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On the Hermitian Positive Definite Solutions of the Nonlinear Matrix Equation $X^s - A^*X^{-t}A = Q$ with Perturbation Estimates

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Abstract In this paper, the Hermitian positive definite solutions of the nonlinear matrix equation $X^s - A^*X^{-t}A = Q$ are studied, where Q is a Hermitian positive definite matrix, s and t are positive integers. The existence of a Hermitian positive definite solution is proved. A sufficient condition for the equation to have a unique Hermitian positive definite solution is given. Some estimates of the Hermitian positive definite solutions are obtained. Moreover, two perturbation bounds for the Hermitian positive definite solutions are derived and the results are illustrated by some numerical examples.

Keywords matrix equation; Hermitian positive definite solution; property; existence; perturbation bound.

MR(2010) Subject Classification 15A24; 65F35

1. Introduction

We consider the matrix equation:

$$X^s - A^*X^{-t}A = Q, \tag{1.1}$$

where Q is an $n \times n$ Hermitian positive definite matrix, s and t are positive integers. Here A^* stands for the conjugate transpose of the matrix A.

The nonlinear matrix equation (1.1) arises in a wide variety of application and research areas, including automatic control, ladder networks, dynamic programming, stochastic filtering and statistics [4, 11, 12]. The Hermitian positive definite solutions of Eq. (1.1) have been studied in some special cases [3–9, 11, 12, 14, 15]. The case that $s = t = 1$, the one arising specifically in the analysis of a stationary Gaussian reciprocal processes over a finite interval [13], has been investigated systematically by many authors, such as Ferrante and Levy [4], Fital and Guo [5], Guo and Lancaster [6], Hasanov and Ivanov [9], Ivanov et al. [12], and Meini [15], and some basic properties, efficient iterative algorithms, and perturbation estimates of the Hermitian positive definite solutions have been obtained. Ivanov et al. [11] studed the equation $X - A^*X^{-2}A = I$. El-Sayed [3] proposed two iterative methods for obtaining the Hermitian positive definite solutions

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of the equation $X - A^*X^{-n}A = Q$ and Hasanov and Ivanov [7] derived some perturbation bounds for these solutions. Liu and Gao [14] studied the equation $X^s - A^TX^{-t}A = I$ and presented the properties and the sensitivity analysis of the symmetric positive definite solutions of this equation. Note that only in the case of $s = t$ can Eq. (1.1) be reduced to the equation $X^s - A^TX^{-t}A = I$ (see [4, Proposition 3.1]). Hence it is necessary to consider Eq. (1.1) which has the more general form.

In this paper, we mainly discuss the basic properties and the sensitivity analysis of the Hermitian positive definite solutions of Eq. (1.1). The rest of the paper is organized as follows. In Section 2, we show that Eq. (1.1) always has a Hermitian positive definite solution and give a sufficient condition for the equation to have a unique solution. Some new estimates of the solutions are also obtained. In Section 3, two perturbation bounds for a Hermitian positive definite solution and the unique solution of Eq. (1.1) are derived respectively. Finally, we give a numerical example in Section 4 to illustrate our theoretical results.

The following notations are used throughout the paper. The symbol $\mathscr{C}^{n\times n}$ and $\mathscr{P}(n)$ denote the set of $n \times n$ complex matrices and positive semi-definite matrices, respectively. A^T denotes the transpose of a matrix A. $\lambda(A)$ stands for an eigenvalue of a square matrix A. Let $\lambda_1(H)$ and $\lambda_n(H)$ denote the maximal and the minimal eigenvalue of an $n \times n$ Hermitian matrix H, respectively. Let $\|\cdot\|$ be the spectral norm and $\|\cdot\|_F$ be the Frobenius norm. The symbol $A \otimes B$ stands for the Kronecker product of matrices A and B. For $A = (a_1, a_2, \dots, a_n) \in \mathscr{C}^{m\times n}$, the symbol $\mathrm{vec}(A)$ stands for a vector defined by $\mathrm{vec}(A) = (a_1^T, a_2^T, \dots, a_n^T)^T$. The notation $B \geq 0$ ($B > 0$) means that B is a Hermitian positive semi-definite (definite) matrix. For two Hermitian matrices B and C, the notation $B \geq C$ ($B > C$) indicates that $B - C \geq 0$ ($B - C > 0$), and $X \in [B, C]$ implies that $B \leq X \leq C$.

