Journal of Mathematical Research with Applications
Nov., 2013, Vol. 33, No. 6, pp. 653–665
DOI:10.3770/j.issn:2095-2651.2013.06.002
Http://jmre.dlut.edu.cn

All Nearly 2-Regular Leaves of Partial 6-Cycle Systems

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Abstract Let K_n be a complete graph on n vertices. In this paper, we find the necessary conditions for the existence of a 6-cycle system of $K_n - L$ for every nearly 2-regular leave L of K_n. This condition is also sufficient when the number of vertices of L is $n-4$.

Keywords complete graph; 6-cycle; nearly 2-regular.

MR(2010) Subject Classification 05C38

1. Introduction

Let $V(G)$ be the vertex set of the graph G and $E(G)$ be the edge set of the graph G. An m-cycle system of a graph G is an ordered pair $(V(G), B)$, where B is a set of edge-disjoint cycles of length m, such that each edge of G is contained in exactly one cycle in B. For convenience, we call B an m-cycle system of a graph G instead of an ordered pair $(V(G), B)$.

There have been many results found on m-cycle systems of G for various graphs G, see surveys [1, 2]. The existence of an m-cycle system of $K_n - I$, where I is a 1-factor (called leave) was solved in [3, 4]. Recently, an m-cycle system of $K_n - L$, where L is a subgraph of K_n (called leave) is considered in several papers. This can alternatively be viewed as a partial m-cycle system of K_n with leave L. A solution of partial 4-cycle system of K_n and a partial 6-cycle system of K_n with a 2-regular leave can be found in [5, 6], respectively. And the existence of a partial 6-cycle system of K_n with a forest leave can be found in [7].

In this paper, we shall consider the existence of a partial 6-cycle system of K_n with a nearly 2-regular leave L. It is an extension of [5–7]. Not only is this result of interest in its own right in the context of history of cycle systems, but, it also arose as a useful tool in studying the cycle systems of the line graphs of complete multipartite graphs [8, 9].

L is said to be nearly 2-regular of K_n if all vertices in L have degree 2 except for one (named ∞) whose degree is greater than 2, and L need not be a spanning subgraph of K_n. The necessity of the existence of a partial 6-cycle system of K_n with a nearly 2-regular leave L can be found in Lemma 1.1.

Lemma 1.1 *Let L be a nearly 2-regular subgraph of a complete graph K_n on n vertices.* $K_n - L$

Received August 28, 2012; Accepted May 17, 2013
Supported by the National Natural Science Foundation of China (Grant No.11071163).
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denotes K_n with a subgraph L removed. The necessary conditions for the existence of a partial 6-cycle systems of K_n with leave L are: (i) n is odd, (ii) 6 divides $|E(K_n - L)|$, and (iii) $n \geq 7$.

Proof The necessity of condition (i) follows from the fact that in a 6-cycle system, each vertex clearly has even degree. The necessity of condition (ii) follows since each 6-cycle has six edges. The necessity of condition (iii) is obvious. □

In this paper, our main result is Theorem 4.1 which shows that Lemma 1.1 is also sufficient when the number of vertices of L is $n - 4$.

The following two theorems are useful in order to prove Theorem 4.1.

Theorem 1.2 ([10]) *There exists a 6-cycle system of $K_{m,n}$ if and only if:*

(i) m *and* n *are even;*

(ii) *6 divides* m *or* n*; and*

(iii) $\min\{m, n\} \geq 4$.

Theorem 1.3 ([6]) *Let L be 2-regular subgraph in the complete graph K_n. There exists a 6-cycle system of $G = K_n - L$ if and only if:*

(i) $|E(K_n - L)|$ *is divisible by 6;*

(ii) n *is odd; and*

(iii) $n \geq 7$.

We will use some symbols in this paper. Let $E(G)$ be the set of edges of the graph G and $V(G)$ be the set of vertices of G. Let (u, v) or (v, u) be the edge with endpoints $\{u, v\}$. For convenience, we can use uv or vu instead of (u, v) or (v, u). If vertex v is an end point of edge uv (or vu), then v and uv are incident. For $v \in G$, the degree of vertex v (in a loopless graph), denoted by $d_G(v)$, is the number of incident edges. The graph union $H = G + F$ between two simple graphs G and F is the graph H defined by $V(H) = V(G) \bigcup V(F)$ and $E(H) = \{uv | uv \in E(G) \text{ or } uv \in E(F)\}$. The graph difference $H = G - F$ between G and its subgraph F is the graph H defined by $V(H) = V(G) \backslash \{v \in V(F) | d_F(v) = d_G(v)\}$ and $E(H) = E(G) \backslash E(F)$. Let $C_s = (a_1, a_2, \ldots, a_s)$ be a cycle of length s. nC_s is a graph union of n C_ss. Let $K_{A,B}$ or $K_{a,b}$ be the complete bipartite graph with vertex set A and B, where $|A| = a, |B| = b$ and $V(K_{A,B}) = A \bigcup B$. Z_n consists of n residual classes. Let $Z_n = \{0, 1, \ldots, n-1\}$. Then $Z_n \setminus Z_m = \{m, m+1, \ldots, n-3, n-2, n-1\}$ when $m \leq n$.

2. 6-cycle systems of $K_n - L$ where $7 \leq n \leq 29$

We shall give some small cases which will be used in the proof of Theorem 4.1.

Lemma 2.1 *There exists a 6-cycle system B of $G = K_7 - [(\infty, 0, 1) + (\infty, 2, 3) + (\infty, 4, 5)]$, where $V(K_7) = Z_6 \cup \{\infty\}$.*

Proof We give the proof by direct construction. Let $B = \{(2, 0, 3, 5, 1, 4), (1, 2, 5, 0, 4, 3)\}$. And B is a 6-cycle system of G. □

Lemma 2.2 *Let $V(K_9)$=$Z_8 \cup \{\infty\}$. There exists a 6-cycle system B of $G = K_9 - L$, where L is a nearly 2-regular subgraph of K_9 and $|E(G)|$ is divisible by 6.*

Proof (i) When L=$(\infty, 0, 1) + (\infty, 2, 3)$, $B = \{(2, 0, 6, 4, 3, 1), (3, 0, 4, 7, 1, 5), (4, 2, 7, 6, \infty, 5), (4, \infty, 7, 5, 6, 1), (5, 0, 7, 3, 6, 2)\}$.

(ii) When $L = (\infty, 0, 1)+(\infty, 2, 3)+(\infty, 4, 5)+(\infty, 6, 7)$, $B = \{(2, 0, 6, 4, 3, 1), (3, 0, 4, 7, 1, 5), (4, 2, 7, 5, 6, 1), (5, 0, 7, 3, 6, 2)\}$. □

Lemma 2.3 *Let $V(K_{11})$=$Z_{10} \cup \{\infty\}$. There exists a 6-cycle system B of $G = K_{11} - L$, where L is a nearly 2-regular subgraph of K_{11} and $|E(G)|$ is divisible by 6.*

Proof (i) When $L = (\infty, 0, 1)+(\infty, 2, 3, 4)+(\infty, 5, 6, 7, 8, 9)$, $B = \{(3, \infty, 8, 2, 5, 0), (6, \infty, 7, 1, 5, 3), (2, 0, 9, 4, 8, 1), (6, 0, 8, 3, 9, 1), (4, 0, 7, 5, 9, 2), (3, 1, 4, 6, 2, 7), (5, 4, 7, 9, 6, 8)\}$.

(ii) When $L = (\infty, 0, 1)+(\infty, 2, 3)+(\infty, 4, 5, 6, 7, 8, 9)$, $B = \{(6, \infty, 7, 1, 5, 3), (3, 1, 4, 6, 2, 7), (5, \infty, 8, 2, 9, 0), (0, 2, 5, 7, 4, 3), (1, 2, 4, 8, 5, 9), (6, 1, 8, 0, 4, 9), (8, 3, 9, 7, 0, 6)\}$.

(iii) When $L = (\infty, 0, 1)+(\infty, 2, 3, 4, 5)+(\infty, 6, 7, 8, 9)$, $B = \{(8, 3, 9, 7, 0, 6), (6, 1, 8, 0, 4, 9), (3, \infty, 7, 2, 1, 5), (4, \infty, 8, 2, 9, 1), (0, 9, 5, 6, 4, 2), (3, 0, 5, 8, 4, 7), (3, 1, 7, 5, 2, 6)\}$.

(iv) When $L = (\infty, 0, 1, 2)+(\infty, 3, 4, 5)+(\infty, 6, 7, 8, 9)$, $B = \{(1, \infty, 7, 3, 0, 8), (4, \infty, 8, 2, 5, 0), (2, 0, 7, 9, 1, 6), (6, 0, 9, 4, 1, 3), (5, 1, 7, 2, 9, 3), (5, 9, 6, 8, 4, 7), (3, 2, 4, 6, 5, 8)\}$.

