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Stability Analysis of n-Dimensional Neutral Differential Equations with Multiple Delays

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Abstract This paper is mainly concerned with stability analysis of neutral differential equations with multiple delays. Some criteria on instability, stability, asymptotic stability and exponential stability are obtained. The criterion on asymptotic stability is necessary and sufficient. Two examples are provided to illustrate the applications of our results. Some previous results are extended.

Keywords stability analysis; neutral differential equation; multiple delays.

MR(2010) Subject Classification 39B82

1. Introduction

The class of equations involving derivatives as well as the function itself are called neutral differential equations or neutral differential difference equations. In the past several decades, neutral differential equations have become important in modeling some real phenomenon such as oscillatory systems with some interconnections between them, coupled systems, the theory of automatic control or population dynamics [1, 2], distributed networks containing lossless transmission [3], etc. Neutral differential equations have been investigated since last 1960's, see [1–24] and the references therein. Stability analysis for neutral differential equations has been the focus of researcher's attention [4–18]. It is necessary to mention the excellent work of Park et al., which are on stability analysis of neutral differential equations. Readers can refer to, for example, Ref. [4–8].

In this paper, we mainly discuss the stability, the asymptotical stability, the exponential stability and the instability for neutral differential equation

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} B_i x(t-\tau_i) + \sum_{i=1}^{m} C_i \dot{x}(t-\tau_i), \quad t > 0 \tag{1.1}$$

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with initial condition

$$x(t) = \phi(t), \quad -\tau_m \le t \le 0, \tag{1.2}$$

where $x(t) \in \mathbb{R}^n$ is the vector, A, B_i and C_i are $n \times n$ constant matrices $(i = 1, 2, \dots, m)$, τ_i is time-delay satisfying $0 < \tau_1 < \tau_2 < \cdots < \tau_m$, $\det C_m \ne 0$, $\phi(t) \in \mathbb{R}^n$ is the given initial function vector which is continuous or has finite discontinuous points on interval $[-\tau_m, 0]$.

Denote by $\{\lambda_j\}$ the set of characteristic roots of equation (1.1), $\mathfrak{Re}(\lambda)$ the real parts of λ and $\Lambda = \sup\{\mathfrak{Re}\lambda_j\}$. In $\mathbb{R}^1$ space, Hale et al [19] obtained the following theorem.

Theorem 1.1 ([19]) *If $\Lambda < 0$, then the zero solution of Eq. (1.1) is exponentially stable.*

For the case $\mathfrak{Re}\lambda_j < 0$ and $\Lambda = 0$, the asymptotical stability of the zero solution of Eq. (1.1) is complex, which was put forward in [19] and [20]. The problem remains unsolved until Ren [18] partly solved the problem in 1999. Ren's results [18] hold only in $\mathbb{R}^1$ space. They are as follows.

Theorem 1.2 ([18]) *If $\mathfrak{Re}\lambda_j < 0$ and $\Lambda = 0$, then the zero solution of Eq. (1.1) is asymptotically stable.*

Furthermore, Ren [18] obtained a necessary and sufficient criterion.

Theorem 1.3([18]) *The zero solution of Eq. (1.1) is asymptotically stable iff the real parts of all the characteristic roots are negative.*

Recently, Eq. (1.1) has been investigated for numerical approximation in $\mathbb{R}^n$ space [8, 15–17, 21, 22]. By methods such as linear the multistep methods, the Lyapunov method, matrix inequality, the Runge-kutta method and the BDFs methods, etc, many results on stability of Eq. (1.1) were derived. In 1998, Zhang and Zhou [15] presented a sufficient condition of asymptotical stability through the spectral radius of modulus matrices. Later in 2004, He and Cao [16] gave some simple delay-independent stability criteria for the asymptotic stability in terms of the spectral radius of modulus matrices. In 2005, Park and Kwon [8] provided a novel stability criterion based on the Lyapunov method. Very recently, Kuang et al. [17] obtained a new sufficient condition of asymptotic stability. The key theoretical bases in these papers are:

- If the zero solution of Eq. (1.1) is asymptotically stable, then the real parts of all characteristic roots of Eq. (1.1) are negative. Namely, $\mathfrak{Re}\lambda_j < 0$, where λ_j is the characteristic roots of Eq. (1.1);
- If there exists a positive number $\gamma > 0$ such that the real parts of all characteristic roots of Eq. (1.1) satisfy: $\Lambda = \sup\{\mathfrak{Re}\lambda_j\} < -\gamma$, then the zero solution of Eq. (1.1) is asymptotically stable.

With regard to the case that $\mathfrak{Re}\lambda_j < 0$ and $\Lambda = 0$ in $\mathbb{R}^n$ $(n \ge 2)$ space, we have not retrieved any results on stability analysis of Eq. (1.1). Motivated by the fact, in this paper we mainly investigate the stability, the asymptotical stability, the instability and the exponential stability of Eq. (1.1). We solve the problem put forward in [19] and [20].

Before presenting our results, we first recall some preliminaries which will be used later.

Definition 1.1 ([19]) *We say the function $x(\phi)(t)$ is a solution of Eq. (1.1) and Eq. (1.2) if*

$x(\phi)(t)$ is defined on $[-\tau_m, \infty]$, the difference

$$x(t) - \sum_{i=1}^{m} C_i x(t - \tau_i)$$

is almost everywhere differentiable and $x(\phi)(t)$ satisfies Eqs. (1.1) and (1.2).

Definition 1.2 ([19]) *The zero solution of Eq. (1.1) is said to be*

(a) Stable iff for any $\varepsilon > 0$, there exists a positive real number $\delta(\varepsilon) > 0$ such that for $\forall\phi \in \mathcal{C} = C([-\tau_m, 0], \mathbb{R}^n)$, when $|\phi| < \delta$, the solution $|x(\phi)(t)|$ of Eqs. (1.1) and (1.2) satisfies: $|x(\phi)(t)| \le \varepsilon$ for $t > 0$;

(b) Asymptotically stable iff the zero solution $x(\phi)(t)$ of Eq. (1.1) is stable and $\lim_{t\to+\infty} |x(\phi)(t)| = 0$;

(c) Exponentially stable iff there exist constants $a, b > 0$ such that the solution $x(\phi)(t)$ of Eqs. (1.1) and (1.2) satisfies: $|x(\phi)(t)| \le a|\phi|e^{-bt}$ for $t > 0$,
where $|\cdot|$ denotes vector norm. Particularly, for $\phi \in \mathcal{C}$, define $|\phi| = \sup_{-r\le\theta\le0} |\phi(\theta)|$.

Lemma 1.1 *Let $x = (x_1, x_2, \ldots, x_n)^T$, $A = (a_{ij})_{n\times n}$. Then the following holds true*

$$|Ax| \le |A| \cdot |x|,$$

where $|x| = (\sum_{i=1}^{n} |x_i|^2)^{1/2}$ and $|A| = (\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2)^{1/2}$.

Lemma 1.2 ([25]) *Suppose that $V \in C^{n\times n}$ and $\rho(V) < 1$. Then $(I - V)^{-1}$ exists, and*

$$(I - V)^{-1} = I + V + V^2 + V^3 + \cdots,$$

where the notation $\rho(V)$ denotes the spectral norm of the matrix V.

Throughout this paper, notation $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, I denotes the unit matrix of appropriate order, $|\cdot|$ denotes either the Euclidean vector norm or the induced matrix 2- norm, $\mathfrak{Re}\lambda$ denotes the real part of λ, and $(\cdot)^T$ denotes the transpose of either the matrix $(\cdot)$ or the vector$(\cdot)$.

2. Existence and exponential growth

Theorem 2.1 *For Eqs. (1.1) and (1.2), there always exists a solution $x(\phi)(t)$ defined on $[0, \infty]$.*

Proof Eq. (1.1) can be written as

$$\Big[e^{-At}x(t) - e^{-At}\sum_{i=1}^{m} C_i x(t - \tau_i)\Big]' = e^{-At}\sum_{i=1}^{m} B_i x(t - \tau_i) + Ae^{-At}\sum_{i=1}^{m} C_i x(t - \tau_i). \tag{2.1}$$

Integrating Eq. (2.1) from 0 to t yields

$$\begin{aligned} x(t) =& e^{At}\Big[x(0) - \sum_{i=1}^{m} C_i\phi(-\tau_i)\Big] + \sum_{i=1}^{m} C_i x(t - \tau_i) + \\ & e^{At}\Big[\sum_{i=1}^{m}\int_0^t e^{-At}B_i x(t - \tau_i)\mathrm{d}t + \sum_{i=1}^{m}\int_0^t Ae^{-At}C_i x(t - \tau_i)\mathrm{d}t\Big]. \end{aligned}$$

For $0 < t < \tau_1$, the following holds true

$$x(t) = e^{At}\Big[\phi(0) - \sum_{i=1}^{m} C_i\phi(-\tau_i)\Big] + \sum_{i=1}^{m} C_i\phi(t-\tau_i) + \\ e^{At}\Big[\sum_{i=1}^{m}\int_0^t e^{-At}B_i\phi(t-\tau_i)\mathrm{d}t + \sum_{i=1}^{m}\int_0^t Ae^{-At}C_i\phi(t-\tau_i)\mathrm{d}t\Big].$$

Repeating the above process respectively on the intervals $[\tau_1, 2\tau_1], \ldots, [n\tau_1, (n+1)\tau_1], \ldots,$ we can obtain the solution $x(\phi)(t)$ of Eqs. (1.1) and (1.2) on $[0, +\infty)$. This completes the proof. □

Using similar method in [19] or [20], we can easily obtain the following theorem, which is the basis of applying the Laplace transform on the solution of Eqs. (1.1) and (1.2).