For convenience of discussion, in the sequel, a solution always means a Hermitian positive definite solution unless otherwise noted.

2. Properties of the solutions

In this section, we discuss the properties of the solutions of Eq. (1.1), including existence, uniqueness and estimates of the solutions.

Note that Ran and Reurings [17] have considered the following more general nonlinear matrix equation:

$$X + A^*\mathscr{F}(X)A = Q \tag{2.1}$$

where $\mathscr{F}$ is a map from $\mathscr{P}(n)$ into $\mathscr{C}^{n\times n}$, and obtained sufficient conditions for the existence of a solution of the equation as follows.

Lemma 2.1 ([17, Lemma 2.2]) *Let $\mathscr{F} : \mathscr{P}(n) \to -\mathscr{P}(n)$ be continuous on $\{X \in \mathscr{P}(n) | X \geq Q\}$.*

(i) If Eq. (2.1) has a positive semi-definite solution $\bar{X}$, then $\bar{X} \geq Q$.

(ii) If there exist a $B \geq Q$ such that

$$Q - B \leq A^*\mathscr{F}(X)A \leq 0 \tag{2.2}$$

for all $X \in [Q, B]$, then Eq.(2.1) has a solution in $[Q, B]$. Moreover, if (2.2) is satisfied for every $X \geq Q$, Then all solutions of Eq. (2.1) are in $[Q, B]$.

Lemma 2.2 ([16]) *If $B > C > 0$ (or $B \geq C > 0$), then $B^\gamma > C^\gamma > 0$ (or $B^\gamma \geq C^\gamma > 0$) for all $\gamma \in (0, 1]$, and $0 < B^\gamma < C^\gamma$ (or $0 < B^\gamma \leq C^\gamma$) for all $\gamma \in [-1, 0)$.*

Theorem 2.1 *Eq. (1.1) has a solution in $[Q^{\frac{1}{s}}, (Q + \lambda_n(Q)^{-\frac{t}{s}} A^*A)^{\frac{1}{s}}]$, and all the solutions are in $[Q^{\frac{1}{s}}, (Q + \lambda_n(Q)^{-\frac{t}{s}} A^*A)^{\frac{1}{s}}]$.*

Proof Let $Y = X^s$. Then Eq. (1.1) is equivalent to

$$Y - A^*Y^{-\frac{t}{s}}A = Q. \tag{2.3}$$

Hence the existence of a solution of Eq. (1.1) is equivalent to that of Eq. (2.3). Let $B = Q + \lambda_n(Q)^{-\frac{t}{s}} A^*A$ and $\mathscr{F}(Y) = -Y^{-\frac{t}{s}}$. Then for all $Y \in [Q, B]$, we have

$$Q - B = -\lambda_n(Q)^{-\frac{t}{s}} A^*A \leq A^*\mathscr{F}(Y)A \leq 0, \tag{2.4}$$

and (2.4) holds for all $Y \geq Q$. According to Lemma 2.1, we know that Eq. (2.3) has a solution in $[Q, Q + \lambda_n(Q)^{-\frac{t}{s}} A^*A]$, and all the solutions of Eq. (2.3) are in $[Q, Q + \lambda_n(Q)^{-\frac{t}{s}} A^*A]$. Then it follows from Lemma 2.2 that Eq. (1.1) has a solution in $[Q^{\frac{1}{s}}, (Q + \lambda_n(Q)^{-\frac{t}{s}} A^*A)^{\frac{1}{s}}]$, and all the solutions are in $[Q^{\frac{1}{s}}, (Q + \lambda_n(Q)^{-\frac{t}{s}} A^*A)^{\frac{1}{s}}]$. □