(v) When $L = (\infty, 0, 1)+(\infty, 2, 3)+(\infty, 4, 5)+(6, 7, 8, 9)$, $B = \{(6, \infty, 9, 4, 3, 0), (7, \infty, 8, 1, 2, 0), (4, 0, 9, 1, 5, 8), (5, 0, 8, 6, 4, 7), (1, 4, 2, 6, 5, 3), (6, 1, 7, 2, 9, 3), (5, 9, 7, 3, 8, 2)\}$.

(vi) When $L = (\infty, 0, 1)+(\infty, 2, 3)+(7, 8, 9)+(\infty, 4, 5, 6)$, $B = \{(5, \infty, 9, 1, 3, 0), (7, \infty, 8, 1, 2, 0), (4, 0, 9, 2, 7, 1), (6, 0, 8, 3, 5, 2), (5, 1, 6, 4, 3, 7), (4, 7, 6, 9, 5, 8), (3, 9, 4, 2, 8, 6)\}$.

(vii) When $L = (\infty, 0, 1) + (\infty, 2, 3) + (\infty, 4, 5) + (\infty, 6, 7, 8)$, $B = \{(9, \infty, 7, 0, 3, 4), (0, 6, 9, 8, 1, 2), (4, 0, 9, 1, 5, 8), (5, 0, 8, 6, 4, 7), (1, 4, 2, 6, 5, 3), (6, 1, 7, 2, 9, 3), (5, 9, 7, 3, 8, 2)\}$. □

Lemma 2.4 *Let $V(K_{13}) = Z_{12} \cup \{\infty\}$. There exists a 6-cycle system B of $G = K_{13} - L$, where L is a nearly 2-regular subgraph of K_{13} and $|E(G)|$ is divisible by 6.*

Proof (i) When $L = (\infty, 0, 1)+(\infty, 2, 3, 4, 5, 6, 7, 8, 9)$, $B = \{(\infty, 10, 1, 11, 5, 3), (\infty, 8, 11, 2, 0, 4), (\infty, 7, 3, 11, 0, 5), (\infty, 11, 4, 9, 0, 6), (3, 0, 8, 10, 2, 1), (7, 0, 10, 6, 4, 1), (5, 1, 9, 3, 8, 2), (6, 1, 8, 5, 10, 3), (6, 2, 7, 5, 9, 11), (4, 2, 9, 10, 11, 7), (4, 8, 6, 9, 7, 10)\}$.

(ii) When $L = (\infty, 0, 1)+(\infty, 2, 3)+(4, 5, 6, 7, 8, 9)$, $B = \{(\infty, 8, 11, 2, 0, 4), (\infty, 7, 3, 11, 0, 5), (\infty, 11, 10, 5, 9, 6), (3, 0, 8, 10, 2, 1), (7, 0, 10, 6, 4, 1), (5, 1, 9, 3, 8, 2), (\infty, 10, 3, 6, 0, 9), (4, 3, 5, 11, 9, 7), (8, 1, 10, 7, 2, 4), (7, 5, 8, 6, 1, 11), (6, 2, 9, 10, 4, 11)\}$.

(iii) When $L = (\infty, 0, 1)+(7, 8, 9)+(\infty, 2, 3, 4, 5, 6)$, $B = \{(\infty, 8, 11, 2, 0, 4), (\infty, 7, 3, 11, 0, 5), (\infty, 10, 3, 6, 0, 9), (3, 0, 8, 10, 2, 1), (7, 0, 10, 6, 4, 1), (5, 1, 9, 3, 8, 2), (3, \infty, 11, 4, 7, 5), (6, 1, 11, 7, 10, 9), (8, 1, 10, 5, 9, 4), (2, 4, 10, 11, 6, 7), (6, 2, 9, 11, 5, 8)\}$.

(iv) When $L = (\infty, 0, 1)+(\infty, 7, 8)+(\infty, 2, 3, 4, 5, 6)$, $B = \{(\infty, 5, 9, 2, 0, 4), (5, 0, 11, 8, 1, 10), (3, 7, 9, 8, 5, 11), (8, 4, 9, 11, 2, 6), (\infty, 10, 3, 6, 0, 9), (3, 0, 8, 10, 2, 1), (7, 0, 10, 6, 4, 1), (5, 1, 9, 3, 8, 2), (3, \infty, 11, 4, 7, 5), (6, 1, 11, 7, 10, 9), (2, 4, 10, 11, 6, 7)\}$.

(v) When $L = (\infty, 0, 1)+(\infty, 2, 3, 4)+(5, 6, 7, 8, 9)$, $B = \{(3, \infty, 11, 4, 7, 5), (6, 1, 11, 7, 10, 9),$

$(2,0,4,5,\infty,7)$, $(6,\infty,8,10,4,2)$, $(0,7,3,9,11,5)$, $(1,7,9,2,5,10)$, $(1,9,4,8,11,2)$, $(8,2,10,11,0,3)$, $(1,5,8,6,11,3)$, $(0,10,6,4,1,8)$, $(\infty,10,3,6,0,9)\}$.

(vi) When $L=(\infty,0,1)+(7,8,9,10)+(\infty,2,3,4,5)$, $B=\{(3,\infty,11,4,7,5)$, $(6,\infty,8,10,4,2)$, $(\infty,10,3,6,0,9)$, $(0,7,3,9,11,5)$, $(1,7,9,2,5,10)$, $(1,9,4,8,11,2)$, $(4,\infty,7,6,8,0)$, $(2,7,11,3,0,10)$, $(5,9,6,10,11,1)$, $(2,0,11,6,1,8)$, $(5,6,4,1,3,8)\}$.

(vii) When $L=(6,7,8)+(\infty,0,1,9)+(\infty,2,3,4,5)$, $B=\{(3,\infty,11,4,7,5)$, $(6,\infty,8,10,4,2)$, $(6,1,11,7,10,9)$, $(0,7,3,9,11,5)$, $(1,7,9,2,5,10)$, $(8,2,10,11,0,3)$, $(0,10,6,4,1,8)$, $(2,7,\infty,1,3,11)$, $(4,\infty,10,3,6,0)$, $(0,9,8,5,1,2)$, $(8,4,9,5,6,11)\}$.

(viii) When $L=(\infty,0,1)+(\infty,2,3,4)+(\infty,5,6,7,8)$, $B=\{(3,\infty,11,4,7,5)$, $(6,1,11,7,10,9)$, $(0,7,3,9,11,5)$, $(1,7,9,2,5,10)$, $(1,9,4,8,11,2)$, $(0,10,6,4,1,8)$, $(\infty,10,3,6,0,9)$, $(7,\infty,6,11,0,2)$, $(5,1,3,11,10,8)$, $(4,5,9,8,6,2)$, $(10,2,8,3,0,4)\}$.

(ix) When $L=(\infty,0,1)+(\infty,2,3)+(4,5,6)+(7,8,9)$, $B=\{(\infty,8,11,2,0,4)$, $(\infty,7,3,11,0,5)$, $(\infty,11,10,5,9,6)$, $(3,0,8,10,2,1)$, $(5,1,9,3,8,2)$, $(\infty,10,3,6,0,9)$, $(8,1,10,7,2,4)$, $(5,8,6,11,4,3)$, $(7,0,10,4,1,11)$, $(6,2,9,4,7,1)$, $(7,5,11,9,10,6)\}$.

(x) When $L=(\infty,0,1)+(\infty,2,3)+(\infty,4,5)+(7,8,9)$, or $L=(\infty,0,1)+(\infty,2,3)+(\infty,4,5)+(\infty,7,8)$, $B=B_1\bigcup B_2\bigcup B_3$. By Lemma 2.1, B_1 is a 6-cycle system of K_7-3C_3, where $3C_3=(\infty,0,1)+(\infty,2,3)+(\infty,4,5)$. B_2 is a 6-cycle system of $K_{6,6}$ by Theorem 1 where $V(K_{6,6})=Z_6\bigcup(Z_{12}\backslash Z_6)$. B_3 is a 6-cycle system of K_7-C_3 by Theorem 2, where $C_3=(\infty,7,8)$, or $(7,8,9)$ and $V(K_7)=(Z_{12}\backslash Z_6)\cup\{\infty\}$.

(xi) When $L=(\infty,0,1,2)+(\infty,4,5,6,7,8,9,10)$, $B=\{(\infty,11,10,5,9,6)$, $(5,1,9,3,8,2)$, $(7,0,10,4,1,11)$, $(6,2,9,4,7,1)$, $(2,7,10,1,8,4)$, $(5,8,6,11,2,0)$, $(\infty,1,3,11,0,8)$, $(10,2,3,4,0,6)$, $(3,0,9,\infty,5,7)$, $(3,\infty,7,9,11,5)$, $(4,11,8,10,3,6)\}$.