Theorem 2.2 *The solution $x(\phi)(t)$ of Eqs. (1.1) and (1.2) is exponential bounded. Namely, there exist constants $\alpha > 0$ and $\gamma > 0$ such that the solution $x(\phi)(t)$ of Eqs. (1.1) and (1.2) satisfies*

$$|x(\phi)(t)| \le \alpha|\phi|e^{\gamma t}.$$

3. Stability, asymptotic stability and instability

Definition 3.1 *If $n \times n$ matrix $X(t)$ satisfies*

$$\dot{X}(t) = AX(t) + \sum_{i=1}^{m} B_iX(t-\tau_i) + \sum_{i=1}^{m} C_i\dot{X}(t-\tau_i), \quad t > 0 \tag{3.1}$$

and $\det X(t) \neq 0$ *for some $t \in [0, +\infty)$, then we say that matrix $X(t)$ is a nonsingular solution matrix of Eq. (1.1).*

If the matrix $X(t)$ satisfies Eq. (3.1) and

$$X(t) = \begin{cases} I, & t = 0, \\ 0, & t < 0, \end{cases}$$

then we say that $X(t)$ is a fundamental solution of Eq. (1.1).

Suppose that $x(t) = e^{\lambda t}\vec{k}$ is a solution of Eq. (1.1), $\vec{k} = (k_1, k_2, \ldots, k_n)^T$. Substituting $x(t) = e^{\lambda t}\vec{k}$ into Eq. (1.1) yields

$$\det\Big(\lambda I - A - \sum_{i=1}^{m} B_ie^{-\lambda\tau_i} - \sum_{i=1}^{m} \lambda C_ie^{-\lambda\tau_i}\Big) = 0.$$

Definition 3.2 *The matrix*

$$H(\lambda) = \Big(\lambda I - A - \sum_{i=1}^{m} B_ie^{-\lambda\tau_i} - \sum_{i=1}^{m} \lambda C_ie^{-\lambda\tau_i}\Big)$$

is said to be the characteristic matrix of Eq. (1.1), and the equation $\det H(\lambda) \triangleq h(\lambda) = 0$ *is said to be the characteristic equation of Eq. (1.1).*

Applying the Laplace transform on both sides of Eq. (3.1) yields the following lemma.

Lemma 3.1 *If $X(t)$ is a fundamental solution of Eq. (1.1), then we have*

$$L(X(t); s) = H^{-1}(s), \tag{3.2}$$

where $H(s)$ is the characteristic matrix of Eq. (1.1), and $L(X(t); s)$ is the Laplace transform of $X(t)$.

Theorem 3.1 *If $X(t)$ is a fundamental solution of Eq. (1.1), then the general solution of Eqs. (1.1) and (1.2) can be written as*

$$\begin{aligned} x(\phi)(t) =& X(t)\Big[\phi(0) - \sum_{i=1}^{m} C_i\phi(-\tau_i)\Big] + \sum_{i=1}^{m}\int_{-\tau_i}^{0} X(t-\tau_i-\theta)B_i\phi(\theta)\mathrm{d}\theta + \\ & \sum_{i=1}^{m}\int_{-\tau_i}^{0} \dot{X}(t-\tau_i-\theta)C_i\phi(\theta)\mathrm{d}\theta + \sum_{i=1}^{m} C_i\hat{\phi}(-\tau_i+t)\omega(-\tau_i+t), \end{aligned} \tag{3.3}$$

where

$$\omega(t) = \begin{cases} 1, & t < 0, \\ 0, & t \geq 0, \end{cases} \quad \text{and} \quad \hat{\phi}(\theta) = \begin{cases} \phi(\theta), & \theta < 0, \\ \phi(0), & \theta \geq 0. \end{cases}$$

Proof Applying the Laplace transform on both side of Eq. (1.1), we get by Lemma 3.1 that

$$\begin{aligned} \bar{x}(s) =& L(X(t); s)\Big[\phi(0) - \sum_{i=1}^{m} C_i\phi(-\tau_i) + \sum_{i=1}^{m} B_i e^{-s\tau_i}\int_{-\tau_i}^{0}\phi(\theta)e^{-s\theta}\mathrm{d}\theta + \\ & s\sum_{i=1}^{m} C_i e^{-s\tau_i}\int_{-\tau_i}^{0}\phi(\theta)e^{-s\theta}\mathrm{d}\theta\Big], \end{aligned}$$

where $X(t)$ is a fundamental solution of Eq. (1.1), and the Laplace transform of $x(t)$ is $\bar{x}(s)$, i.e., $L(x(t); s) = \bar{x}(s)$, $L(X(t); s) = \bar{X}(s)$. By the definitions of $\hat{\phi}(\theta)$ and $\omega(\theta)$, we get

$$\begin{aligned} & \sum_{i=1}^{m} B_i e^{-s\tau_i}\int_{-\tau_i}^{0}\phi(\theta)e^{-s\theta}\mathrm{d}\theta = \sum_{i=1}^{m} B_i\int_{0}^{\tau_i}\phi(\theta-\tau_i)e^{-s\theta}\mathrm{d}\theta \\ =& \sum_{i=1}^{m} B_i\Big[\int_{0}^{\tau_i}\hat{\phi}(\theta-\tau_i)\omega(\theta-\tau_i)e^{-s\theta}\mathrm{d}\theta + \sum_{i=1}^{m}\int_{\tau_i}^{+\infty}\hat{\phi}(\theta-\tau_i)\omega(\theta-\tau_i)e^{-s\theta}\mathrm{d}\theta\Big] \\ =& \sum_{i=1}^{m} B_i\int_{0}^{+\infty}\hat{\phi}(\theta-\tau_i)\omega(\theta-\tau_i)e^{-s\theta}\mathrm{d}\theta = L\Big(\sum_{i=1}^{m} B_i\hat{\phi}(\theta-\tau_i)\omega(\theta-\tau_i); s\Big) \end{aligned}$$

and

$$\sum_{i=1}^{m} C_i e^{-s\tau_i}\int_{-\tau_i}^{0}\phi(\theta)e^{-s\theta}\mathrm{d}\theta = L\Big(\sum_{i=1}^{m} C_i\hat{\phi}(\theta-\tau_i)\omega(\theta-\tau_i); s\Big).$$

By the formula $L(\dot{X}(t); s) = s\bar{X}(s) - X(0) = sL(X(t); s) - X(0)$, we get

$$sL(X(t); s) = L(\dot{X}(t); s) + X(0).$$

Thus, it follows

$$\bar{x}(s) = L(X(t); s)\Big[\phi(0) - \sum_{i=1}^{m} C_i\phi(-\tau_i)\Big] + L(X(t); s)L\Big(\sum_{i=1}^{m} B_i\hat{\phi}(\theta-\tau_i)\omega(\theta-\tau_i); s\Big) +$$

$$[L(\dot{X}(t);s)+X(0)]L\Big(\sum_{i=1}^{m}C_i\hat{\phi}(\theta-\tau_i)\omega(\theta-\tau_i);s\Big).$$

Applying the inverse Laplace transform and Convolution Theorem yields

$$\begin{aligned}x(\phi)(t)=&X(t)\Big[\phi(0)-\sum_{i=1}^{m}C_i\phi(-\tau_i)\Big]+\int_0^t X(t-\theta)\sum_{i=1}^{m}B_i\hat{\phi}(\theta-\tau_i)\omega(\theta-\tau_i)\mathrm{d}\theta+\\&\int_0^t \dot{X}(t-\theta)\sum_{i=1}^{m}C_i\hat{\phi}(\theta-\tau_i)\omega(\theta-\tau_i)\mathrm{d}\theta+\sum_{i=1}^{m}C_i\hat{\phi}(t-\tau_i)\omega(t-\tau_i).\end{aligned}$$

Set $\theta-\tau_i=u$. Then, it follows

$$\begin{aligned}x(\phi)(t)=&X(t)\Big[\phi(0)-\sum_{i=1}^{m}C_i\phi(-\tau_i)\Big]+\sum_{i=1}^{m}\int_{-\tau_i}^{0}X(t-\tau_i-\theta)B_i\phi(\theta)\mathrm{d}\theta+\\&\sum_{i=1}^{m}\int_{-\tau_i}^{0}\dot{X}(t-\tau_i-\theta)C_i\phi(\theta)\mathrm{d}\theta+\sum_{i=1}^{m}C_i\hat{\phi}(-\tau_i+t)\omega(-\tau_i+t).\end{aligned}$$

This completes the proof. $\Box$

Lemma 3.2 *Suppose that λ is an m-multiple characteristic root of Eq. (1.1). Then there exists a vector $\vec{\alpha}=(k_1,k_2,\ldots,k_n)^T$ such that $\vec{\alpha}t^ke^{\lambda t}$ $(k=0,1,\ldots,m-1)$ are the solutions of Eq. (1.1).*