Theorem 2.2 *If $\|A\|^2 < \frac{s}{t}\lambda_n^{\frac{s+t}{s}}(Q)$, then Eq. (1.1) has a unique solution.*

Proof Suppose that Eq. (1.1) has two different solutions $X, Y \in [Q^{\frac{1}{s}}, (Q + \lambda_n(Q)^{-\frac{t}{s}} A^*A)^{\frac{1}{s}}]$. Since

$$X^s - Y^s = \sum_{k=0}^{s-1} X^k(X - Y)Y^{s-1-k}$$

and

$$Y^{-t} - X^{-t} = X^{-t}(X^t - Y^t)Y^{-t} = \sum_{k=1}^{t} X^{-k}(X - Y)Y^{k-t-1}, \tag{2.5}$$

we have

$$\|X^s - Y^s\|_F = \|(\sum_{k=0}^{s-1} Y^{s-1-k} \otimes X^k)\mathrm{vec}(X - Y)\| \geq s\lambda_n^{\frac{s-1}{s}}(Q)\|X - Y\|_F \tag{2.6}$$

and

$$\|X^{-t} - Y^{-t}\|_F = \|(\sum_{k=1}^{t} Y^{k-t-1} \otimes X^{-k})\mathrm{vec}(X - Y)\| \leq \frac{t}{\lambda_n^{\frac{t+1}{s}}(Q)}\|X - Y\|_F. \tag{2.7}$$

If $\|A\|^2 < \frac{s}{t}\lambda_n^{\frac{s+t}{s}}(Q)$, then by (2.6) and (2.7), we get

$$\begin{aligned}\|X - Y\|_F &\leq \frac{1}{s\lambda_n^{\frac{s-1}{s}}(Q)}\|X^s - Y^s\|_F = \frac{1}{s\lambda_n^{\frac{s-1}{s}}(Q)}\|A^*(X^{-t} - Y^{-t})A\|_F \\ &\leq \frac{\|A\|^2}{\frac{s}{t}\lambda_n^{\frac{s+t}{s}}(Q)}\|X - Y\|_F < \|X - Y\|_F.\end{aligned}$$

This is a contradiction. Hence Eq. (1.1) has a unique solution. □

Corollary 2.1 ([14, Theorem 2.4]) *If $\|A\|^2 < \frac{s}{t}$, then the equation $X^s - A^T X^{-t} A = I$ has a unique solution.*

The following theorem is a direct generalization of Liu and Gao's Theorem 2.5 in [14]. Its proof is similar, and is omitted here.

Theorem 2.3 *Every solution X of Eq. (1.1) satisfies*

$$\alpha I \le X \le \beta I, \tag{2.8}$$

where α and β are solutions of the following system of equations:

$$\begin{cases} \alpha^s = \lambda_n(Q) + \frac{1}{\beta^t}\lambda_n(A^*A), \\ \beta^s = \lambda_1(Q) + \frac{1}{\alpha^t}\lambda_1(A^*A). \end{cases}$$

Next we shall derive a new estimate which is sharper than the estimate (2.8).

Lemma 2.3 *Let $f(x) = x^t(x^s - \theta), \theta > 0, x \ge 0$. Then*

(i) f is decreasing on $[0, (\frac{t}{s+t}\theta)^{\frac{1}{s}}]$ and increasing on $[(\frac{t}{s+t}\theta)^{\frac{1}{s}}, +\infty)$;

(ii) $f_{\min} = f((\frac{t}{s+t}\theta)^{\frac{1}{s}}) = -\frac{s}{s+t}(\frac{t}{s+t})^{\frac{t}{s}}\theta^{\frac{t}{s}+1}$.

Consider two polynomial equations as follows:

$$x^{s+t} - \lambda_n(Q)x^t - \lambda_n(A^*A) = 0, \tag{2.9}$$

$$x^{s+t} - \lambda_1(Q)x^t - \lambda_1(A^*A) = 0. \tag{2.10}$$

Then from Lemma 2.3 it follows that both Eqs. (2.9) and (2.10) have unique positive solutions, denoted by $\tilde{\alpha}$ and $\tilde{\beta}$, respectively. Moreover, one can easily see that $\tilde{\alpha} \le \tilde{\beta}$.

