}$ is strictly positive and approaches $+\infty$ as $f(v_\epsilon, t) \to 0$. Consequently v_ϵ is strictly increasing and does never touch the curve $\widetilde{C}$. Therefore v_ϵ is global. Moreover, we can see $\lim_{t\to+\infty} v_\epsilon(t) = +\infty$. This means that v_ϵ is a one to one from $[0,+\infty)$ to $[\epsilon,+\infty)$. Now let w_ϵ be the inverse function of v_ϵ defined from $[\epsilon,+\infty)$ to $[0,+\infty)$. It is easy to see that w_ϵ satisfies the following Cauchy problem

$$\begin{aligned} &\frac{\mathrm{d}w_\epsilon}{\mathrm{d}X} = k + \lambda X^q w_\epsilon^{-\frac{1}{p-1}}, \\ &w_\epsilon(\epsilon) = 0. \end{aligned}$$

Therefore, the problem (2.4) has a unique global solution for any $\epsilon > 0$.

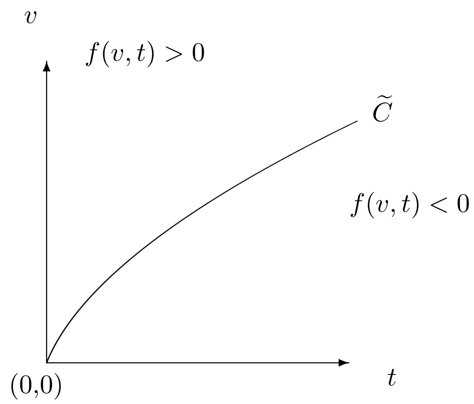

Figure 2 Trajectories $v(t)$

Lemma 2.2 *The problem (2.2) has a unique global solution.*

Proof We will prove the uniqueness at first. Assume that there exist two solutions Y_1 and Y_2 of (2.2) such that $Y_1 \not\equiv Y_2$. Define $M = \sup\{r > 0; Y_1(X) = Y_2(X) \text{ for } 0 \le X < r\}$ and let N be close to M, such that $N > M$. Without loss of generality we assume $Y_1(N) > Y_2(N)$. Set $Z(X) = Y_1(X) - Y_2(X)$, then there exists some $X_0 \in (M, N]$ satisfying

$$0 \le Z(N) - Z(M) = \lambda X_0^q (Y_1^{-\frac{1}{p-1}}(X_0) - Y_2^{-\frac{1}{p-1}}(X_0))(N - M) < 0,$$

which is a contradiction and then $Y_1 \equiv Y_2$.

It remains to prove the existence. Let Y_ϵ be the solution of (2.3). As $\{Y_\epsilon\}$ is a decreasing sequence as $\epsilon \to 0^+$, there exists some function $Y(X)$ such that $\lim_{\epsilon\to 0^+} Y_\epsilon(X) = Y(X)$ with $Y(0) = 0$. In order to prove that Y is the solution of (2.2), we firstly prove that $Y(X)$ is strictly positive for any $X > 0$. Let Y^ϵ be the solution of (2.4). Since Y_ϵ and Y^ϵ satisfy the same equation in $(\epsilon, +\infty)$. Taking any $\delta_0 \in (0, +\infty)$ and using the fact that $\{Y^\epsilon\}$ is an increasing sequence as $\epsilon \to 0^+$, we get

$$Y(\delta_0) = \lim_{\epsilon\to 0} Y_\epsilon(\delta_0) \geq \lim_{\epsilon\to 0} Y^\epsilon(\delta_0) \geq Y^{\frac{\delta_0}{2}}(\delta_0) > 0$$

and then Y is strictly positive on $(0, +\infty)$.