(xii) When $L=(\infty,0,1,2,3)+(\infty,4,5,6,7,8,9)$, $B=\{(\infty,11,10,5,9,6)$, $(\infty,1,3,11,0,8)$, $(\infty,2,10,9,11,5)$, $(3,5,7,\infty,10,6)$, $(5,1,9,3,8,2)$, $(7,0,10,4,1,11)$, $(6,2,9,4,7,1)$, $(2,7,10,1,8,4)$, $(5,8,6,11,2,0)$, $(0,6,4,3,7,9)$, $(3,10,8,11,4,0)\}$.

(xiii) When $L=(\infty,0,1,2)+(\infty,3,4,5)+(6,7,8,9)$, $B=\{(2,3,1,4,8,0)$, $(7,5,8,6,11,2)$, $(3,8,11,9,0,6)$, $(9,2,6,1,7,3)$, $(1,\infty,7,11,0,10)$, $(4,0,7,9,1,11)$, $(11,\infty,10,2,5,3)$, $(5,10,6,\infty,8,1)$, $(4,\infty,9,10,8,2)$, $(3,0,5,9,4,10)$, $(5,6,4,7,10,11)\})$.

(xiv) When $L=(\infty,0,1,2)\cup(\infty,3,4,5)\cup(\infty,6,7,8)$, $B=\{(2,3,1,4,8,0)$, $(7,5,8,6,11,2)$, $(3,8,11,9,0,6)$, $(9,8,1,5,10,4)$, $(1,\infty,7,11,0,10)$, $(9,2,6,1,7,3)$, $(11,\infty,10,2,5,3)$, $(0,5,9,6,10,3)$, $(4,\infty,9,10,8,2)$, $(5,6,4,7,10,11)$, $(4,0,7,9,1,11)\}$.

(xv) When $L=(\infty,0,1)+(\infty,2,3)+(\infty,4,5)+(\infty,6,7)+(\infty,8,9)+(\infty,10,11)$, $B=B_1\bigcup B_2\bigcup B_3$. By Lemma 2.1, B_1 is a 6-cycle system of K_7-3C_3, where $3C_3=(\infty,0,1)+(\infty,2,3)+(\infty,4,5)$ and $V(K_7)=Z_6\cup\{\infty\}$. By Lemma 2.1, B_2 is a 6-cycle system of K_7-3C_3, where $3C_3=(\infty,6,7)+(\infty,8,9)+(\infty,10,11)$ and $V(K_7)=(Z_{12}\backslash Z_6)\cup\{\infty\}$. By Theorem 1, B_3 is a 6-cycle system of $K_{6,6}$ where $V(K_{6,6})=Z_6\bigcup(Z_{12}\backslash Z_6)$.

(xvi) When $L=(\infty,0,1,2,3,9)\cup(\infty,4,5,6,7,8)$, $B=\{(\infty,11,10,5,9,6)$, $(7,0,10,4,1,11)$, $(2,7,10,1,8,4)$, $(5,8,6,11,2,0)$, $(\infty,2,10,9,11,5)$, $(3,5,7,\infty,10,6)$, $(3,10,8,11,4,0)$, $(1,\infty,3,8,2,9)$, $(0,11,3,4,9,8)$, $(0,9,7,3,1,6)$, $(5,1,7,4,6,2)\}$. $\Box$

Let $A=\{(3,8,11,9,0,6)$, $(9,2,6,1,7,3)$, $(4,0,7,9,1,11)$, $(3,0,5,9,4,10)$, $(0,11,7,13,1,12)$,

$(2,4,7,10,8,0)$, $(5,11,13,12,2,\infty)$, $(1,\infty,12,6,13,10)$, $(4,6,9,13,2,8)$, $(7,8,13,0,10,12)\}$. A will be used in Lemma 2.5.

Lemma 2.5 *Let $V(K_{15}) = Z_{14} \cup \{\infty\}$. There exists a 6-cycle system B of $G = K_{15} - L$, where L is a nearly 2-regular subgraph of K_{15} and $|E(G)|$ is divisible by 6.*

Proof (i) When $L = (\infty,0,1,2,3) + (\infty,4,5,6,7) + (9,10,11,12,13)$, $B = A \cup B_1$ where $B_1 = \{(7,5,8,6,11,2), (11,\infty,10,2,5,3), (5,10,6,\infty,8,1), (12,5,13,4,1,3), (4,3,13,\infty,9,12)\}$.

(ii) When $L = (\infty,0,1,2,3) + (\infty,4,5,6,7) + (\infty,8,9,10,11)$, $B = A \cup B_2$ where $B_2 = \{(10,6,\infty,9,12,5), (5,2,10,\infty,13,3), (3,11,12,8,1,4), (1,3,12,4,13,5), (5,8,6,11,2,7)\}$.

(iii) When $L = (\infty,0,1,2,3) + (\infty,4,5,6)$, $B = A \cup B_3$ where $B_3 = \{(7,5,8,6,11,2), (11,\infty,10,2,5,3), (10,6,7,\infty,13,5), (9,10,11,12,3,13), (1,8,\infty,9,12,4), (3,4,13,12,5,1)\}$. □

Lemma 2.6 *Let $V(K_{17}) = Z_{16} \cup \{\infty\}$. There exists a 6-cycle system B of $G = K_{17} - L$, where L is a nearly 2-regular subgraph of K_{17} and $|E(G)|$ is divisible by 6.*

Proof (i) When $L = (\infty,0,1,2,3) + (\infty,4,5,6,7)$, $B = \{(9,2,6,1,7,3), (11,\infty,10,2,5,3), (4,0,7,9,1,11), (1,\infty,8,2,13,3), (7,5,8,6,11,2), (3,8,11,9,0,6), (6,\infty,9,12,7,13), (12,5,10,7,4,2), (5,0,14,12,1,15), (8,0,13,14,9,15), (4,8,1,10,15,6), (5,1,13,15,7,14), (11,14,6,10,9,13), (10,6,14,11,13,9), (13,4,14,15,11,5), (4,1,14,\infty,5,9), (11,7,8,9,6,12), (10,8,13,12,3,4), (0,3,14,2,\infty,15), (0,11,10,13,\infty,12), (2,0,10,12,4,15)\}$.

(ii) When $L = 4C_4$, $B = B_1 \bigcup B_2 \bigcup B_3$. Let $K_{17} - 4C_4 = (K_{13} - 3C_4) + K_{4,6} + (K_{4,6} + K_5 - C_4)$, where $V(K_{13}) = (Z_{16} \backslash Z_4) \cup \{\infty\}$, $V(K_{4,6}) = Z_4 \bigcup (Z_{16} \backslash Z_{10})$, and $V(K_{4,6} + K_5 - C_4) = Z_{10} \cup \{\infty\}$. B_1 is a 6-cycle system of $K_{13} - 3C_4$ by Lemma 2.4 (xiii) and (xiv). B_2 is a 6-cycle system of $K_{4,6}$ by Theorem 1. $B_3 = \{(0,\infty,2,9,1,8), (0,9,3,8,2,5), (1,5,3,6,2,4), (1,\infty,3,7,0,6), (2,0,4,3,1,7)\}$ is a 6-cycle system of $K_5 + K_{4,6} - C_4$ where $C_4 = (0,1,2,3)$, $V(K_5) = Z_4 \cup \{\infty\}$ and $V(K_{4,6}) = Z_4 \bigcup (Z_{10} \backslash Z_4)$. (When $C_4 = (\infty,1,2,3)$, the method can also be used). □

The method in Lemma 2.6 will be used in Lemma 2.7.

Lemma 2.7 *Let $V(K_{19}) = Z_{18} \cup \{\infty\}$. There exists a 6-cycle system B of $G = K_{19} - L$, where L is a nearly 2-regular subgraph of K_{19} and $|E(G)|$ is divisible by 6.*

Proof (i) When $L = C_5 \bigcup C_5 \bigcup C_5$, $B = B_1 \bigcup B_2 \bigcup B_3$. B_1 is a 6-cycle system of $K_{15} - 3C_5$ by (i) and (ii) of Lemma 2.5. B_2 is a 6-cycle system of $K_{4,12}$ by Theorem 1. $B_3 = \{(0,4,1,\infty,3,2), (2,4,3,5,0,1), (3,0,\infty,2,5,1)\}$ is a 6-cycle system of $K_5 + K_{4,2}$, where $V(K_5) = Z_4 \bigcup \{\infty\}$ and $V(K_{4,2}) = Z_4 \bigcup \{4,5\}$.

(ii) When $L = (\infty,0,1,2,3) + (\infty,4,5,6)$, $B = B_1 \bigcup B_2 \bigcup B_3$. B_1 is a 6-cycle system of $K_{15} - L$ by Lemma 2.5(iii). B_2 is a 6-cycle system of $K_{4,12}$ by Theorem 1. $B_3 = \{(0,4,1,\infty,3,2), (2,4,3,5,0,1), (3,0,\infty,2,5,1)\}$ is a 6-cycle system of $K_5 + K_{4,2}$ where $V(K_5) = Z_4 \bigcup \{\infty\}$ and $V(K_{4,2}) = Z_4 \bigcup \{4,5\}$. □

Lemma 2.8 *Let $V(K_{21}) = Z_{20} \bigcup \{\infty\}$. There exists a 6-cycle system B of $G = K_{21} - L$, where*

L is a nearly 2-regular subgraph of K_{21} and $|E(G)|$ is divisible by 6.