Proof Since λ is an m-multiple characteristic root of Eq. (1.1), so there exists a vector $\vec{\alpha}=(k_1,k_2,\ldots,k_n)^T$ such that

$$H(\lambda)\vec{\alpha}e^{\lambda t}=0, \tag{3.4}$$

where $H(\lambda)$ is the characteristic matrix of Eq. (1.1). Taking the kth-order $(k=0,1,2,\ldots,m-1)$ derivative of λ on both sides of Eq. (3.4) yields

$$\sum_{i=0}^{k}P_k^iH^{(k-i)}(\lambda)t^ie^{\lambda t}\vec{\alpha}=0, \tag{3.5}$$

here $P_k^j=\frac{k!}{j!(k-j)!}$. Now we prove that $x(t)=\vec{\alpha}t^ke^{\lambda t}$ $(k=0,1,\ldots,m-1)$ are the solutions of Eq. (1.1). Substituting $x(t)=\vec{\alpha}t^ke^{\lambda t}$ into the left of Eq. (1.1) yields

$$\begin{aligned}&\dot{x}(t)-Ax(t)-\sum_{i=1}^{m}B_ix(t-\tau_i)-\sum_{i=1}^{m}C_i\dot{x}(t-\tau_i)\\&=\Big[\lambda t^ke^{\lambda t}I+kt^{k-1}e^{\lambda t}I-At^ke^{\lambda t}-\sum_{i=1}^{m}B_i(t-\tau_i)^ke^{\lambda(t-\tau_i)}-\\&\quad\sum_{i=1}^{m}C_i\lambda(t-\tau_i)^ke^{\lambda(t-\tau_i)}-\sum_{i=1}^{m}C_ik(t-\tau_i)^{k-1}e^{\lambda(t-\tau_i)}\Big]\vec{\alpha}\\&=\Big[\lambda t^kI+kt^{k-1}I-At^k-\sum_{i=1}^{m}B_i(t-\tau_i)^ke^{-\lambda\tau_i}-\\&\quad\sum_{i=1}^{m}C_i\lambda(t-\tau_i)^ke^{-\lambda\tau_i}-\sum_{i=1}^{m}C_ik(t-\tau_i)^{k-1}e^{-\lambda\tau_i}\Big]e^{\lambda t}\vec{\alpha}.\end{aligned} \tag{3.6}$$

Inserting

$$(t-\tau_i)^k = P_k^0 t^k + P_k^1 t^{k-1}(-\tau_i) + P_k^2 t^{k-2}(-\tau_i)^2 + \cdots + P_k^k(-\tau_i)^k \tag{3.7}$$

and

$$(t-\tau_i)^{k-1} = P_{k-1}^0 t^{k-1} + P_{k-1}^1 t^{k-2}(-\tau_i) + P_{k-1}^2 t^{k-3}(-\tau_i)^2 + \cdots + P_{k-1}^{k-1}(-\tau_i)^{k-1} \tag{3.8}$$

into the right side of Eq. (3.6) yields

$$\begin{aligned}&\dot{x}(t) - Ax(t) - \sum_{i=1}^{m} B_i x(t-\tau_i) - \sum_{i=1}^{m} C_i \dot{x}(t-\tau_i)\\ &= \Big[t^k H(\lambda) + P_k^1 t^{k-1} H'(\lambda) + P_k^2 t^{k-2} H''(\lambda) + \cdots + P_k^{k-1} t H^{(k-1)}(\lambda) + H^{(k)}(\lambda)\Big]\vec{\alpha} e^{\lambda t}.\end{aligned}$$

By the equality (3.5) we have

$$\dot{x}(t) - Ax(t) - \sum_{i=1}^{m} B_i x(t-\tau_i) - \sum_{i=1}^{m} C_i \dot{x}(t-\tau_i) = 0.$$

Thus $x(t) = \vec{\alpha} t^k e^{\lambda t}$ $(k = 0, 1, \ldots, m-1)$ are solutions of Eq. (1.1).

Theorem 3.2 *Suppose that $h(\lambda) = 0$ is the characteristic equation of Eq. (1.1).*

(1) If the real parts of all roots of $h(\lambda) = 0$ are negative, then the zero solution of Eq. (1.1) is asymptotically stable;

(2) If the real parts of all roots of $h(\lambda) = 0$ are non-positive and there exist finite single roots with zero real parts, then the zero solution of Eq. (1.1) is stable, but is not asymptotically stable;

(3) If the real parts of all roots of $h(\lambda) = 0$ are non-negative and there exist multiple roots with zero real parts, or there exists a root with positive real parts, then the zero solution of Eq. (1.1) is unstable.

Proof (1) First, it is easy to check that if $M(t)$ is a nonsingular solution matrix of Eq. (1.1). Then for any nonsingular constant matrix G, $M(t)G$ is a nonsingular solution matrix of Eq. (1.1). Secondly, the conditions that $0 < \tau_1 < \tau_2 < \cdots < \tau_m$ and $\det C_m \neq 0$ ensure that Eq. (1.1) possesses at least n characteristic roots.

Case (i) Let λ_i $(i = 1, 2, \ldots, n)$ be different roots of $h(\lambda) = 0$ with $\mathfrak{Re}\lambda_i < 0$. Then there exist nonzero characteristic vectors: $\vec{\alpha}_i = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in})^T$ such that $x_i(t) = e^{\lambda_i t}\vec{\alpha}_i$ are solutions of Eq. (1.1), $i = 1, 2, \ldots, n$. Thus

$$M(t) = \begin{pmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{pmatrix}$$

is a solution matrix of Eq. (1.1). There exists a nonsingular constant matrix

$$K = \begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{pmatrix}$$

such that $M(t)K|_{t=0} = I$. Extend the matrix $M(t)K$ to $\bar{M}(t)K$ such that

$$\bar{M}(t)K = \begin{cases} M(t)K, & t \geq 0, \\ O, & t < 0, \end{cases}$$

where O denotes the n-order zero matrix. Thus, the matrix $X(t) = \bar{M}(t)K$ is a fundamental solution of Eq. (1.1), where

$$\bar{M}(t)K = \begin{pmatrix} \sum\limits_{i=1}^{n} \alpha_{1i}k_{i1}e^{\lambda_i t} & \sum\limits_{i=1}^{n} \alpha_{1i}k_{i2}e^{\lambda_i t} & \cdots & \sum\limits_{i=1}^{n} \alpha_{1i}k_{in}e^{\lambda_i t} \\ \sum\limits_{i=1}^{n} \alpha_{2i}k_{i1}e^{\lambda_i t} & \sum\limits_{i=1}^{n} \alpha_{2i}k_{i2}e^{\lambda_i t} & \cdots & \sum\limits_{i=1}^{n} \alpha_{2i}k_{in}e^{\lambda_i t} \\ \cdots & \cdots & \cdots & \cdots \\ \sum\limits_{i=1}^{n} \alpha_{ni}k_{i1}e^{\lambda_i t} & \sum\limits_{i=1}^{n} \alpha_{ni}k_{i2}e^{\lambda_i t} & \cdots & \sum\limits_{i=1}^{n} \alpha_{ni}k_{in}e^{\lambda_i t} \end{pmatrix}, \quad t > 0.$$

By computation, we get

$$\dot{X}(t) = \begin{pmatrix} \sum\limits_{i=1}^{n} \lambda_i\alpha_{1i}k_{i1}e^{\lambda_i t} & \sum\limits_{i=1}^{n} \lambda_i\alpha_{1i}k_{i2}e^{\lambda_i t} & \cdots & \sum\limits_{i=1}^{n} \lambda_i\alpha_{1i}k_{in}e^{\lambda_i t} \\ \sum\limits_{i=1}^{n} \lambda_i\alpha_{2i}k_{i1}e^{\lambda_i t} & \sum\limits_{i=1}^{n} \lambda_i\alpha_{2i}k_{i2}e^{\lambda_i t} & \cdots & \sum\limits_{i=1}^{n} \lambda_i\alpha_{2i}k_{in}e^{\lambda_i t} \\ \cdots & \cdots & \cdots & \cdots \\ \sum\limits_{i=1}^{n} \lambda_i\alpha_{ni}k_{i1}e^{\lambda_i t} & \sum\limits_{i=1}^{n} \lambda_i\alpha_{ni}k_{i2}e^{\lambda_i t} & \cdots & \sum\limits_{i=1}^{n} \lambda_i\alpha_{ni}k_{in}e^{\lambda_i t} \end{pmatrix}, \quad t > 0.$$

Since $\mathfrak{Re}\lambda_i < 0$, $i = 1, 2, \ldots, n$, we have $|X(t)| \to 0$ and $|\dot{X}(t)| \to 0$ as $t \to +\infty$. By the formula (3.2), we know that the zero solution of Eq. (1.1) is asymptotically stable.

Case (ii) Without loss of generality, let λ_1 be a k-multiple $(2 \leq k \leq n)$ root of the characteristic equation $h(\lambda) = 0$ with $\mathfrak{Re}\lambda_1 < 0$. Suppose further that the other $(n-k)$ roots of $h(\lambda) = 0$ are λ_{k+1}, λ_{k+2}, $\cdots$ and λ_n, which are different from each other with negative real parts. By Lemma 3.2, there exist vector functions $\vec{\alpha}_i(t) = (\alpha_{1i}(t), \alpha_{2i}(t), \ldots, \alpha_{ni}(t))^T$ such that $x_1(t) = e^{\lambda_1 t}\vec{\alpha}_0(t)$, $x_2(t) = e^{\lambda_1 t}\vec{\alpha}_1(t), \ldots, x_k(t) = e^{\lambda_1 t}\vec{\alpha}_{k-1}(t)$ are solutions of Eq. (1.1), where $\alpha_{ji}(t)$ are polynomials in t with degree i, $i = 0, 2, \ldots, k-1$, $j = 1, 2, \ldots, n$. There exist vectors $\vec{\alpha}_{k+1}, \ldots, \vec{\alpha}_n$ such that $x_{k+1}(t) = e^{\lambda_{k+1} t}\vec{\alpha}_{k+1}, \ldots, x_n(t) = e^{\lambda_n t}\vec{\alpha}_n$ are solutions of Eq. (1.1). Thus,

$$M(t) = \begin{pmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{pmatrix}$$

is a nonsingular solution matrix of Eq.(1.1). The remainder of the proof is similar to that of Case (i).