Now we will prove that the function Y is a solution of the problem (2.2). In fact, since Y_ϵ is the solution of (2.3), for any test function $\varphi \in D((0, +\infty))$ we have

$$\int_0^{+\infty} Y_\epsilon(X)\varphi'(X)\mathrm{d}X + \int_0^{+\infty} (k + \lambda X^q Y_\epsilon^{-\frac{1}{p-1}}(X))\varphi(X)\mathrm{d}X = 0.$$

By letting $\epsilon \to 0$ we get

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = k + \lambda X^q Y^{-\frac{1}{p-1}}(X) \text{ in } D'([0, +\infty)). \tag{2.6}$$

Then for any $0 < \alpha < \beta$, we deduce $\int_\alpha^\beta |\frac{\mathrm{d}Y}{\mathrm{d}X}|\mathrm{d}X$ is finite, therefore $Y \in W^{1,\gamma}((\alpha, \beta))$ for any $\gamma \in \mathbb{N} - \{0\}$. Then Y and $\frac{\mathrm{d}Y}{\mathrm{d}X}$ are continuous in (α, β). Therefore (2.6) holds in the usual sense in $(0, +\infty)$. $\square$

Let Y be the solution of the problem (2.2). For the problem

$$\frac{\mathrm{d}\phi}{\mathrm{d}\eta} = Y^{\frac{1}{p-1}}(\phi(\eta)), \quad \phi(0) = 0, \tag{2.7}$$

there exists a unique maximal solution defined in $(-\infty, \beta)$ such that $\lim_{\eta\to\beta^-} \phi(\eta) = +\infty$. In fact we have $\phi'(0) = 0$ by (2.7), then we can prolong solution ϕ by 0 on $(-\infty, 0)$. On the other hand, as ϕ is strictly increasing, we obtain $\lim_{\eta\to\beta^-} \phi(\eta) = +\infty$ if β is finite; while if $\beta = +\infty$, it is easy enough to use (2.7) to get also $\lim_{\eta\to\beta^-} \phi(\eta) = +\infty$.

Noting that the solution of (2.7) defined in $(-\infty, \beta)$ satisfies

$$\begin{aligned} &(|\phi'|^{p-2}\phi')' - k\phi' - \lambda\phi^q = 0 \text{ in } (-\infty, \beta), \\ &\phi(0) = 0, \quad \phi'(0) = 0. \end{aligned} \tag{2.8}$$

Next, we will prove the solution of (2.8) is global. For that we need the asymptotic behavior of the solution $Y(X)$ to the problem (2.2).

Lemma 2.3 *Let Y be the solution of (2.2). Then we have*

(i) $Y(X) \approx [\frac{\lambda p}{(p-1)(q+1)}]^{\frac{p-1}{p}} X^{\frac{(p-1)(q+1)}{p}}$ *as* $X \to 0$ *if* $(p-1)q < 1$;

(ii) $Y(X) \approx [\frac{\lambda p}{(p-1)(q+1)}]^{\frac{p-1}{p}} X^{\frac{(p-1)(q+1)}{p}}$ *as* $X \to +\infty$ *if* $(p-1)q > 1$;

(iii) $Y(X) \approx kX$ *as* $X \to +\infty$ *if* $k > 0$, $(p-1)q < 1$;

(iv) $Y(X) \approx kX$ *as* $X \to 0$ *if* $k > 0$, $(p-1)q > 1$;

(v) $Y(X) \approx (-\frac{k}{\lambda})^{1-p} X^{(p-1)q}$ *as* $X \to +\infty$ *if* $k < 0$, $(p-1)q < 1$;

(vi) $Y(X) \approx (-\frac{k}{\lambda})^{1-p} X^{(p-1)q}$ *as* $X \to 0$ *if* $k < 0$, $(p-1)q > 1$,
here $f(x) \approx g(x)$ *as* $x \to c$ *means that* $\frac{f(x)}{g(x)} \to 1$ *as* $x \to c$.