Proof (i) When $L=2C_5+2C_4$, $B=B_1\bigcup B_2$. B_1 is a 6-cycle system of $K_{10,6}$ by Theorem 1. $B_2=\{(1,\infty,2,11,3,10)$, $(\infty,5,11,6,10,4)$, $(7,\infty,8,11,9,10)$, $(6,\infty,9,3,4,0)$, $(0,10,5,8,1,11)$, $(2,10,8,3,7,0)$, $(4,11,7,5,3,1)$, $(0,3,6,1,9,5)$, $(1,5,2,8,9,7)$, $(2,7,8,4,9,6)$, $(0,9,2,4,6,8)\}$ is a 6-cycle system of $K_{11}+K_{10,2}-(C_5+C_4)$, where $C_5+C_4=(\infty,0,1,2,3)+(4,5,6,7)$, $V(K_{11})=Z_{10}\bigcup\{\infty\}$ and $V(K_{10,2})=Z_{10}\bigcup\{10,11\}$ (The construction can also be used for other form of $C_5\bigcup C_4$. When $V(K_{11})=(Z_{20}\backslash Z_{10})\bigcup\{\infty\}$, $V(K_{10,2})=Z_{10}\bigcup\{18,19\}$, there also exists a 6-system B_2 of $K_{11}+K_{10,2}-(C_5+C_4)$, where $V(C_5+C_4)=(Z_{20}\backslash Z_{10})\bigcup\{\infty\}$).

(ii) When $L=4C_5+C_4$, $B=B_1\bigcup B_2\bigcup B_3^i$, $i=1,2,3$. B_1 is a 6-cycle system of $K_{15}-3C_5$ by Lemma 2.5(i). B_2 is a 6-cycle system of $K_{6,12}$ by Theorem 1 where $V(K_{6,12})=K(Z_6\bigcup\{i|8\le i\le 19\}$. Let $B_3^1=\{(1,\infty,5,0,4,3)$, $(2,\infty,4,5,3,6)$, $(0,6,1,4,2,7)$, $(1,5,2,0,3,7)\}$, $B_3^2=\{(1,\infty,5,0,7,2)$, $(1,7,4,0,2,3)$, $(3,7,5,1,4,6)$, $(2,4,3,0,6,5)\}$, and $B_3^3=\{(0,6,5,1,\infty,2)$, $(0,\infty,4,3,2,7)$, $(1,7,4,5,3,6)$, $(0,5,2,4,1,3)\}$. When $C_5+C_4=(\infty,0,1,2,3)+(4,6,5,7)$, B_3^1 is a 6-cycle system of $K_7+K_{6,2}-(C_5+C_4)$, where $V(K_7)=Z_6\bigcup\{\infty\}$ and $V(K_{6,2})=Z_6\bigcup\{6,7\}$. When $C_5+C_4=(\infty,0,1,6,2)+(\infty,4,5,3)$, B_3^2 is a 6-cycle system of $K_7+K_{6,2}-(C_5+C_4)$, where $V(K_7)=Z_6\bigcup\{\infty\}$, $V(K_{6,2})=Z_6\bigcup\{6,7\}$. When $C_5+C_4=(0,1,2,6,4)+(\infty,5,7,3)$, B_3^3 is a 6-cycle system of $K_7+K_{6,2}-(C_5+C_4)$, where $V(K_7)=Z_6\bigcup\{\infty\}$, $V(K_{6,2})=Z_6\bigcup\{6,7\}$. $\Box$

Lemma 2.9 *Let $V(K_{23})=Z_{22}\bigcup\{\infty\}$. There exists a 6-cycle system B of $G=K_{23}-L$, where L is a nearly 2-regular subgraph of K_{23} and $|E(G)|$ is divisible by 6.*

Proof (i) When $L=3C_5+C_4$, B_1 is a 6-cycle system of $K_{19}-3C_5$ by Lemma 2.7(i). B_2 is a 6-cycle system of $K_{4,12}$ by Theorem 1. $B_3=\{(0,\infty,2,9,1,8)$, $(0,9,3,8,2,5)$, $(1,5,3,6,2,4)$, $(1,\infty,3,7,0,6)$, $(2,0,4,3,1,7)\}$ is a 6-cycle system of $K_5+K_{4,6}-C_4$, where $C_4=(0,1,2,3)$ (The construction can also be used when $C_4=(\infty,1,2,3)$).

(ii) When $L=5C_5$, let $B=B_1\bigcup B_2\bigcup B_3$ where B_1 is a 6-cycle system of $K_{15}-3C_5$ by Lemma 2.5(i) and $V(K_{15})=(Z_{22}\backslash Z_8)\bigcup\{\infty\}$, B_2 is a 6-cycle system of $K_{8,12}$ by Theorem 1 and $B_3=\{(1,\infty,2,9,3,8)$, $(6,2,7,1,9,0)$, $(5,\infty,6,9,7,8)$, $(4,9,5,7,0,3)$, $(0,8,4,1,5,2)$, $(4,0,5,3,1,6)$, $(2,8,6,3,7,4)\}$ is a 6-cycle system of $K_9+K_{8,2}-2C_5$, where $2C_5=(\infty,0,1,2,3)+(\infty,4,5,6,7)$, $V(K_9)=Z_8\bigcup\{\infty\}$, and $V(K_{8,2})=Z_8\bigcup\{8,9\}$. $\Box$

Lemma 2.10 *Let $V(K_{25})=Z_{24}\bigcup\{\infty\}$. There exists a 6-cycle system of $G=K_{25}-L$, where L is a nearly 2-regular subgraph of K_{25} and $|E(G)|$ is divisible by 6.*

Proof (i) When $L=2C_5+2C_4$, $B=B_1\bigcup B_2\bigcup B_3$. B_1 is a 6-cycle system of $K_{21}-(2C_5+2C_4)$ by Lemma 2.8(i). B_2 is a 6-cycle system of $K_{4,3}+K_4$, where $V(K_4)=\{20,21,22,23\}$ and $V(K_{4,3})=\{20,21,22,23\}\bigcup\{\infty,18,19\}$. $B_3=\{(20,\infty,23,18,22,21)$, $(21,\infty,22,20,23,19)$, $(20,19,22,23,21,18)\}$ is a 6-cycle system of $K_{4,18}$ by Theorem 1.

(ii) When $L=4C_5\bigcup C_4$, $B=B_1\bigcup B_2\bigcup B_3$. B_1 is a 6-cycle system of $K_{15}-3C_5$ by Lemma 2.5(i) where $V(K_{15})=(Z_{24}\backslash Z_{10})\bigcup\{\infty\}$. B_2 is a 6-cycle system of $K_{10,12}$ by Theorem

1 where $V(K_{10,12}) = Z_{10} \bigcup (Z_{24} \backslash Z_{12})$. $B_3 = \{(1, \infty, 5, 0, 11, 3), (4, \infty, 7, 3, 2, 0), (8, \infty, 9, 7, 0, 6), (3, 0, 9, 4, 2, 10), (8, 0, 10, 5, 1, 4), (5, 3, 9, 6, 4, 11), (6, 3, 8, 7, 1, 10), (7, 4, 10, 8, 5, 2), (7, 5, 9, 11, 1, 6), (8, 1, 9, 2, 6, 11), (8, 2, 11, 7, 10, 9)\}$ is a 6-cycle system of $K_{10,3}+K_{10}-(C_4+C_5)$, where $V(K_{10,3}) = Z_{10} \bigcup (\{\infty, 10, 11\})$, $V(K_{10}) = Z_{10}$, and $C_4 + C_5 = (\infty, 0, 1, 2) + (\infty, 3, 4, 5, 6)$ (The construction can also be used for other form of $C_4 + C_5$). $\square$

Lemma 2.11 *Let $V(K_{29}) = Z_{28} \bigcup \{\infty\}$. There exists a 6-cycle system B of $G = K_{29} - (4C_5 + 2C_4)$, where L is a nearly 2-regular subgraph of K_{29} and $|E(G)|$ is divisible by 6.*

Proof Let $B = B_1 \bigcup B_2 \bigcup B_3$. B_1 is a 6-cycle system of $K_{25} - (4C_5 + C_4)$ by Lemma 2.10(ii) where $V(K_{25}) = (Z_{28} \backslash Z_4) \bigcup \{\infty\}$. B_2 is a 6-cycle system of $K_{4,8}$ by Theorem 1 where $V(K_{4,8}) = Z_4 \bigcup (Z_{28} \backslash Z_{10})$. $B_2 = \{(0, \infty, 2, 9, 1, 8), (0, 9, 3, 8, 2, 5), (1, 5, 3, 6, 2, 4), (1, \infty, 3, 7, 0, 6), (2, 0, 4, 3, 1, 7)\}$ is a 6-cycle system of $K_5 + K_{4,6} - C_4$, where $C_4 = (0, 1, 2, 3)$, $V(K_5) = Z_4 \bigcup \{\infty\}$, and $V(K_{4,6}) = Z_4 \bigcup (Z_{10} \backslash Z_4)$ (The construction can also be used when $C_4 = (\infty, 1, 2, 3)$). $\square$

3. Some special cases

In this section, we provide some special 6-cycle systems which will be used in the proof of Theorem 4.1.