(2) First, without loss of generality, we suppose that $\lambda_j = \beta i$ is a characteristic root of Eq. (1.1) with $i^2 = -1$, where $\beta \in \mathbb{R}$ and $\beta \neq 0$. Suppose that other $n-1$ characteristic roots of Eq. (1.1) are $\lambda_1, \lambda_2, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_n$, with $\mathfrak{Re}\lambda_l < 0$ $(l = 1, 2, \ldots, j-1, j+1, \ldots, n)$. Then there exist nonzero characteristic vectors: $\vec{c}_1 = (c_{11}, c_{21}, \ldots, c_{n1})^T$,

$\vec{c}_2 = (c_{12}, c_{22}, \ldots, c_{n2})^T, \ldots, \vec{c}_n = (c_{1n}, c_{2n}, \ldots, c_{nn})^T$ such that the matrix

$$M(t) = \begin{pmatrix} e^{\lambda_1 t}c_{11} & \cdots & e^{\lambda_{j-1}t}c_{1,j-1} & e^{\beta ti}c_{1j} & e^{\lambda_{j+1}t}c_{1,j+1} & \cdots & e^{\lambda_n t}c_{1n} \\ e^{\lambda_1 t}c_{21} & \cdots & e^{\lambda_{j-1}t}c_{2,j-1} & e^{\beta ti}c_{2j} & e^{\lambda_{j+1}t}c_{2,j+1} & \cdots & e^{\lambda_n t}c_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e^{\lambda_1 t}c_{n1} & \cdots & e^{\lambda_{j-1}t}c_{n,j-1} & e^{\beta ti}c_{nj} & e^{\lambda_{j+1}t}c_{n,j+1} & \cdots & e^{\lambda_n t}c_{nn} \end{pmatrix}$$

is a nonsingular solution matrix of Eq. (3.1), where the vector $\vec{c}_l$ is the characteristic vector with regard to the characteristic root λ_l $(l = 1, 2, \ldots, n)$. There exists a nonsingular constant matrix K satisfying

$$K = \begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{pmatrix}$$

such that $M(t)K|_{t=0} = I$. Extend the matrix $M(t)K$ to $\bar{M}(t)K$ such that

$$\bar{M}(t)K = \begin{cases} M(t)K, & t \geq 0, \\ O, & t < 0. \end{cases}$$

Thus, the matrix $X(t) = \bar{M}(t)K$ is a fundamental solution of Eq. (1.1), where

$$\bar{M}(t)K = \begin{pmatrix} \sum\limits_{m=1}^{n} c_{1m}k_{m1}e^{\lambda_m t} & \sum\limits_{m=1}^{n} c_{1m}k_{m2}e^{\lambda_m t} & \cdots & \sum\limits_{m=1}^{n} c_{1m}k_{mn}e^{\lambda_m t} \\ \sum\limits_{m=1}^{n} c_{2m}k_{m1}e^{\lambda_m t} & \sum\limits_{m=1}^{n} c_{2m}k_{m2}e^{\lambda_m t} & \cdots & \sum\limits_{m=1}^{n} c_{2m}k_{mn}e^{\lambda_m t} \\ \cdots & \cdots & \cdots & \cdots \\ \sum\limits_{m=1}^{n} c_{nm}k_{m1}e^{\lambda_m t} & \sum\limits_{m=1}^{n} c_{nm}k_{m2}e^{\lambda_m t} & \cdots & \sum\limits_{m=1}^{n} c_{nm}k_{mn}e^{\lambda_m t} \end{pmatrix}, \quad t > 0.$$

Since $\mathfrak{Re}\lambda_m < 0$, $m = 1, 2, \ldots, j-1, j+1, \ldots, n$, it holds that

$$|X(t)| = |\bar{M}(t)K| = \left| \begin{pmatrix} |\sum\limits_{m=1}^{n} c_{1m}k_{m1}e^{\lambda_m t}| & |\sum\limits_{m=1}^{n} c_{1m}k_{m2}e^{\lambda_m t}| & \cdots & |\sum\limits_{m=1}^{n} c_{1m}k_{mn}e^{\lambda_m t}| \\ |\sum\limits_{m=1}^{n} c_{2m}k_{m1}e^{\lambda_m t}| & |\sum\limits_{m=1}^{n} c_{2m}k_{m2}e^{\lambda_m t}| & \cdots & |\sum\limits_{m=1}^{n} c_{2m}k_{mn}e^{\lambda_m t}| \\ \cdots & \cdots & \cdots & \cdots \\ |\sum\limits_{m=1}^{n} c_{nm}k_{m1}e^{\lambda_m t}| & |\sum\limits_{m=1}^{n} c_{nm}k_{m2}e^{\lambda_m t}| & \cdots & |\sum\limits_{m=1}^{n} c_{nm}k_{mn}e^{\lambda_m t}| \end{pmatrix} \right|$$

tends to

$$\begin{aligned} &\left| \begin{pmatrix} |c_{1j}k_{j1}e^{\lambda_j t}| & |c_{1j}k_{j2}e^{\lambda_j t}| & \cdots & |c_{1j}k_{jn}e^{\lambda_j t}| \\ |c_{2j}k_{j1}e^{\lambda_j t}| & |c_{2j}k_{j2}e^{\lambda_j t}| & \cdots & |c_{2j}k_{jn}e^{\lambda_j t}| \\ \cdots & \cdots & \cdots & \cdots \\ |c_{nj}k_{j1}e^{\lambda_j t}| & |c_{nj}k_{j2}e^{\lambda_j t}| & \cdots & |c_{nj}k_{jn}e^{\lambda_j t}| \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} |c_{1j}k_{j1}| & |c_{1j}k_{j2}| & \cdots & |c_{1j}k_{jn}| \\ |c_{2j}k_{j1}| & |c_{2j}k_{j2}| & \cdots & |c_{2j}k_{jn}| \\ \cdots & \cdots & \cdots & \cdots \\ |c_{nj}k_{j1}| & |c_{nj}k_{j2}| & \cdots & |c_{nj}k_{jn}| \end{pmatrix} \right| \neq 0 \end{aligned} \tag{3.9}$$

as $t \to +\infty$. In fact, there exists at least a nonzero element $c_{\mu j}k_{j\nu}$ in the matrix

$$\begin{pmatrix} c_{1j}k_{j1} & c_{1j}k_{j2} & \cdots & c_{1j}k_{jn} \\ c_{2j}k_{j1} & c_{2j}k_{j2} & \cdots & c_{2j}k_{jn} \\ \cdots & \cdots & \cdots & \cdots \\ c_{nj}k_{j1} & c_{nj}k_{j2} & \cdots & c_{nj}k_{jn} \end{pmatrix}, \tag{3.10}$$

where $\mu, \nu \in \{1, 2, \dots, n\}$. Otherwise, if all elements in the first column of the matrix (3.10) are equal to zero, then by the vector $\vec{c}_j = (c_{1j}, c_{2j}, \dots, c_{nj})^T \neq 0$, we have $k_{j1} = 0$. If all elements in the second column of the matrix (3.10) are equal to zero, then by vector $\vec{c}_j = (c_{1j}, c_{2j}, \dots, c_{nj})^T \neq 0$, we have $k_{j2} = 0$. Continuing this process, we can obtain $k_{j3} = 0, k_{j4} = 0, \dots, k_{jn} = 0$. Thus, all elements of j row of the matrix K are zero. This contradicts the fact that the matrix K is nonsingular. Therefore, $|X(t)| \to$ a nonzero finite number as $t \to +\infty$. By using similar arguments, we can prove that $|\dot{X}(t)| \to$ a nonzero finite number as $t \to +\infty$. By the formula (3.2), the zero solution of Eq. (1.1) is stable, but not asymptotically stable.

Secondly, without loss of generality, suppose that Eq. (1.1) possesses several different characteristic roots with zero real parts. Using the similar arguments used above, we can prove that the zero solution of Eq. (1.1) is stable, but not asymptotically stable.

(3) Without loss of generality, assume that $\lambda_1 = \beta i$ is a k-multiple characteristic root of Eq. (1.1) with $i^2 = -1$, then there exist function vectors $\vec{c}_1(t), \vec{c}_2(t), \dots, \vec{c}_k(t)$ such that $e^{\beta t i}\vec{c}_1(t)$, $e^{\beta t i}\vec{c}_2(t), \dots, e^{\beta t i}\vec{c}_k(t)$ are solution vectors of Eq. (1.1), where all elements of the vector $\vec{c}_j(t)$, $j = 1, \cdots, k$, are polynomials in t with degree $j-1$. Since the other characteristic roots of Eq. (1.1) have negative real parts, we can choose $n-k$ different roots $\lambda_{k+1}, \lambda_{k+2}, \dots, \lambda_n$ with $\mathfrak{Re}\lambda_j < 0$, $j = k+1, k+2, \dots, n$. Then there exist vectors $\vec{c}_{k+1}, \vec{c}_{k+2}, \dots, \vec{c}_n$ such that matrix

$$M(t) = (e^{\beta t i}\vec{c}_1, e^{\beta t i}\vec{c}_2(t), e^{\beta t i}\vec{c}_3(t), \dots, e^{\beta t i}\vec{c}_k(t), e^{\lambda_{k+1}t}\vec{c}_{k+1}, e^{\lambda_{k+2}t}\vec{c}_{k+2}, \dots, e^{\lambda_n t}\vec{c}_n)$$

is a nonsingular solution matrix of Eq. (3.1). There exists a nonsingular constant matrix K such that $M(t)K|_{t=0} = I$. Extend the matrix $M(t)K$ to $\bar{M}(t)K$ such that

$$\bar{M}(t)K = \begin{cases} M(t)K, & t \geq 0, \\ O, & t < 0, \end{cases}$$

where O denotes the n-order zero matrix. Thus, the matrix $X(t) = \bar{M}(t)K$ is a fundamental solution of Eq. (1.1). As the proof in (2), we can prove that $|X(t)| \to \infty$ and $|\dot{X}(t)| \to +\infty$ as $t \to +\infty$. Therefore, the zero solution of Eq. (1.1) is unstable. Obviously, if a characteristic root of Eq. (1.1) has positive real part, then the zero solution of Eq. (1.1) is unstable. This completes the proof. $\square$

The following corollary is necessary and sufficient, which is an extension of [18].