Proof We only prove conclusions (i), (ii), and conclusions (iii)–(vi) can be obtained in a similar way. We consider $\widetilde{Y}(X) = AX^{\alpha}$, where $A, \alpha > 0$ are determined later. It is easy to see that $\widetilde{Y}$ is a supersolution of (2.2) (resp., subsolution) if and only if

$$A\alpha X^{\alpha-1} \geq k + \lambda X^q A^{-\frac{1}{p-1}} X^{-\frac{\alpha}{p-1}}, \tag{2.9}$$

respectively,

$$A\alpha X^{\alpha-1} \leq k + \lambda X^q A^{-\frac{1}{p-1}} X^{-\frac{\alpha}{p-1}}. \tag{2.10}$$

Let $\alpha = \frac{(p-1)(q+1)}{p}$. Then (2.9) and (2.10) become

$$\frac{(p-1)(q+1)}{p} A \geq kX^{1-\frac{(p-1)(q+1)}{p}} + \lambda A^{-\frac{1}{p-1}},$$

respectively,

$$\frac{(p-1)(q+1)}{p} A \leq kX^{1-\frac{(p-1)(q+1)}{p}} + \lambda A^{-\frac{1}{p-1}}.$$

Noting that $(p-1)q < 1$ implies that $\frac{(p-1)(q+1)}{p} < 1$. Then $\widetilde{Y}$ is a supersolution of (2.2) (resp., sub-solution) in the neighborhood 0 for all $A > [\frac{\lambda p}{(p-1)(q+1)}]^{\frac{p-1}{p}}$ (resp., $A < [\frac{\lambda p}{(p-1)(q+1)}]^{\frac{p-1}{p}}$). Consequently

$$Y(X) \approx [\frac{\lambda p}{(p-1)(q+1)}]^{\frac{p-1}{p}} X^{\frac{(p-1)(q+1)}{p}} \quad \text{as} \quad X \to 0.$$

Noting that $(p-1)q > 1$ implies that $\frac{(p-1)(q+1)}{p} > 1$. Then $\widetilde{Y}$ is a supersolution of (2.2) (resp., sub-solution) in the neighborhood $+\infty$ for all $A > [\frac{\lambda p}{(p-1)(q+1)}]^{\frac{p-1}{p}}$ (resp., $A < [\frac{\lambda p}{(p-1)(q+1)}]^{\frac{p-1}{p}}$). Consequently

$$Y(X) \approx [\frac{\lambda p}{(p-1)(q+1)}]^{\frac{p-1}{p}} X^{\frac{(p-1)(q+1)}{p}} \quad \text{as} \quad X \to +\infty. \ \Box$$

Proof of Theorem 1.1 As long as $\phi(\eta) \neq 0$ (consequently $Y(\phi(x)) \neq 0$), we rewrite the equality (2.7) as

$$Y^{-\frac{1}{p-1}}(\phi(\eta))\phi'(\eta) = 1. \tag{2.11}$$

Integrating (2.11) on $(\eta_1, \eta) \subset (0, \beta)$ yields

$$\eta - \eta_1 = \int_{\phi(\eta_1)}^{\phi(\eta)} Y^{-\frac{1}{p-1}}(s)\mathrm{d}s. \tag{2.12}$$

By letting $\eta \to \beta$ in (2.12), we obtain $\beta = \infty$ if and only if

$$\int^{+\infty} Y^{-\frac{1}{p-1}}(s)\mathrm{d}s = \infty. \tag{2.13}$$

By (ii), (iii), (v) in Lemma 2.3, we know that the formula (2.13) holds, then the solution of (2.8) is global and the equation (1.1) admits a finite travelling-wave solution $u(x,t) = \phi((kt-x))$. Combining Lemma 2.3 and $\frac{\mathrm{d}\phi}{\mathrm{d}\eta} = Y^{\frac{1}{p-1}}(\phi(\eta))$, and integrating, we obtain (1.3) in Theorem 1.1.
$\Box$

3. Interfaces

In order to study the interface, we first study the self-similar solutions of the equation (1.1) with special initial data

$$u(x,0) = C(-x)_+^{\frac{p}{p-1-q}} \quad \text{for} \quad C > 0. \tag{3.1}$$

Lemma 3.1 *Let u be the solution of the equation (1.1) with $u(x,0)$ given by (3.1). Then there exists a function $f : \mathbb{R} \to [0,\infty)$ such that*

$$u(x,t) = t^{\frac{1}{1-q}} f\Big(\frac{x}{t^{\frac{p-1-q}{p(1-q)}}}\Big). \tag{3.2}$$