Lemma 3.1 *(i) There exists a 6-cycle system B_1 of $G = K_7 + K_{6,2} + (\infty, a) + (\infty, b) - [(0, a) + (0, b)] - (\infty, 1, 2)$, where $V(K_{6,2}) = \{c, 1, 2, 3, 4, 5\} \bigcup \{a, b\}$ and $V(K_7) = \{1, 2, 3, 4, 5, c, \infty\}$.*

(ii) There exists a 6-cycle system B_2 of $G = K_7 + K_{6,2} + (\infty, n-2) + (\infty, n-3) - [(0, n-2) + (0, n-3)] - (1, 2, 3)$, where $V(K_7) = Z_6 \bigcup \{\infty\}$ and $V(K_{6,2}) = Z_6 \bigcup \{n-2, n-3\}$.

Proof We give the proof by direct construction of B_i, $i = 1, 2$, in the following two cases.

(i) Let $B_1 = \{(c, \infty, 3, 5, a, 4), (4, \infty, 5, c, 1, b), (3, c, 2, 5, 4, 1), (a, \infty, b, 2, 4, 3), (1, a, 2, 3, b, 5)\}$.

(ii) Let $B_2 = \{(3, n{-}3, 5, n{-}2, 4, 0), (1, n{-}3, \infty, 4, 2, n{-}2), (3, n{-}2, \infty, 0, 1, 5), (2, 0, 5, 4, 1, \infty), (3, 4, n-3, 2, 5, \infty)\}$. $\square$

Lemma 3.2 *(i) There exists a 6-cycle system B_1 of $G = K_7 + K_{6,4} + (\infty, n-2) + (\infty, n-3) + (\infty, n-4) + (\infty, n-5) - [(5, n-2) + (5, n-3) + (4, n-4) + (4, n-5)] - (\infty, 1, 2)$, where $V(K_7) = Z_6 \bigcup \{\infty\}$ and $V(K_{6,4}) = Z_6 \bigcup \{n-2, n-3, n-4, n-5\}$.*

(ii) There exists a 6-cycle system B_2 of $G = K_7 + K_{6,4} + (\infty, n-2) + (\infty, n-3) + (\infty, n-4) + (\infty, n-5) - [(5, n-2) + (5, n-3) + (4, n-4) + (4, n-5)] - (1, 2, 3)$, where $V(K_7) = Z_6 \bigcup \{\infty\}$ and $V(K_{6,4}) = Z_6 \bigcup \{n-2, n-3, n-4, n-5\}$.

(iii) There exists a 6-cycle system B_3 of $G = K_7 + K_{6,4} - [(1, 2) + (2, 3) + (3, 4) + (0, 1)] + (0, 4)$, where $V(K_7) = \{1, 2, 3, n{-}2, n{-}3, n{-}4, \infty\}$ and $V(K_{6,4}) = \{1, 2, 3, n{-}2, n{-}3, n{-}4\} \bigcup \{0, 4, 5, 6\}$.

Proof We give the proof by direct construction of B_i, $1 \leq i \leq 3$ by three cases as follows.

(i) Let $B_1 = \{(0, n-2, \infty, n-3, 4, 3), (n-2, 4, 1, 5, 3, 2), (n-2, 3, n-4, 0, n-3, 1), (5, n-4, \infty, 0, 2, n-5), (n-3, 3, n-5, 0, 4, 2), (1, 0, 5, 4, \infty, n-5), (\infty, 5, 2, n-4, 1, 3)\}$.

(ii) Let $B_2 = \{(0, n-2, \infty, n-3, 4, 3), (1, n-4, 2, 5, 3, \infty), (n-2, 3, n-4, 0, n-3, 1),$

$(5, n-4, \infty, 0, 2, n-5)$, $(n-3, 3, n-5, 0, 4, 2)$, $(1, 0, 5, 4, \infty, n-5)$, $(2, n-2, 4, 1, 5, \infty)\}$.

(iii) Let $B_3 = \{(3, 1, n-2, 2, n-3, n-4)$, $(n-3, 1, n-4, 2, \infty, n-2)$, $(4, n-2, n-4, \infty, n-3, 0)$, $(n-2, 3, n-3, 5, n-4, 0)$, $(6, n-4, 4, 1, \infty, 3)$, $(5, 1, 6, 2, 0, 3)$, $(4, 2, 5, n-2, 6, n-3)\}$. $\square$

Lemma 3.3 *(i) There exists a 6-cycle system B_1 of $G = K_7 + K_{6,6} + (\infty, n-2) + (\infty, n-3) + (\infty, n-4) + (\infty, n-5) + (\infty, n-6) + (\infty, n-7) - [(3, n-2) + (3, n-3) + (4, n-4) + (4, n-5) + (5, n-6) + (5, n-7)] - (\infty, 1, 2)$, where $V(K_7) = Z_6 \bigcup \{\infty\}$ and $V(K_{6,6}) = Z_6 \bigcup \{n-2, n-3, n-4, n-5, n-6, n-7\}$.*

(ii) There exists a 6-cycle system B_2 of $G = K_7 + K_{6,6} + [(\infty, n-2) + (\infty, n-3) + (\infty, n-4) + (\infty, n-5) + (\infty, n-6) + (\infty, n-7)] - [(3, n-2) + (3, n-3) + (4, n-4) + (4, n-5) + (5, n-6) + (5, n-7)] - (1, 2, 3)$, where $V(K_7) = Z_6 \bigcup \{\infty\}$ and $V(K_{6,6}) = Z_6 \bigcup \{n-2, n-3, n-4, n-5, n-6, n-7\}$.

(iii) There exists a 6-cycle system B_3 of $G = K_7 + K_{6,6} + (\infty, n-2) + (\infty, n-3) + (\infty, n-4) + (\infty, n-5)] - [(3, n-2) + (3, n-3) + (4, n-4) + (4, n-5) + (6, \infty)] - [(\infty, 0) + (0, 1) + (1, 2) + (2, 6)$, where $V(K_7) = Z_6 \bigcup \{\infty\}$ and $V(K_{6,6}) = Z_6 \bigcup \{n-2, n-3, n-4, n-5, 6, 7\})$.

(iv) There exists a 6-cycle system B_4 of $G = K_7 + K_{6,8} + [(\infty, n-2) + (\infty, n-3) + (\infty, n-4) + (\infty, n-5) + (\infty, n-6) + (\infty, n-7) + (\infty, n-8) + (\infty, n-9)] - [(3, n-2) + (3, n-3) + (4, n-4) + (4, n-5) + (5, n-6) + (5, n-7) + (2, n-8) + \{2, n-9\}] - (\infty, 1, 2)$, where $V(K_7) = Z_6 \bigcup \{\infty\}$ and $V(K_{6,8}) = Z_6 \bigcup \{n-2, n-3, n-4, n-5, n-6, n-7, n-8, n-9\}$.

(v) There exists a 6-cycle system B_5 of $G = K_7 + K_{6,8} + [(\infty, n-2) + (\infty, n-3) + (\infty, n-4) + (\infty, n-5) + (\infty, n-6) + (\infty, n-7) - [(3, n-2) + (3, n-3) + (4, n-4) + (4, n-5) + (5, n-6) + (5, n-7)] - [(\infty, 0) + (1, 2) + (0, 1) + (2, 6) + (6, \infty)]$, where $V(K_7) = Z_6 \bigcup \{\infty\}$ and $V(K_{6,8}) = Z_6 \bigcup \{6, 7, n-2, n-3, n-4, n-5, n-6, n-7\}$.

(vi) There exists a 6-cycle system B_6 of $G = K_7 + K_{6,4} + [(\infty, 8) + (\infty, 9)] - [(5, 8) + (5, 9)] - [(1, 7) + (1, 2) + (2, 3) + (3, 6)] + (6, 7)$, where $V(K_7) = Z_6 \bigcup \{\infty\}$ and $V(K_{6,4}) = Z_6 \bigcup \{6, 7, 8, 9\}$.

Proof We give the proof by direct construction of B_i, $1 \le i \le 6$ by six cases as follows.