Theorem 2.4 *Every solution X of Eq. (1.1) satisfies*

$$\tilde{\alpha} I \le X \le \tilde{\beta} I, \tag{2.11}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are the unique positive solutions of Eqs. (2.9) and (2.10), respectively.

Proof By Theorem 3.3.16 (d) of Horn and Johnson [10], we have

$$\lambda(A^*X^{-t}A) \le \lambda(X^{-t})\lambda_1(A^*A). \tag{2.12}$$

Moreover, if A is nonsingular, then we have

$$\begin{aligned} \lambda(X^{-t}) &= \lambda((A^{-1})^*A^*X^{-t}AA^{-1}) \le \lambda(A^*X^{-t}A)\lambda_1((A^*A)^{-1}) \\ &= \lambda(A^*X^{-t}A)(\lambda_n(A^*A))^{-1}, \end{aligned}$$

which yields

$$\lambda(A^*X^{-t}A) \ge \lambda(X^{-t})\lambda_n(A^*A). \tag{2.13}$$

If A is singular, the inequality (2.13) still holds since $\lambda_n(A^*A) = 0$.

From $\lambda(X)^s = \lambda(X^s) = \lambda(Q + A^*X^{-t}A)$, we get

$$\lambda_n(Q) + \lambda(A^*X^{-t}A) \le \lambda(X)^s \le \lambda_1(Q) + \lambda(A^*X^{-t}A). \tag{2.14}$$

Then by (2.12)–(2.14), we obtain

$$\lambda_n(Q) + (\lambda(X))^{-t}\lambda_n(A^*A) \le \lambda(X)^s \le \lambda_1(Q) + (\lambda(X))^{-t}\lambda_1(A^*A),$$

which implies

$$(\lambda(X))^{s+t} - \lambda_1(Q)(\lambda(X))^t \le \lambda_1(A^*A)$$

and

$$(\lambda(X))^{s+t} - \lambda_n(Q)(\lambda(X))^t \ge \lambda_n(A^*A).$$

Then according to Lemma 2.3, we conclude that $\tilde{\alpha} \le \lambda(X) \le \tilde{\beta}$, from which it follows that $\tilde{\alpha}I \le X \le \tilde{\beta}I$. □

Remark 2.1 From Theorem 2.3, we have $\alpha^s = \lambda_n(Q) + \frac{1}{\beta^t}\lambda_n(A^*A) \le \lambda_n(Q) + \frac{1}{\alpha^t}\lambda_n(A^*A)$, i.e., $\alpha^{s+t} - \lambda_n(Q)\alpha^t \le \lambda_n(A^*A)$. Then according to Lemma 2.3, we know that $\alpha \le \tilde{\alpha}$. Similarly, we get $\beta \ge \tilde{\beta}$. Therefore, the estimate (2.11) is sharper than the estimate (2.8).

Theorem 2.5 *Suppose X is a solution of Eq. (1.1). Then for any eigenvalue of A, the following inequality holds.*

$$(\lambda_n(X))^t[(\lambda_n(X))^s - \lambda_1(Q)] \le |\lambda(A)|^2 \le (\lambda_1(X))^t[(\lambda_1(X))^s - \lambda_n(Q)]. \tag{2.15}$$

Proof Let v be an eigenvector corresponding to $\lambda(A)$ with $\|v\|_2 = 1$. Then we have

$$v^T X^s v - v^T A^* X^{-t} A v = v^T Q v,$$

which yields

$$|\lambda(A)|^2 v^T X^{-t} v = v^T (X^s - Q) v.$$

From which it follows that

$$(\lambda_n(X))^s - \lambda_1(Q) \le |\lambda(A)|^2 v^T X^{-t} v \le (\lambda_1(X))^s - \lambda_n(Q). \tag{2.16}$$