Moreover $\operatorname{supp}(f) = (-\infty, L]$ *with $L < +\infty$, i.e.,*

$$f(y) = 0 \quad \textit{for any} \quad y \geq L. \tag{3.3}$$

Proof We put

$$u_k(x,t) = ku(k^{-\frac{p-1-q}{p}}x, k^{q-1}t),$$

where $k > 0$ and $u(x,t)$ is the solution of (1.1) with $u(x,0) = C(-x)_+^{\frac{p}{p-1-q}}$, $C > 0$. Then $u_k(x,t)$ solves the problem

$$\begin{aligned} &u_{kt} = (|u_{kx}|^{p-2}u_{kx})_x - \lambda u_k^q, && x \in \mathbb{R}, t > 0, \\ &u_k(x,0) = C(-x)_+^{\frac{p}{p-1-q}}, && x \in \mathbb{R}. \end{aligned}$$

By the uniqueness of the solution to the equation (1.1) with $u(x,0)$ given by (3.1), we conclude that $u_k(x,t) = u(x,t)$ for any $k > 0$. Now, given $\tau > 0$, we choose $k = \tau^{\frac{1}{1-q}}$ and so we have that $u(x,\tau) = \tau^{\frac{1}{1-q}} u(\tau^{-\frac{p-1-q}{p(1-q)}}x, 1)$. Finally, given $y \in \mathbb{R}$, we define $f(y) = u(y,1)$. Making $t = \tau$, we obtain (3.2). In order to prove (3.3), we consider the Cauchy problem

$$\begin{aligned} &\overline{u}_t = (|\overline{u}_x|^{p-2}\overline{u}_x)_x, && x \in \mathbb{R}, t > 0, \\ &\overline{u}(x,0) = C(-x)_+^{\frac{p}{p-1-q}}, && x \in \mathbb{R}. \end{aligned}$$

Using the comparison principle, we have that $0 \leq u(x,t) \leq \overline{u}(x,t)$ for any $(x,t) \in \mathbb{R} \times [0,+\infty)$. Since the operator $(|u_x|^{p-2}u_x)_x$ has a finite propagation property for $p > 2$, we know that for any $t \geq 0$ we have that $\sup\{x : \overline{u}(x,t) > 0\} < +\infty$ and hence $\sup\{x : u(x,t) > 0\} < +\infty$ is also finite. Choosing $t = 1$, we obtain (3.3). □

For the equation (1.1) with $u(x,0) = C(-x)_+^{\frac{p}{p-1-q}}$, $C > 0$, we know from Lemma 3.1 that the interface $\zeta^+(t) = Lt^{\frac{p-1-q}{p(1-q)}}$. And so the behavior of $\zeta^+(t)$ is determined by the sign of L.

Lemma 3.2 *Let C_0, $u(x,0)$ and f be given by (1.4), (3.1) and (3.2), respectively. Let $L \in \mathbb{R}$ be defined by $L = \sup\{y : f(y) > 0\}$. Then we have (i) $C = C_0$ implies $L = 0$; (ii) $C < C_0$ implies $L < 0$; (iii) $C > C_0$ implies $L > 0$.*

Proof If $C = C_0$, it follows from the uniqueness of solutions to the equation (1.1) with $u(x,0)$ given by (3.1) that $u(x,t) = C_0(-x)_+^{\frac{p}{p-1-q}}$ and so the conclusion (i) holds. When $C \neq C_0$, we divide the proof into three cases.

Case 1 $(p-1)q < 1$. By Theorem 1.1, there exists a family of travelling-wave solutions to the equation (1.1) of the form $u(x,t;k) = \phi((kt-x))$ for arbitrary $k \in \mathbb{R}$. Now let $C < C_0$ and take $k < 0$. By (1.3) there exists $M > 0$ such that

$$\phi(\eta) > C\eta^{\frac{p}{p-1-q}} \quad \text{for} \quad 0 < \eta < M.$$

On the other hand, by the continuity of ϕ and u, there exists $t_0 > 0$ such that

$$u(-M,t;k) \geq u(-M,t) \quad \text{for any} \quad t \in [0,t_0].$$

Then we compare $u(x,t;k)$ and $u(x,t)$ in the region $[-M,M] \times [0,t_0]$ and obtain

$$u(x,t;k) \geq u(x,t) \quad \text{in} \quad [-M,M] \times [0,t_0].$$

Thus, since $k < 0$, the conclusion (ii) holds. Finally, if $C > C_0$, we choose $k > 0$ and by a similar argument we obtain (iii).