(i) Let $B_1 = \{(0, n-2, \infty, n-3, 1, 3)$, $(5, n-4, \infty, 0, n-7, 4)$, $(2, n-5, 5, n-2, 4, n-6)$, $(1, n-4, 3, 4, 0, n-6)$, $(3, n-6, \infty, 5, 0, n-5)$, $(1, n-5, \infty, 3, 2, n-2)$, $(n-7, 3, 5, n-3, 0, 2)$, $(1, 0, n-4, 2, n-3, 4)$, $(5, 1, n-7, \infty, 4, 2)\}$.

(ii) Let $B_2 = \{(2, n-6, 4, n-2, 1, \infty)$, $(5, n-4, \infty, 0, n-7, 4)$, $(n-3, 1, n-5, 2, n-2, \infty)$, $(1, n-4, 3, 4, 0, n-6)$, $(3, n-6, \infty, 5, 0, n-5)$, $(3, \infty, n-5, 5, n-2, 0)$, $(n-7, 3, 5, n-3, 0, 2)$, $(1, 0, n-4, 2, n-3, 4)$, $(5, 1, n-7, \infty, 4, 2)\}$.

(iii) Let $B_3 = \{(0, 7, 5, 3, 1, n-2)$, $(1, 7, 3, n-4, 2, n-3)$, $(2, 7, 4, 3, n-5, 5)$, $(0, 6, \infty, 4, 5, n-3)$, $(4, n-3, \infty, n-4, 5, n-2)$, $(0, n-4, 1, \infty, n-5, 2)$, $(2, n-2, \infty, 3, 0, 4)$, $(2, 3, 6, 1, 5, \infty)$, $(4, 6, 5, 0, n-5, 1)\}$.

(iv) Let $B_4 = \{(0, n-2, \infty, 4, n-8, 2)$, $(5, n-4, \infty, 0, n-7, 4)$, $(2, n-5, 5, n-2, 4, n-6)$, $(1, n-4, 3, 4, 0, n-6)$, $(3, \infty, 5, 0, n-5)$, $(1, n-5, \infty, 3, 2, n-2)$, $(n-7, \infty, n-3, 5, n-9, 1)$, $(1, 0, n-4, 2, n-3, 4)$, $(0, n-3, 1, 3, 5, n-8)$, $(1, n-8, 3, n-9, 2, 5)$, $(0, n-9, 4, 2, n-7, 3)$.

(v) Let $B_5 = \{(0, 7, 5, 3, 1, n-2)$, $(1, 7, 3, n-4, 2, n-3)$, $(2, 7, 4, 3, n-5, 5)$, $(0, 6, \infty, 4, 5, n-3)$, $(4, n-3, \infty, n-4, 5, n-2)$, $(0, n-4, 1, \infty, n-5, 2)$, $(2, n-2, \infty, 3, 0, 4)$, $(0, n-5, 1, n-6, \infty, 5)$,

$(2, \infty, n-7, 0, n-6, 3)$, $(2, n-6, 4, 6, 1, n-7)$, $(4, 1, 5, 6, 3, n-7)\}$.

(vi) Let $B_6 = \{(2,6,5,4,1,0), (4,6,7,2,5,0), (1,5,7,3,4,8), (0,7,4,9,\infty,3), (8,3,9,2,4,\infty), (1,\infty,2,8,0,9)\}$. $\square$

Set $A = \{(2,3,1,4,8,0), (7,5,8,6,11,2), (3,8,11,9,0,6), (9,8,1,5,10,4), (1,\infty,7,11,0,10), (9,2,6,1,7,3), (11,\infty,10,2,5,3), (0,5,9,6,10,3), (4,\infty,9,10,8,2), (5,6,4,7,10,11), (4,0,7,9,1,11)\}$. A will be used in Lemmas 3.4(ii) and 3.4(iii).

Lemma 3.4 *(i) There exists a* 6*-cycle system* B_1 *of* $G = K_{13} + K_{12,2} + [(\infty,12)+(\infty,13)] - [(1,12)+(1,13)] - 3C_4$*, where* $V(K_{13}) = Z_{12}\bigcup\{\infty\}$ *and* $V(K_{12,2}) = (Z_{12}, \{12,13\})$*.*

(ii) There exists a 6*-cycle system* B_2 *of* $G = K_{13} + K_{12,4} + [(\infty,12)+(\infty,13)+(\infty,14)+(\infty,15)] - [(6,12)+(6,13)+(0,14)+(0,15)] - 3C_4$*, where* $V(K_{13}) = Z_{12}\bigcup\{\infty\}$ *and* $V(K_{12,4}) = Z_{12}\bigcup\{12,13,14,15\}$*.*

(iii) There exists a 6*-cycle system* B_3 *of* $G = K_{13} + K_{12,6} + [(\infty,12)+(\infty,13)+(\infty,14)+(\infty,15)+(\infty,16)+(\infty,17)] - [(0,12)+(0,13)+(5,14)+(5,15)+(8,16)+(8,17)] - 3C_4$*, where* $V(K_{13}) = Z_{12}\bigcup\{\infty\}$ *and* $V(K_{12,6}) = Z_{12}\bigcup\{12,13,14,15,16,17\}$*.*

Proof We give the proof by direct construction of B_i, $1 \le i \le 3$ by three cases as follows.

(i) $B_1 = \{(2,3,1,4,8,0), (7,5,8,6,11,2), (3,8,11,9,0,6), (9,8,1,5,10,4), (1,\infty,7,11,0,10), (11,\infty,10,2,5,3), (4,\infty,9,10,8,2), (12,8,13,5,0,7), (12,0,13,4,7,9), (4,0,3,13,10,12), (11,4,6,5,9,1), (11,5,12,6,9,13), (2,9,3,10,7,13), (1,7,3,12,2,6), (10,6,13,1,12,11)\}$.

(ii) $B_2 = A\bigcup\{(12,0,13,2,14,1), (14,0,15,2,12,3), (13,1,15,5,12,4), (13,3,15,4,14,5), (12,6,13,8,14,7), (14,6,15,8,12,9), (13,7,15,11,12,10), (13,9,15,10,14,11)\}$.

(iii) $B_3 = A\bigcup\{(12,0,13,1,15,4), (14,0,15,2,16,4), (16,0,17,2,12,3), (13,2,14,5,15,3), (14,3,17,9,12,6), (12,1,16,7,13,5), (13,4,17,7,15,6), (16,8,17,11,12,10), (7,12,8,15,10,14), (13,8,14,11,15,9), (10,13,11,16,5,17), (6,16,9,14,1,17)\}$. $\square$

Set $D = \{(2,5,11,4,14,\infty), (0,3,4,7,1,5), (13,6,16,12,17,\infty), (9,n-3,13,14,11,3), (10,12,9,14,16,\infty), (11,13,16,2,15,\infty), (1,4,9,16,8,\infty), (0,2,6,12,1,9), (1,3,7,13,2,10), (3,5,9,15,4,12), (4,6,10,16,5,13), (5,7,11,17,6,14), (6,8,12,0,7,15), (0,4,10,3,8,11), (0,17,1,6,9,8), (2,4,n-3,3,n-2,7)\}$. D will be used in Lemmas 3.5(i)–3.5(iii).

Lemma 3.5 *(i) There exists a* 6*-cycle system* B_1 *of* $G = K_{19} + K_{18,2} + [(\infty,n-2)+(\infty,n-3)] - 3C_5 - [(17,n-2)+(17,n-3)]$*, where* $3C_5 = (\infty,0,1,2,3)+(4,5,6,7,8)+(9,10,11,12,13)$*,* $V(K_{19}) = Z_{18}\bigcup\{\infty\}$ *and* $V(K_{18,2}) = (Z_{18}, \{n-2, n-3\})$*.*

(ii) There exists a 6*-cycle system* B_2 *of* $G = K_{19} + K_{18,4} + [(\infty,n-2)+(\infty,n-3)+(\infty,n-4)+(\infty,n-5)] - [(17,n-2)+(17,n-3)+(8,n-4)+(8,n-5)] - 3C_5$*, where* $3C_5 = (\infty,0,1,2,3)+(\infty,4,5,6,7)+(9,10,11,12,13), V(K_{19}) = Z_{18}\bigcup\{\infty\}$ *and* $V(K_{18,4}) = (Z_{18}, \{n-2,n-3,n-4,n-5\})$*.*

(iii) There exists a 6*-cycle system* B_3 *of* $G = K_{19} + K_{18,6} + [(\infty,n-2)+(\infty,n-3)+(\infty,n-4)+(\infty,n-5)+(\infty,n-6)+(\infty,n-7)] - [(17,n-2)+(17,n-3)+(8,n-4)+(8,n-5)+(13,n-6)+(13,n-7)] - 3C_5$*, where* $V(K_{18,6}) = Z_{18}\bigcup\{n-2,n-3,n-4,n-5,n-6,n-7\}$*,* $V(K_{19}) = Z_{18}\bigcup\{\infty\}$ *and* $3C_5 = (\infty,0,1,2,3)+(\infty,4,5,6,7)+(\infty,9,10,11,12)$*.*

Proof We give the proof by direct construction of B_i, $1 \le i \le 3$ by three cases as follows.