Corollary 3.1 *Suppose that $h(\lambda) = 0$ is the characteristic equation of Eq. (1.1). Then the zero solution of Eq. (1.1) is asymptotically stable iff the real parts of all characteristic roots of Eq. (1.1) are negative.*

Proof Sufficiency. See the proof of Theorem 3.1 (1).

Necessity. Suppose that Eq. (1.1) possesses a characteristic root with zero real part. Then by Theorem 3.1 (2), the zero solution of Eq. (1.1) is stable but not asymptotically stable. This is a contradiction. If Eq. (1.1) has a characteristic root with positive real parts, then by Theorem 3.1 (3), the zero solution of Eq. (1.1) is unstable. This is also a contradiction. Thus, the necessity of the theorem holds. This completes the proof. □

Remark 3.1 Corollary 3.1 holds in n-dimensional space and is an extension of [18]. The methods used in this paper are different from ones in [18]. Corollary 3.1 is also an answer to the problem put forward in [19] and [20] .

Next, we will investigate the stability of the perturbed equation

$$\dot{x}(t) = Ax(t) + B(t)x(t) + \sum_{i=1}^{m} B_i x(t-\tau_i) + \sum_{i=1}^{m} C_i \dot{x}(t-\tau_i), \quad t > 0, \tag{3.11}$$

$$x(t) = \phi(t), \quad -\tau_m \leq t \leq 0, \tag{3.12}$$

where $x(t), A, B_i, C_i, \tau_i$ and $\phi(t)$ are the same meanings as in Eq. (1.1), $B(t)$ is an $n \times n$ matrix depending on time t. We have the following theorem.

Theorem 3.3 *Suppose that $h(\lambda) = 0$ is the characteristic equation of Eq. (1.1), and that $B(t)$ is bounded, i.e., there exists some $M > 0$ such that $|B(t)| \leq M$.*

(1) If all roots of $h(\lambda) = 0$ have negative part, then the zero solution of Eq. (3.11) is asymptotically stable;

(2) If all roots of $h(\lambda) = 0$ are not positive, there is at least a single root with zero real part and $\int_0^{+\infty} |B(t)|\mathrm{d}t$ is bounded, then the zero solution of Eq. (3.11) is stable, but not asymptotically stable;

(3) If all roots of $h(\lambda) = 0$ are not positive, and there exists at least a multiple root with zero real parts for $h(\lambda) = 0$, or there exists at least a root with positive real part for $h(\lambda) = 0$, then the zero solution of Eq. (3.11) is unstable.

Proof Let $X(t)$ be a fundamental solution of Eq. (1.1). Then the general solution of equation (3.11) can be written as

$$\begin{aligned} x(\phi)(t) =& X(t)\Big[\phi(0) - \sum_{i=1}^{m} C_i\phi(-\tau_i)\Big] + \sum_{i=1}^{m}\int_{-\tau_i}^{0} X(t-\tau_i-\theta)B_i\phi(\theta)\mathrm{d}\theta + \\ & \sum_{i=1}^{m}\int_{-\tau_i}^{0} \dot{X}(t-\tau_i-\theta)C_i\phi(\theta)\mathrm{d}\theta + \sum_{i=1}^{m} C_i\hat{\phi}(-\tau_i+t)\omega(-\tau_i+t) + \\ & \int_0^t B(t-\theta)X(\theta)x(\theta)\mathrm{d}\theta, \end{aligned} \tag{3.13}$$

where

$$\omega(t) = \begin{cases} 1, & t < 0, \\ 0, & t \geq 0, \end{cases} \quad \text{and} \quad \hat{\phi}(\theta) = \begin{cases} \phi(\theta), & \theta < 0, \\ \phi(0), & \theta \geq 0. \end{cases}$$

Thus, we get

$$|x(\phi)(t)| \le \Big|X(t)\Big[\phi(0)-\sum_{i=1}^{m}C_i\phi(-\tau_i)\Big]+\sum_{i=1}^{m}\int_{-\tau_i}^{0}X(t-\tau_i-\theta)B_i\phi(\theta)\mathrm{d}\theta+ \sum_{i=1}^{m}\int_{-\tau_i}^{0}\dot{X}(t-\tau_i-\theta)C_i\phi(\theta)\mathrm{d}\theta+\sum_{i=1}^{m}C_i\hat{\phi}(-\tau_i+t)\omega(-\tau_i+t)\Big|+ \int_0^t|B(t-\theta)||X(\theta)||x(\theta)|\mathrm{d}\theta.$$

Applying Gronwall inequality, we get

$$|x(t)|\le M_0\exp\Big\{\int_0^t|X(t-\theta)||B(\theta)|\mathrm{d}\theta\Big\}=M_0\exp\Big\{\int_0^t|X(\theta)|\cdot|B(t-\theta)|\mathrm{d}\theta\Big\}, \tag{3.14}$$

where

$$M_0=\Big|X(t)\Big[\phi(0)-\sum_{i=1}^{m}C_i\phi(-\tau_i)\Big]+\sum_{i=1}^{m}\int_{-\tau_i}^{0}X(t-\tau_i-\theta)B_i\phi(\theta)\mathrm{d}\theta+ \sum_{i=1}^{m}\int_{-\tau_i}^{0}\dot{X}(t-\tau_i-\theta)C_i\phi(\theta)\mathrm{d}\theta+\sum_{i=1}^{m}C_i\hat{\phi}(-\tau_i+t)\omega(-\tau_i+t)\Big|.$$

(1) If all roots of $h(\lambda)=0$ have negative parts, then using similar method in Theorem 3.2, we can easily prove that $|X(t)|\to 0$ and $|\dot{X}(t)|\to 0$ as $t\to+\infty$. By inequality (3.14), we know that Eq. (3.11) is asymptotically stable. The proof of (2) and (3) is similar to that of Theorem 3.2. This completes the proof. □

Now we investigate the perturbed equation

$$\dot{x}(t)=Ax(t)+\sum_{i=1}^{m}B_ix(t-\tau_i)+\sum_{i=1}^{m}C_i\dot{x}(t-\tau_i)+f(t,x(t)),\quad t>0, \tag{3.15}$$

$$x(t)=\phi(t),\quad -\tau_i\le t\le 0, \tag{3.16}$$

where $f(t,x(t))\in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$, $f(t,0)\equiv 0$, $x(t),A,B_i,C_i,\tau_i,\phi(t)$ are the same meanings as in Eqs. (1.1) and (1.2). The following theorem holds.

Theorem 3.4 *Suppose that $h(\lambda)=0$ is the characteristic equation of Eq. (1.1).*

(1) If all roots of $h(\lambda)=0$ have negative real parts, and $|f(t,x(t))|\le\varepsilon|x(t)|$, then the zero solution of Eq. (3.15) is asymptotically stable;

(2) If all roots of $h(\lambda)=0$ are not positive, there exists at least a single root of $h(\lambda)=0$ with zero real part, $|f(t,x(t))|\le|B(t)||x(t)|$, and $\int_0^{+\infty}|B(t)|\mathrm{d}t$ is bounded, then the zero solution of Eq. (3.15) is stable, but not asymptotically stable;

(3) If all roots of $h(\lambda)=0$ are not positive, there exists at least a k-multiple root with zero real part, $|f(t,x(t))|\le|B(t)||x(t)|$, $\int_0^{+\infty}|B(t)|\mathrm{d}t$ is bounded, or there exists a root with positive real part, then the zero solution of Eq. (3.15) is unstable.

Proof The proof is similar to that of Theorem 3.3.

4. Exponential stability

In this section and after this section, we assume that delays τ_i $(i = 1, 2, \ldots, m)$ are rational numbers.

Lemmas 4.1 and 4.2 will be used in the proof of Lemma 4.3, which is a key lemma in the proof of Theorem 4.1.

Lemma 4.1 *Let the $h(\lambda) = 0$ be the characteristic equation of Eq. (1.1). Denote $\alpha_0 = \sup\{\mathfrak{Re}\lambda : h(\lambda) = 0\}$. Then,*

$$\alpha_0 = \max\Big\{\mathfrak{Re}\lambda : \det\Big(I - \sum_{i=1}^{m} C_i e^{-\lambda\tau_i}\Big) = 0\Big\},$$

where $\mathfrak{Re}\lambda$ denotes the real parts of λ.

Proof Let all roots of $h(\lambda) = 0$ be $\{\lambda_j\}$. Since $0 < \tau_1 < \tau_2 < \cdots < \tau_m$ and $\det C_m \neq 0$, there exist constants $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \leq \mathfrak{Re}\lambda_j \leq \beta.$$

For the delay τ_i, there exists a positive integer M such that $\tau_i M = T_i$ with T_i being positive integer, $i = 1, 2, \ldots, m$. Thus,

$$\det\Big(I - \sum_{i=1}^{m} C_i e^{-\lambda\tau_i}\Big) = 0$$

can be written as

$$\det\Big(I - \sum_{i=1}^{m} C_i (e^{-\lambda/M})^{T_i}\Big) = 0.$$

Denote $e^{-\lambda/M} = u$. Then,

$$\det\Big(I - \sum_{i=1}^{m} C_i u^{T_i}\Big) = 0 \tag{4.1}$$

is an algebraic equation in u, and Eq. (4.1) possesses only finite nonzero roots. Without loss of generality, suppose these nonzero roots are $u_1, u_2, \ldots, u_l$. Therefore,

$$e^{-\lambda/M} = u_\nu, \quad \nu = 1, 2, \ldots, l.$$

That is

$$\lambda = -M[\ln|u_\nu| + 2k\pi i], \quad \nu = 1, 2, \ldots, l;\ k = \pm1, \pm2, \ldots.$$

Thus,

$$\max\Big\{\mathfrak{Re}\lambda : \det\Big(I - \sum_{i=1}^{m} C_i e^{-\lambda\tau_i}\Big) = 0\Big\}$$

is well defined.