Case 2 $(p-1)q = 1$. In this case function f can be made explicit and so

$$u(x,t) = C\Big[\big(((\frac{p-1}{p-2})^{p-1}C^{p-2} - \lambda(\frac{p-2}{p-1})C^{q-1})t - x\big)_+\Big]^{\frac{p-1}{p-2}}.$$

It is easy to check that the assertions (ii) and (iii) follow.

Case 3 $(p-1)q > 1$. Given $0 \neq k \in \mathbb{R}$ and $\xi \in \mathbb{R}$, again by Theorem 1.1 and under the transformations $x \to x + \xi$, there exists a family of travelling-wave solutions to the equation in (1.1) of the form $u(x,t;k,\xi) = \phi((kt - x + \xi)$. If $C < C_0$, by (1.3) there exists $M > 0$ such that

$$\phi(\eta) > C\eta^{\frac{p}{p-1-q}} \quad \text{for} \quad \eta > M.$$

Let $\xi = M$. $u(x,t;k,M)$ is a solution of (1.1) with $u(x,0) = \phi((-x+M))$. Besides, from (1.3), we obtain that

$$\phi((-x+M)) \geq C(-x)_+^{\frac{p}{p-1-q}} \quad \text{for any} \quad x \in \mathbb{R}.$$

Then by the comparison principle we have that

$$u(x,t;k,M) \geq u(x,t) \quad \text{for any} \quad x \in \mathbb{R}, \quad t \geq 0.$$

Finally, choosing $k < -M < 0$, we obtain (ii).

If $C > C_0$, we choose $k > 0$. By (1.3) we have that

$$\phi(\eta) < C\eta^{\frac{p}{p-1-q}} \quad \text{for} \quad \eta > M$$

for some $M > 0$. Let $K = \max\{\phi(\eta) : 0 \leq \eta \leq M\}$ and $\xi = \max\{M, (\frac{K}{C})^{\frac{p-1-q}{p}}\}$. $u(x,t;k,-\xi) = \phi((kt - x - \xi))$ is a solution of (1.1) with $u(x,0) = \phi((-x-\xi))$. In addition

$$\phi((-x-\xi)) \leq C(-x)_+^{\frac{p}{p-1-q}} \quad \text{for any} \quad x \in \mathbb{R}.$$

Then by the comparison result $u(x,t;k,-\xi) \leq u(x,t)$ for any $x \in \mathbb{R}$, $t \geq 0$. We select $k > \xi > 0$ and see that (iii) holds. $\Box$

Proof of Theorem 1.2 Let u be any continuous solution of the equation (1.1). Assume that there exist $x_0 \in (-\infty, 0)$ and $C \in (0, C_0)$ such that

$$u(x,0) \leq C(-x)_+^{\frac{p}{p-1-q}} \quad \text{if} \quad x \in [x_0, 0]. \tag{3.4}$$

Let C_1 satisfy $C < C_1 < C_0$, and let $\overline{v}$ be the solution of the equation (1.1) with $u(x,0) = C_1(-x)_+^{\frac{p}{p-1-q}}$ for $x \in \mathbb{R}$. From the continuity of u, $\overline{v}$ and the inequality (3.4), we deduce the existence of a time $t_0 > 0$ such that $u(x_0,t) \le \overline{v}(x_0,t)$ for any $t \in [0,t_0]$. Then we are allowed to apply the comparison principle for the solution to the equation (1.1) on the set $Q_0 = (x_0,+\infty) \times (0,t_0)$ and thus we have

$$u(x,t) \le \overline{v}(x,t) \quad \text{for any} \quad (x,t) \in \overline{Q}. \tag{3.5}$$

By Lemma 3.2 and (3.5), the conclusion (i) of Theorem 1.2 is immediate. The proof of the assertion (ii) is similar to the previous one. □

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