(i) $B_1 = D \bigcup \{(12,14,17,3,6,\infty),\ (4,\infty,n-3,2,11,n-2),\ (7,\infty,n-2,0,10,n-3),\ (16,4,17,2,8,n-3),\ (9,2,14,0,15,n-2),\ (12,2,n-2,6,n-3,5),\ (8,5,17,13,1,15),\ (10,5,n-2,14,8,17),\ (8,10,14,15,12,n-2),\ (13,10,n-2,1,11,15),\ (13,n-2,16,7,14,3),\ (1,8,13,0,n-3,14),\ (6,0,16,1,n-3,11),\ (12,7,17,16,15,n-3),\ (3,16,11,9,17,15),\ (9,7,10,15,5,\infty)\}$.

(ii) $B_2 = D \bigcup \{(9,7,10,15,5,\infty),\ (n-3,\infty,n-4,1,13,0),\ (n-2,\infty,n-5,1,14,0),\ (6,0,n-5,2,9,11),\ (10,0,16,n-5,4,n-2),\ (15,0,n-4,2,n-3,14),\ (11,1,n-3,7,12,n-2),\ (8,1,n-2,5,n-4,4),\ (15,1,16,7,n-4,11),\ (2,11,n-5,10,13,n-2),\ (n-3,6,n-4,17,10,5),\ (14,3,n-4,15,17,2),\ (n-5,13,n-4,16,17,9),\ (16,11,n-3,15,13,3),\ (12,2,8,7,17,5),\ (8,5,n-5,7,14,10),\ (15,3,n-5,6,n-2,8),\ (16,4,17,13,8,n-3),\ (8,17,n-5,15,n-2,14),\ (9,n-2,16,15,12,n-4),\ (n-3,12,n-5,14,n-4,10)\}$.

(iii) $B_3 = D \bigcup \{(n-3,\infty,n-4,1,13,0),\ (n-2,\infty,n-5,1,14,0),\ (6,0,n-5,2,9,11),\ (10,0,16,n-5,4,n-2),\ (15,0,n-4,2,n-3,14),\ (11,1,n-3,7,12,n-2),\ (5,\infty,n-7,0,n-6,10),\ (6,\infty,n-6,14,3,n-4),\ (8,1,n-6,12,14,7),\ (15,1,n-7,3,n-5,13),\ (16,1,n-2,8,13,n-4),\ (17,13,n-2,15,10,14),\ (8,2,n-6,3,17,10),\ (8,4,n-7,5,15,17),\ (14,2,17,n-6,15,n-7),\ (8,n-7,12,5,n-5,14),\ (13,3,16,17,n-5,9),\ (17,4,n-6,16,11,n-7),\ (n-3,8,n-6,11,15,10),\ (6,n-7,10,n-4,7,n-5),\ (n-3,6,n-6,7,17,5),\ (6,3,15,8,5,n-2),\ (11,2,n-7,7,9,n-4),\ (n-3,11,n-5,10,7,16),\ (n-4,5,n-6,9,n-2,14),\ (15,n-3,10,13,12,n-5),\ (9,17,n-4,15,16,n-7),\ (2,n-2,16,4,n-4,12)\}$. $\square$

4. The main result

Now we are in the position to prove Theorem 4.1.

We will assume $|V(L)| \ge n-4$. Since if $|V(L)| \le n-5$, we can get a C_6 such that $V(C_6) \cap V(L) \subseteq \{\infty\}$ and $V(C_6) \subseteq V(K_n)$. Let $L^* = C_6 + L$. A 6-cycle system of $K_n - L$ is equal to a 6-cycle system of $K_n - L^*$ and C_6. If $|V(L^*)| \le n-5$, repeat the process until $|V(L^*)| \ge n-4$.

Theorem 4.1 *Let L be a nearly 2-regular graph in the complete graph K_n and $|V(L)| = n-4$. There exists a 6-cycle system of $K_n - L$ for positive integer n satisfying the following three conditions: (i) n is odd, (ii) 6 divides $|E(K_n - L)|$ and (iii) $n \ge 7$.*

Proof We use induction method to prove the result. When $n \le 13$, the proof can be seen in Lemma 2.1-2.4. Suppose that for any odd $z < n$ and any nearly 2-regular leave $L_1 \subseteq L$ of K_z for which 6 divides $|E(K_z - L_1)|$, there exists a 6-cycle system of $K_z - L_1$. In the following, we shall show there exists a 6-cycle system of $K_n - L$ by five cases.

Case 1 L contains at least one C_3.

Without lose of generality, suppose $C_3 = (1,2,3)$, or $C_3 = (\infty,1,2)$. Let $L = L_1 + C_3$. We can get $K_n - L = (K_{n-6} - L_1) + K_{6,n-7} + (K_7 - C_3)$ where $V(K_{n-6}) = (Z_{n-1} \backslash Z_6) \bigcup \{\infty\}$, $V(K_{6,n-7}) = Z_6 \bigcup (Z_{n-1} \backslash Z_6)$ and $V(K_7) = Z_6 \bigcup \{\infty\}$. Since 6 divides $|E(K_n - L)|$, $|E(K_{6,n-7})|$ and $|E(K_7 - C_3)|$, then 6 divides $|E(K_{n-6} - L_1)|$. By induction or Theorem 2, B_1 is a 6-cycle system of $K_{n-6} - L_1$. B_2 is a 6-cycle system of $K_{6,n-7}$ by Theorem 1. B_3 is a 6-cycle system of $K_7 - C_3$ by Theorem 2, where $V(C_3) \subseteq V(K_7)$. Then $B_1 \bigcup B_2 \bigcup B_3$ is a 6-cycle system of $K_n - E(L)$.

Case 2 L contains at least one C_m, m $(m \geq 6)$.

Let $C_m = (0, 1, 2, 3, 4, \ldots, m-2, \infty) \in L$. Then $C_m = C_{m-3} + [(0,1) + (1,2) + (2,3) + (3,4)] - (0,4)$ where $C_{m-3} = (0, 4, 5, 6, \ldots, m-2, \infty)$. Let $L = (L - C_m) + C_{m-3} + [(0,1) + (1,2) + (2,3) + (3,4)] - (0,4) = L_1 + [(0,1) + (1,2) + (2,3) + (3,4)] - (0,4)$ where $L_1 = (L - C_m) + C_{m-3}$. $K_n - L = (K_{n-6} - L_1) + K_{6,n-11} + [K_7 + K_{6,4} - (0,1) - (1,2) - (2,3) - (3,4) + (0,4)]$, where $V(K_{n-6}) = (Z_{n-4} \backslash Z_4) \bigcup \{0, \infty\}$, $V(K_{6,n-11}) = A \bigcup (Z_{n-4} \backslash Z_7)$, $V(K_7) = A \bigcup \{\infty\}$, $K_{6,4} = A \bigcup \{0, 4, 5, 6\}$ and $A = \{1, 2, 3, n-2, n-3, n-4\}$. Since 6 divides $|E(K_n - L)|$, $|E(K_{6,n-11})|$ and $|E[K_7 + K_{6,4} - (0,1) - (1,2) - (2,3) - (3,4) + (0,4)]|$, then 6 divides $|E(K_{n-6} - L_1)|$. By induction and Theorem 2, B_1 is a 6-cycle system of $K_{n-6} - L_1$. B_2 is a 6-cycle system of $K_{6,n-11}$ by Theorem 1. B_3 is a 6-cycle system of $K_7 + K_{6,4} - (1,2) - (2,3) - (3,4) - (0,1) + (0,4)$ by Lemma 3.2(iii). Then $B_1 \bigcup B_2 \bigcup B_3$ is a 6-cycle system of $K_n - L$.

Case 3 L contains only cycles of length 4.

When $n < 21$, since 6 divides $|E(K_n - L)|$, there exist the following cases:

(1) $n = 13$, L contains three 4-cycles. This case has been constructed in Lemma 2.4(iii) and 2.4(iv).