The characteristic equation $h(\lambda) = 0$ can be written as

$$\lambda^n \det\Big(I - \sum_{i=1}^{m} C_i e^{-\lambda\tau_i}\Big) + \lambda^{n-1} R_{n-1}(e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \ldots, e^{-\lambda\tau_m}) + \cdots +$$
$$\lambda R_1(e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \ldots, e^{-\lambda\tau_m}) + R_0(e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \ldots, e^{-\lambda\tau_m}) = 0,$$

where $R_i(Z_1, Z_2, \dots, Z_m)$ $(i = 0, 1, 2, \dots, n-1)$ are polynomials in $Z_1, Z_2, \dots, Z_m$. Let $\{\lambda_j'\}$ be a subsequence of characteristic roots $\{\lambda_j\}$ satisfying

$$|\lambda_j'| \to \infty, \quad j \to \infty.$$

Thus, it follows

$$\det\Big(I - \sum_{i=1}^{m} C_i e^{-\lambda_j' \tau_i}\Big) \to 0, \quad j \to \infty.$$

Otherwise, $|h(\lambda_j')| \gg 0$ as $j \to \infty$. This contradicts $h(\lambda_j') = 0$. Then there exists an $n_0 \in \{1, 2, \dots, l\}$ such that

$$\mathfrak{Re}\lambda_j' \to -M[\ln |u_{n_0}| + 2k\pi i], \quad j \to \infty.$$

On the other hand, there exists a subsequence $\{\lambda_j''\}$ of all roots $\{\lambda_j\}$ of $h(\lambda) = 0$ satisfying

$$|\lambda_j''| \to \infty, \quad j \to \infty$$

such that

$$\mathfrak{Re}(\lambda_j'') \to \max\Big\{\mathfrak{Re}\lambda : \det\Big(I - \sum_{i=1}^{m} C_i e^{-\lambda \tau_i}\Big) = 0\Big\}, \quad j \to \infty.$$

Otherwise, $|h(\lambda_j'')| \gg 0$ as $j \to \infty$. This is a contradiction. Thus, it follows

$$\alpha_0 = \max\Big\{\mathfrak{Re}\lambda : \det\Big(I - \sum_{i=1}^{m} C_i e^{-\lambda \tau_i}\Big) = 0\Big\}.$$

This completes the proof. $\Box$

Lemma 4.2 *If $\alpha > 0$, then $\int_{(\alpha)} e^{\lambda(t+jr)}\lambda^{-1}\mathrm{d}\lambda = 1$; If $\alpha \le 0$, then $\int_{(\alpha)} e^{\lambda(t+jr)}\lambda^{-1}\mathrm{d}\lambda \le 1$, where r is a constant, t, j satisfy $t + jr > 0$, $\int_{(\alpha)} = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{\alpha - iT}^{\alpha + iT}$.*

Proof Denote $\sigma = t + jr$, then it holds

$$\int_{(\alpha)} e^{\lambda(t+jr)}\lambda^{-1}\mathrm{d}\lambda = \frac{1}{2\pi i}\int_L e^{\lambda\sigma}\lambda^{-1}\mathrm{d}\lambda,$$

where $L = \{\alpha + iu : -\infty \le u \le \infty\}$. On the line L, we get

$$\begin{aligned} e^{\lambda(t+jr)}\lambda^{-1} &= \frac{e^{(\alpha+iu)\sigma}}{\alpha + iu} = \frac{e^{\alpha\sigma}e^{i\sigma u}(\alpha - iu)}{\alpha^2 + u^2} \\ &= \frac{e^{\alpha\sigma}(\alpha\cos\sigma u + u\sin\sigma u - iu\cos\sigma u + i\alpha\sin\sigma u)}{\alpha^2 + u^2} \\ &= \frac{\alpha\cos\sigma u + u\sin\sigma u}{\alpha^2 + u^2}e^{\alpha\sigma} + ie^{\alpha\sigma}\frac{-u\cos\sigma u + \alpha\sin\sigma u}{\alpha^2 + u^2}. \end{aligned}$$

Denote $A(u) = \frac{\alpha\cos\sigma u + u\sin\sigma u}{\alpha^2+u^2}e^{\alpha\sigma}$, $B(u) = e^{\alpha\sigma}\frac{-u\cos\sigma u + \alpha\sin\sigma u}{\alpha^2+u^2}$, and $f(\lambda) = e^{\lambda(t+jr)}\lambda^{-1}$. Then it holds

$$\begin{aligned} \int_L f(\lambda)\mathrm{d}\lambda &= \Big(\int_L A(u)\mathrm{d}\alpha - B(u)\mathrm{d}u\Big) + \Big(i\int_L A(u)\mathrm{d}u + B(u)\mathrm{d}\alpha\Big) \\ &= \int_{-\infty}^{\infty} -B(u)\mathrm{d}u + i\int_{-\infty}^{\infty} A(u)\mathrm{d}u. \end{aligned}$$

Since $B(u)$ is an odd function, we have $\int_{-\infty}^{\infty} B(u)\mathrm{d}u = 0$. By computation, we get

$$\begin{aligned}\int_{-\infty}^{\infty} A(u)\mathrm{d}u &= \int_{-\infty}^{\infty} \frac{\alpha\cos\sigma u + u\sin\sigma u}{\alpha^2+u^2} e^{\alpha\sigma}\mathrm{d}u \\ &= \int_{-\infty}^{\infty} \frac{\alpha\cos\sigma u}{\alpha^2+u^2} e^{\alpha\sigma}\mathrm{d}u + \int_{-\infty}^{\infty} \frac{u\sin\sigma u}{\alpha^2+u^2} e^{\alpha\sigma}\mathrm{d}u\end{aligned}$$

and

$$\int_{-\infty}^{\infty} \frac{\alpha\cos\sigma u}{\alpha^2+u^2} e^{\alpha\sigma}\mathrm{d}u = e^{\alpha\sigma}\int_{-\infty}^{\infty} \frac{\cos\sigma\alpha\frac{u}{\alpha}}{1+(\frac{u}{\alpha})^2}\mathrm{d}\frac{u}{\alpha}.$$

By the fact that

$$\int_{-\infty}^{\infty} \frac{\cos mx}{1+x^2}\mathrm{d}x = \pi e^{-m}, \quad m>0,$$

we get

$$\int_{-\infty}^{\infty} \frac{\cos\sigma\alpha\frac{u}{\alpha}}{1+(\frac{u}{\alpha})^2}\mathrm{d}\frac{u}{\alpha} = \pi e^{-\sigma\alpha}, \quad \sigma\alpha>0 \text{ or } \alpha(t+jr)>0.$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{\alpha\cos\sigma u}{\alpha^2+u^2} e^{\alpha\sigma}\mathrm{d}u = \pi.$$

Now we will solve $\int_{-\infty}^{\infty} \frac{u\sin\sigma u}{\alpha^2+u^2} e^{\alpha\sigma}\mathrm{d}u$. Denote $f(z) = \frac{z\alpha e^{-\sigma\alpha}}{\alpha^2+z^2}$. It is obvious that $f(z)$ has two primary zero points: $z=\pm i\alpha$.

If $\alpha>0$, then

$$\operatorname{Res} f(z) = \frac{i\alpha e^{-\sigma\alpha}}{2i\alpha} = \frac{e^{-\sigma\alpha}}{2},$$

where $\operatorname{Res} f(z)$ denotes the residue of the function $f(z)$. Thus, it follows

$$\int_{-\infty}^{\infty} \frac{u\sin\sigma u}{\alpha^2+u^2}\mathrm{d}u = 2\pi i\frac{e^{-\sigma\alpha}}{2} = \pi i e^{-\sigma\alpha}.$$

Comparing the real part and imaginary part of $f(z)$, we get

$$\int_{-\infty}^{\infty} \frac{u\sin\sigma u}{\alpha^2+u^2}\mathrm{d}u = \pi e^{-\sigma\alpha}.$$

Namely,

$$\int_{-\infty}^{\infty} \frac{u\sin\sigma u}{\alpha^2+u^2} e^{\alpha\sigma}\mathrm{d}u = \pi.$$

Therefore, it follows that

$$\int_{-\infty}^{\infty} A(u)\mathrm{d}u = \int_{-\infty}^{\infty} \frac{\alpha\cos\sigma u + u\sin\sigma u}{\alpha^2+u^2} e^{\alpha\sigma} = 2\pi.$$

So it holds that

$$\int_{(\alpha)} e^{\lambda(t+jr)}\lambda^{-1}\mathrm{d}\lambda = \frac{1}{2\pi i}\int_L e^{\lambda\sigma}\lambda^{-1}\mathrm{d}\lambda = \frac{1}{2\pi}\cdot 2\pi = 1.$$

If $\alpha<0$, then $\sigma\alpha<0$. It follows

$$\int_{-\infty}^{\infty} \frac{\alpha\cos\sigma u}{\alpha^2+u^2}\mathrm{d}u = \pi e^{\sigma\alpha}.$$

That is

$$\int_{-\infty}^{\infty} \frac{\alpha \cos \sigma u}{\alpha^2 + u^2} e^{\sigma\alpha} \mathrm{d}u = \pi e^{2\sigma\alpha}.$$

By the fact

$$\int_{-\infty}^{\infty} \frac{u \sin \sigma u}{\alpha^2 + u^2} \mathrm{d}u = \pi e^{\alpha\sigma},$$

we can get

$$\int_{(\alpha)} e^{\lambda(t+jr)} \lambda^{-1} \mathrm{d}\lambda = \frac{1}{2\pi}(\pi e^{2\sigma\alpha} + \pi e^{\alpha\sigma}) = \frac{1}{2}(e^{2\sigma\alpha} + e^{\alpha\sigma}) < 1.$$

This completes the proof. □

The idea used in the proof of the following lemma, which plays an important role in proving Theorem 4.1, mainly comes from [19].