(2) $n = 17$, L contains one 4-cycle, or four 4-cycles. When L contains one 4-cycle, we can obtain this 6-cycle system by Theorem 2. When L contains four 4-cycles, the proof can be seen in Lemma 2.6(ii). So in the following we assume $n \geq 21$. As long as L meets the condition of the induction, we can construct a 6-cycle system. So we only consider $3C_4 = (\infty, 0, 1, 2) + (\infty, 3, 4, 5) + (\infty, 6, 7, 8)$. Let $L = L_1 + 3C_4$. So $K_n - E(L) = (K_{n-12} - L_1) + K_{12,n-13} + (K_{13} - 3C_4)$, where $V(K_{n-12}) = (Z_{n-1} \backslash Z_{12}) \bigcup \{\infty\}$, $V(K_{12,n-13}) = Z_{12} \bigcup Z_{n-1} \backslash Z_{12}$ and $V(K_{13}) = Z_{12} \bigcup \{\infty\}$. Since 6 divides $|E(K_n - L)|$, $|E(K_{12,n-13})|$ and $|E(K_{13} - 3C_4)|$, then 6 divides $|E(K_{n-12} - L_1)|$. By induction or Theorem 2, B_1 is a 6-cycle system of $K_{n-12} - L_1$. B_2 is a 6-cycle system of $K_{12,n-13}$ by Theorem 1. B_3 is a 6-cycle system of $K_{13} - 3C_4$ by Lemma 2.4(iii) and 2.4(iv), where $V(3C_4) \subseteq V(K_{13})$. Then $B_1 \bigcup B_2 \bigcup B_3$ is a 6-cycle system of $K_n - E(L)$.

Case 4 L contains only cycles of length 5.

When $n < 25$, since 6 divides $|E(K_n - E(L))|$, there exist the following cases:

(1) $n = 15$, L contains three 5-cycles. This case has been constructed in Lemma 2.5(i);

(2) $n = 17$, L contains two 5-cycles. This case has been constructed in Lemma 2.6(i);

(3) $n = 19$, L contains three 5-cycles. This case has been constructed in Lemma 2.7(i);

(4) $n = 23$, L contains five 5-cycles. This case has been constructed in Lemma 2.9(ii).

So in the following we assume $n \geq 25$. Let $L = L_1 + 3C_5$. So $K_n - L = (K_{n-18} - L_1) + K_{18,n-19} + (K_{19} - 3C_5)$, where $V(K_{n-18}) = (Z_{n-1} \backslash Z_{18}) \bigcup \{\infty\}$, $V(K_{18,n-19}) = Z_{18} \bigcup (Z_{n-1} \backslash Z_{18})$ and $V(K_{19}) = Z_{18} \bigcup \{\infty\}$. Since 6 divides $|E(K_n - L)|$, $|E(K_{18,n-19})|$ and $|E(K_{19} - 5C_5)|$, then

6 divides $|E(K_{n-12} - L_1)|$. By induction or Theorem 2, B_1 is a 6-cycle system of $K_{n-18} - L_1$. B_2 is a 6-cycle system of $K(18, n-19)$ by Theorem 1. B_3 is a 6-cycle system of $K_{19} - 3C_5$ by Lemma 2.7(i), where $V(3C_5) \subseteq V(K_{19})$. Then $B_1 \bigcup B_2 \bigcup B_3$ is a 6-cycle system of $K_n - E(L)$.

Case 5 L contains only cycles of length 4 and 5.

Case 5.1 L contains one 5-cycle.

Since 6 divides $|E(K_n - L)|$, we can get the following two cases:

(1) When $n = 12k+3$ or $12k+7$, there are one 5-cycle and $3m+1$ 4-cycles ($k \geq 1, 0 \leq m \leq k$. $m, k \in N$).

We give the proof by induction on k. When $k = 1$, and $n = 15$ or 19, we give the proof in Lemma 2.5(iii) and 2.7(ii). Suppose that for $k = n_0$ and any nearly 2-regular leave L_1, there exists a 6-cycle system of $K_{12n_0+7} - L_1$. When $k = n_0 + 1$, let $L_1 = L - 3C_4$. We can get $K_{12k+7} - L = K_{12n_0+19} - L = (K_{12n_0+7} - L_1) + K_{12,12n_0+6} + (K_{13} - 3C_4)$, where $V(L_1) \subseteq V(K_{12n_0+7}) = Z_{12n_0+6} \bigcup \{\infty\}$, $V(K_{12,12n_0+6}) = A \bigcup Z_{12n_0+6}$, $V(3C_4) \subseteq V(K_{13}) = A \bigcup \{\infty\}$ and $A = \{12n_0 + i | 7 \leq i \leq 18.\ \ i \in N\}$. Since 6 divides $|E(K_n - L)|$, $|E(K_{12,12n_0+6})|$ and $|E(K_{13} - 3C_4)|$, then 6 divides $|E(K_{12n_0+19} - L_1)|$. By induction or Theorem 2, B_1 is a 6-cycle system of $K_{12n_0+7} - L_1$, where $L_1 = L - 3C_4$. B_1 is a 6-cycle system of $K_{12,12n_0+6}$ by Theorem 1. B_2 is a 6-cycle system of $K_{13} - 3C_4$ by Lemma 2.4(xiii) and 2.4(iv). Then $B_1 \bigcup B_2 \bigcup B_3$ is a 6-cycle system of $K_{12n_0+19} - E(L)$.

The proof of $n = 12k + 3$ is similar to that of the case $n = 12k + 7$. Since $n = 15$, there exists a 6-cycle system by Lemma 2.5(iii).

(2) When $n = 12k + 11$, there are one 5-cycle and $3m + 2$ 4-cycles ($k \geq 0, 0 \leq m \leq k$. $m, k \in N$). The proof of this case is similar to that of the case 5.1(1). Since $n = 11$, there exists a 6-cycle system by Lemma 2.3(iv).

Case 5.2 L contains two 5-cycles.

Since 6 divides $|E(K_n - L)|$, we can get the following two cases:

(1) When $n = 12k+5$, there are two 5-cycles and $3m$ 4-cycles ($k \geq 1, 0 \leq m \leq k$. $m, k \in N$). The proof is similar to that of the case 5.1(1). Since $n = 17$, there exists a 6-cycle system by Lemma 2.6(i).

(2) When $n = 12k+1$ or $12k+9$, there are two 5-cycles and $3m+2$ 4-cycles ($k \geq 1, 0 \leq m \leq k$. $m, k \in N$). The proof is similar to that of the case 5.1(1). Since $n = 25$ or 21, there exists a 6-cycle system by Lemma 2.10(i) or 2.8(i).

Case 5.3 L contains three 5-cycles.

Since 6 divides $|E(K_n - L)|$, we can get the following two cases:

(1) When $n = 12k+3$ or $12k+7$, there are three 5-cycles and $3m$ 4-cycles ($k \geq 1, 0 \leq m \leq k$. $m, k \in N$). The proof is similar to that of the case 5.1(1). Since $n = 15$ or 19, there exists a 6-cycle system by Lemma 2.5(i) or 2.7(i).

(2) When $n = 12k + 11$, there are three 5-cycle and $3m + 1$ 4-cycles ($k \geq 1, 0 \leq m \leq k$. $m, k \in N$). The proof is similar to that of the case 5.1(1). Since $n = 23$, there exists a 6-cycle

system by Lemma 2.9(i).

Case 5.4 L contains at least four 5-cycles.

When $n < 31$, since 6 divides $|E(K_n - L)|$, we can get the following three cases:

(1) $n = 21$, L contains four 5-cycles and one 4-cycles. The 6-cycle system of this case has been constructed in Lemma 2.8(ii).

(2) $n = 25$, L contains four 5-cycles and one 4-cycles. The 6-cycle system of this case has been constructed in Lemma 2.10(ii).

(3) $n = 29$, L contains four 5-cycles and two 4-cycles. The 6-cycle system of this case has been constructed in Lemma 2.11. So in the following we assume $n \geq 31$.

Let $L = L_1 + 4C_5 + C_4$. So $K_n - L = (K_{n-24} - L_1) + K_{24,n-25} + (K_{25} - 4C_5 - C_4)$, where $V(L_1) \subseteq V(K_{n-24}) = (Z_{n-1} \backslash Z_{24}) \bigcup \{\infty\}, V(K_{24,n-25}) = Z_{24} \bigcup (Z_{n-1} \backslash Z_{24})$ and $V(4C_5 \bigcup C_4) \subseteq V(K_{25}) = Z_{24} \bigcup \{\infty\}$. Since 6 divides $|E(K_n - L)|$, $|E(K_{24,n-25})|$ and $|E(K_{25} - 4C_5 - C_4)|$, then 6 divides $|E(K_{n-24} - L_1)|$. By induction B_1 is a 6-cycle system of $K_{n-24} - L_1$. B_2 is a 6-cycle system of $K_{24,n-25}$ by Theorem 1 where $V(K_{24,n-25}) = \{n - i, 2 \leq i \leq 25, i \in N.\} \bigcup Z_{n-25}$. B_3 is a 6-cycle system of $K_{25} - 4C_5 - C_4$ by Lemma 2.10(ii). Then $B_1 \bigcup B_2 \bigcup B_3$ is a 6-cycle system of $K_n - E(L)$. $\square$

Our method will result in complicated classification for the cases of $n - 3 \leq |V(L)| \leq n$. To solve the remaining cases, a new method different from ours is needed.

Acknowledgements We appreciate the anonymous referees for their valuable comments which help to improve the paper.

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