Lemma 4.3 *Let $H(\lambda)$ be the characteristic matrix of Eq. (1.1). If $\alpha_0 = \sup\{\mathfrak{Re}\lambda : \det H(\lambda) = 0\}$, then for any $\alpha > \alpha_0$, there exists a constant $k = k(\alpha)$ such that the fundamental solution $X(t)$ of Eq. (1.1) satisfies*

$$|X(t)| \le k e^{\alpha t}$$

and

$$|\dot{X}(t)| \le k e^{\alpha t},$$

where $t \ge 0$, $t \ne N\tau_i$, $i = 1, 2, \ldots, m$, and N is any positive integer.

Proof The characteristic equation of Eq. (1.1)

$$\det H(\lambda) = \det\Big(\lambda I - \sum_{i=1}^{m} C_i \lambda e^{-\lambda\tau_i} - A - \sum_{i=1}^{m} B_i e^{-\lambda\tau_i}\Big) = 0$$

can be written as

$$\begin{aligned}\det H(\lambda) =& \lambda^n \det\Big(I - \sum_{i=1}^{m} C_i e^{-\lambda\tau_i}\Big) + \lambda^{n-1} R_{n-1}(e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \ldots, e^{-\lambda\tau_m}) + \cdots + \\ & \lambda R_1(e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \ldots, e^{-\lambda\tau_m}) + R_0(e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \ldots, e^{-\lambda\tau_m}) \\ =& 0,\end{aligned}$$

where $R_i(Z_1, Z_2, \ldots, Z_m)$ is polynomial in Z_j, $i = 0, 1, \ldots, n-1$, $j = 1, 2, \ldots, m$. By Theorems 2.5 and 3.1, there exists a real number d satisfying

$$X(t) = \int_{(d)} e^{\lambda t} H^{-1}(\lambda) \mathrm{d}\lambda,$$

where $\int_{(d)} = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{d-iT}^{d+iT}$.

First, we will prove

$$X(t) = \int_{(\alpha)} e^{\lambda t} H^{-1}(\lambda) \mathrm{d}\lambda$$

for any $\alpha > \alpha_0$. Consider the integral of $e^{\lambda t} H^{-1}(\lambda)$ along the closed curve $\Gamma = L_1 M_1 L_2 M_2$ (see the following figure)

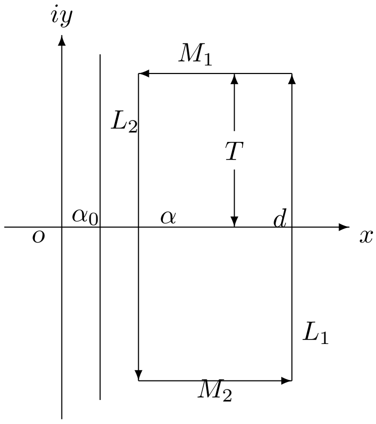

Figure 4.1 The closed curve Γ

where

$$L_1 = \{d + iu : -T \le u \le T\}, \quad L_2 = \{\alpha + iu : -T \le u \le T\},$$

$$M_1 = \{\nu + iT : \alpha \le \nu \le d\}, \quad M_2 = \{\nu - iT : \alpha \le \nu \le d\}.$$

Denote $S = \{\lambda \in C : \alpha \le \mathfrak{Re}\lambda \le d\}$. $\det H(\lambda) \ne 0$ as $\lambda \in S$. Thus, there exists no zero point in the rectangle Γ. So every element of $H^{-1}(\lambda)$ is analytic in Γ. Thus, it follows

$$\begin{aligned}&\int_\Gamma e^{\lambda t} H^{-1}(\lambda)\mathrm{d}\lambda \\ &= \int_{L_1} e^{\lambda t} H^{-1}(\lambda)\mathrm{d}\lambda + \int_{M_1} e^{\lambda t} H^{-1}(\lambda)\mathrm{d}\lambda + \int_{L_2} e^{\lambda t} H^{-1}(\lambda)\mathrm{d}\lambda + \int_{M_2} e^{\lambda t} H^{-1}(\lambda)\mathrm{d}\lambda \\ &= 0.\end{aligned}$$

We will prove: $\int_{M_1} e^{\lambda t} H^{-1}(\lambda)\mathrm{d}\lambda \to 0$ and $\int_{M_2} e^{\lambda t} H^{-1}(\lambda)\mathrm{d}\lambda \to 0$ as $T \to \infty$. Since $\lambda \in S$, we have $H(\lambda) \ne 0$. In consequence, $H^{-1}(\lambda)$ exists. Let $H^*(\lambda)$ be the adjoint matrix of $H(\lambda)$, and denote $h(\lambda) = \det H(\lambda)$. Then it follows that

$$H^{-1}(\lambda) = \frac{1}{h(\lambda)} H^*(\lambda).$$

Since the most large power in λ of every element of $H^{-1}(\lambda)$ is less than that of $h(\lambda)$, and $e^{-\lambda t}$ is bounded as $\lambda \in S$, we have

$$\int_{M_2} e^{\lambda t} H^{-1}(\lambda)\mathrm{d}\lambda \to 0$$

and

$$\int_{M_1} e^{\lambda t} H^{-1}(\lambda)\mathrm{d}\lambda \to 0$$

as $T \to \infty$. Thus, it follows

$$X(t) = \int_{(\alpha)} e^{\lambda t} H^{-1}(\lambda)\mathrm{d}\lambda.$$

Secondly, we will prove

$$\Big| \int_{(\alpha)} e^{\lambda t} H^{-1}(\lambda)\mathrm{d}\lambda \Big| \le k e^{\alpha t}.$$

It is obvious that

$$H^{-1}(\lambda)=\Big[\frac{1}{\lambda^n\det(I-\sum\limits_{i=1}^m e^{-\lambda\tau_i}C_i)}-\frac{\lambda^{n-1}R_{n-1}+\cdots+\lambda R_1+R_0}{\lambda^n\det(I-\sum\limits_{i=1}^m e^{-\lambda\tau_i}C_i)h(\lambda)}\Big]H^*(\lambda).$$

On the line $\mathfrak{Re}\lambda=\alpha$, it holds

$$\begin{aligned}&\Big|\int_{(\alpha)}\frac{\lambda^{n-1}R_{n-1}+\cdots+\lambda R_1+R_0}{\lambda^n\det(I-\sum\limits_{i=1}^m e^{-\lambda\tau_i}C_i)h(\lambda)}H^*(\lambda)e^{\lambda t}\mathrm{d}\lambda\Big|\\&\le e^{\alpha t}\int_{(\alpha)}\Big|\frac{\lambda^{n-1}R_{n-1}+\cdots+\lambda R_1+R_0}{\lambda^n\det(I-\sum\limits_{i=1}^m e^{-\lambda\tau_i}C_i)h(\lambda)}H^*(\lambda)\Big|\mathrm{d}\lambda.\end{aligned}$$

By Lemma 4.1, it follows that $\det(I-\sum_{i=1}^m e^{-\lambda\tau_i}C_i)\neq 0$ for $\lambda\in S$. For every element of

$$\frac{\lambda^{n-1}R_{n-1}+\cdots+\lambda R_1+R_0}{\lambda^n\det(I-\sum\limits_{i=1}^m e^{-\lambda\tau_i}C_i)h(\lambda)}H^*(\lambda),$$

the power of denominator in λ is at least 2 greater than the power of numerator. Thus, for

$$\Big|\frac{\lambda^{n-1}R_{n-1}+\cdots+\lambda R_1+R_0}{\lambda^n\det(I-\sum\limits_{i=1}^m e^{-\lambda\tau_i}C_i)h(\lambda)}H^*(\lambda)\Big|,$$

the power of the denominator in λ is at least 2 greater than the power of numerator. Thus, the integral

$$\int_{(\alpha)}\Big|\frac{\lambda^{n-1}R_{n-1}+\cdots+\lambda R_1+R_0}{\lambda^n\det(I-\sum\limits_{i=1}^m e^{-\lambda\tau_i}C_i)h(\lambda)}H^*(\lambda)\Big|\mathrm{d}\lambda$$

is convergent. Therefore, there exists a constant $k_1>0$ such that

$$\int_{(\alpha)}\Big|\frac{\lambda^{n-1}R_{n-1}+\cdots+\lambda R_1+R_0}{\lambda^n\det(I-\sum\limits_{i=1}^m e^{-\lambda\tau_i}C_i)h(\lambda)}H^*(\lambda)\Big|\mathrm{d}\lambda\le k_1.$$

It follows

$$\Big|\int_{(\alpha)}\frac{\lambda^{n-1}R_{n-1}+\cdots+\lambda R_1+R_0}{\lambda^n\det(I-\sum\limits_{i=1}^m e^{-\lambda\tau_i}C_i)h(\lambda)}H^*(\lambda)e^{\lambda t}\mathrm{d}\lambda\Big|\le k_1e^{\alpha t}.$$

Thirdly, we will prove that there exists a constant $k_2>0$ such that

$$\Big|\int_{(\alpha)}\frac{1}{\lambda^n\det(I-\sum\limits_{i=1}^m e^{-\lambda\tau_i}C_i)}H^*(\lambda)e^{\lambda t}\mathrm{d}\lambda\Big|\le k_2e^{\alpha t}.$$

It suffices to prove that there exists a constant $k'>0$ satisfying

$$\int_{(\alpha)}\frac{1}{\lambda\det(I-\sum\limits_{i=1}^m e^{-\lambda\tau_i}C_i)}e^{\lambda t}\mathrm{d}\lambda\le k'e^{\alpha t}.$$

By Lemma 4.1, we know that $[\det(I-\sum_{i=1}^m C_ie^{-\lambda\tau_i})]^{-1}$is analytic in S and is a periodic function. Assume that the period of $[\det(I-\sum_{i=1}^m C_ie^{-\lambda\tau_i})]^{-1}$ is T_1. Denote $T_0=\frac{2\pi}{T_1}=\omega$. The function

$[\det(I-\sum_{i=1}^{m}C_ie^{-\lambda\tau_i})]^{-1}$ possesses an absolutely convergent Fourier series

$$\Big[\det\Big(I-\sum_{i=1}^{m}e^{-\lambda\tau_i}C_i\Big)\Big]^{-1}=\sum_{j=-\infty}^{\infty}D_je^{jT_0\lambda},$$

where $\lambda\in S$, $D_j=\frac{1}{T_1}\int_{-\frac{T_1}{2}}^{\frac{T_1}{2}}[\det(I-\sum_{i=1}^{m}e^{-\lambda\tau_i}C_i)]^{-1}e^{jT_0\lambda}\mathrm{d}\lambda$. If $\lambda=\beta+i\omega$, $\alpha\le\beta\le d$, then we get

$$\Big|\sum_{j=-\infty}^{\infty}D_je^{jT_0\lambda}\Big|<\infty.$$

Therefore, if $t>0$, $T_0j+t\ge 0$, then it holds that

$$\begin{aligned}\int_{(\alpha)}\frac{1}{\lambda\det(I-\sum\limits_{i=1}^{m}C_ie^{-\lambda\tau_i})}e^{\lambda t}\mathrm{d}\lambda&=\sum_{j=-\infty}^{\infty}D_j\int_{(\alpha)}e^{\lambda T_0j}e^{\lambda t}\lambda^{-1}\mathrm{d}\lambda\\&=\sum_{j=-\infty}^{\infty}D_j\int_{(\alpha)}e^{\lambda(T_0j+t)}\lambda^{-1}\mathrm{d}\lambda.\end{aligned}$$

By Lemma 4.2, it holds

$$\int_{(\alpha)}e^{\lambda(T_0j+t)}\lambda^{-1}\mathrm{d}\lambda=1,\quad \alpha(T_0j+t)>0.$$

Furthermore, it holds that

$$0<\int_{(\alpha)}e^{\lambda(T_0j+t)}\lambda^{-1}\mathrm{d}\lambda<1,\quad \alpha(T_0j+t)\le 0.$$

Thus, we get

$$\Big|\sum_{j=-\infty}^{\infty}D_j\int_{(\alpha)}e^{\lambda(T_0j+t)}\lambda^{-1}\mathrm{d}\lambda\Big|\le 2\sum_{\alpha(T_0j+t)>0}|D_j|.$$

It is obvious that

$$\sum_{\alpha(T_0j+t)>0}2|D_j|\le e^{\alpha t}\sum_{\alpha(T_0j+t)>0}2|D_j|e^{\alpha T_0j}\le e^{\alpha t}2\sum_{j=-\infty}^{\infty}|D_j|e^{\alpha T_0j}.$$

Set $k'=2\sum_{j=-\infty}^{\infty}|D_j|e^{\alpha T_0j}$. Then we get

$$\Big|\int_{(\alpha)}\frac{1}{\lambda\det(I-\sum\limits_{i=1}^{n}e^{-\lambda\tau_i}C_i)}e^{\lambda t}\mathrm{d}\lambda\Big|\le k'e^{\alpha t}.$$

So it holds

$$\int_{(\alpha)}\frac{1}{\lambda\det(I-\sum\limits_{i=1}^{n}e^{-\lambda\tau_i}C_i)}e^{\lambda t}\mathrm{d}\lambda\le k'e^{\alpha t}.$$

Thus, there exists a constant $k_2>0$ such that

$$\Big|\int_{(\alpha)}\frac{1}{\lambda^n\det(I-\sum\limits_{i=1}^{n}e^{-\lambda\tau_i}C_i)}H^*(\lambda)e^{\lambda t}\mathrm{d}\lambda\Big|\le k_2e^{\alpha t}.$$

Denote $k_3 = k_1 + k_2$, it holds

$$|X(t)| \le k_3 e^{\alpha t}.$$

Finally, we will prove that

$$|\dot{X(t)}| \le k e^{\alpha t}, \quad t \ge 0$$

on $[t-r, t]$. Let $\dot{X(t)} = Y(t)$. Then Eq. (3.1)

$$\dot{X}(t) = AX(t) + \sum_{i=1}^{m} B_i X(t-\tau_i) + \sum_{i=1}^{m} C_i \dot{X}(t-\tau_i)$$

can be reduced to

$$Y(t) - \sum_{i=1}^{m} C_i Y(t-\tau_i) = P(t), \tag{4.2}$$

where $P(t) = AX(t) + \sum_{i=1}^{m} B_i X(t-\tau_i)$. Since $|X(t)| \le k_3 e^{\alpha t}$, it follows that

$$|P(t)| \le \Big(|A| + \sum_{i=1}^{m} B_i\Big) k_3 e^{\alpha t}.$$

Let $Y(t) = Z(t)e^{\alpha t}$. Then Eq. (4.2) can be written as

$$Z(t) - \sum_{i=1}^{n} C_i e^{-\alpha\tau_i} Z(t-\tau_i) = P(t)e^{-\alpha t}. \tag{4.3}$$

Since $|P(t)e^{-\alpha t}|$ is bounded, the solutions of Eq. (4.3) are bounded for $t \ge 0$. There exists a constant $k_4 > 0$ satisfying $Z(t) \le k_4$. Therefore,

$$|\dot{X}(t)| = |Y(t)| = |Z(t)| \cdot e^{\alpha t} \le k_4 e^{\alpha t}.$$

Choose $k = \max\{k_3, k_4\}$, then it follows that $|\dot{X}(t)| \le k e^{\alpha t}$ and $|X(t)| \le k e^{\alpha t}$. This completes the proof. □

By Theorem 3.1 and Lemma 4.3, we can easily get the following theorem.

Theorem 4.1 *Let $h(\lambda) = 0$ be the characteristic equation of Eq. (1.1). If $\alpha_0 = \sup\{\mathfrak{Re}\lambda : h(\lambda) = 0\}$, then, for any $\alpha > \alpha_0$, there exists a constant $k = k(\alpha)$ such that the solution $x(\varphi)(t)$ of Eqs. (1.1) and (1.2) satisfies*

$$|x(\varphi)(t)| \le k e^{\alpha t} |\varphi|.$$

If $\alpha_0 < 0$, then the zero solution of Eq. (1.1) is exponentially stable.

Corollary 4.2 *If all roots of the characteristic equation $h(\lambda) = 0$ for Eq. (1.1) satisfies*

(1) The real parts of all roots are negative;

(2) $\rho(\sum_{i=1}^{m} C_i e^{-iy\tau_i}) < 1$, $\forall y \in \mathbb{R}$.

Then the zero solution of Eq. (1.1) is exponentially stable.

Proof By condition (1), $\alpha_0 \le 0$ holds. By condition (2) and Lemma 1.2, the matrix

$$\Big(I - \sum_{i=1}^{m} C_i e^{-iy\tau_i}\Big)^{-1}$$

exists. So for any $y \in \mathbb{R}$, we have

$$\det\Big(I - \sum_{i=1}^{m} C_i e^{-iy\tau_i}\Big) \neq 0.$$

Thus, the equation

$$\det\Big(I - \sum_{i=1}^{m} C_i e^{-\lambda\tau_i}\Big) = 0$$

has no root on imaginary axis. By Lemma 4.1, we get $\alpha_0 < 0$. By Theorem 4.1, it follows that the zero solution of Eq. (1.1) is exponentially stable.

5. Numerical examples

In this section, we give two examples to illustrate the applications of our results.

Example 5.1 Consider the following equation

$$\dot{x}(t) = \begin{pmatrix} -\frac{1}{6}\ln 2 - \frac{1}{6}\pi i & 0 \\ 0 & -\ln\frac{3}{2} \end{pmatrix} x(t-1) + \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \dot{x}(t-2). \tag{5.1}$$

The characteristic equation of Eq. (5.1) is

$$\big(\lambda - (-\frac{1}{6}\ln 2 - \frac{1}{6}\pi i)e^{-\lambda} - \frac{1}{3}\lambda e^{-2\lambda}\big)\big(\lambda - (-\ln\frac{3}{2})e^{-\lambda} - \frac{1}{3}\lambda e^{-2\lambda}\big) = 0. \tag{5.2}$$

Eq. (5.2) possesses two roots in all: $\lambda_1 = -\ln\frac{3}{2}$ and $\lambda_2 = -\ln 2 - \pi i$. Obviously, $\mathfrak{Re}\lambda_1 < 0$, $\mathfrak{Re}\lambda_2 < 0$. By the conclusion (1) in Theorem 3.2, the zero solution of Eq. (5.1) is asymptotically stable. By the Theorem 4.1, the zero solution of Eq. (5.1) is exponentially stable.

Example 5.2 Consider the neutral differential equation

$$\ddot{x}(t) - \ddot{x}(t-1) + 2x(t-1) + x(t-2) = 0 \tag{5.3}$$

or

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} x(t-1) + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} x(t-2) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \dot{x}(t-1). \tag{5.4}$$

The characteristic equation of Eqs. (5.3) or (5.4) is

$$\lambda^2(1 - e^{-\lambda}) + e^{-2\lambda} + 2e^{-\lambda} = 0. \tag{5.5}$$

We can check that $\lambda = 0.7820 \pm 0.8629i$ are the roots of Eq. (5.5). $\mathfrak{Re}(\lambda) > 0$ holds. By the conclusion (3) in Theorem 3.2, the zero solution of Eqs. (5.3) or (5.4) is unstable.

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