From the left inequality of (2.16), we get

$$(\lambda_n(X))^s - \lambda_1(Q) \le |\lambda(A)|^2 (\lambda_n(X))^{-t}, \tag{2.17}$$

and from its right one, we obtain

$$|\lambda(A)|^2 (\lambda_1(X))^{-t} \le (\lambda_1(X))^s - \lambda_n(Q). \tag{2.18}$$

Then it follows from (2.17) and (2.18) that the inequality (2.15) holds. □

3. Perturbation estimates

Assume that Eq. (1.1) is perturbed to:

$$\tilde{X}^s - \tilde{A}^* \tilde{X}^{-t} \tilde{A} = \tilde{Q}. \tag{3.1}$$

Let $\Delta X = \tilde{X} - X$, $\Delta A = \tilde{A} - A$, and $\Delta Q = \tilde{Q} - Q$.

Theorem 3.1 *Suppose X and $\tilde{X}$ are the solutions of Eqs. (1.1) and (3.1), respectively. If*

$$\gamma = s\cdot\min\{\lambda_n(Q),\lambda_n(\tilde{Q})\}^{\frac{s-1}{s}} - \|A\|^2\sum_{k=1}^{t}\|\tilde{Q}^{-1}\|^{\frac{k}{s}}\|Q^{-1}\|^{\frac{t+1-k}{s}} > 0,$$

then

$$\|\Delta X\| \le \frac{1}{\gamma}[\|\Delta Q\| + \|\tilde{Q}^{-1}\|^{\frac{t}{s}}\|\Delta A\|(2\|A\| + \|\Delta A\|)] \tag{3.2}$$

and

$$\frac{\|\Delta X\|}{\|X\|} \le \frac{1}{\gamma}[\frac{\|\Delta Q\|}{\|Q\|}\|Q\|^{1-\frac{1}{s}} + \frac{\|\Delta A\|}{\|A\|}\frac{\|\tilde{Q}^{-1}\|^{\frac{t}{s}}\|A\|^2}{\|Q\|^{\frac{1}{s}}}(\frac{\|\Delta A\|}{\|A\|}+2)]. \tag{3.3}$$

Proof Firstly, by (2.5), we get

$$\begin{aligned}\Delta Q &= \tilde{X}^s - X^s - \tilde{A}^*\tilde{X}^{-t}\tilde{A} + A^*X^{-t}A\\ &= \tilde{X}^s - X^s - \Delta A^*\tilde{X}^{-t}\Delta A - \Delta A^*\tilde{X}^{-t}A - A^*\tilde{X}^{-t}\Delta A + A^*(X^{-t}-\tilde{X}^{-t})A\\ &= \tilde{X}^s - X^s - \Delta A^*\tilde{X}^{-t}\Delta A - \Delta A^*\tilde{X}^{-t}A - A^*\tilde{X}^{-t}\Delta A + A^*\sum_{k=1}^{t}\tilde{X}^{-k}\Delta X X^{k-t-1}A. \end{aligned}\tag{3.4}$$

Then, we have

$$\begin{aligned}&\|\tilde{X}^s - X^s + A^*\sum_{k=1}^{t}\tilde{X}^{-k}\Delta X X^{k-t-1}A\|\\ &\ge \|\tilde{X}^s - X^s\| - \|A\|^2\sum_{k=1}^{t}\|\tilde{X}^{-1}\|^{k}\|\Delta X\|\|X^{-1}\|^{t+1-k}\\ &\ge \|\sum_{k=0}^{s-1}\tilde{X}^{k}\Delta X X^{s-1-k}\| - \|A\|^2\sum_{k=1}^{t}\|\tilde{Q}^{-1}\|^{\frac{k}{s}}\|\Delta X\|\|Q^{-1}\|^{\frac{t+1-k}{s}}\\ &\ge [s\cdot\min\{\lambda_n(Q),\lambda_n(\tilde{Q})\}^{\frac{s-1}{s}} - \|A\|^2\sum_{k=1}^{t}\|\tilde{Q}^{-1}\|^{\frac{k}{s}}\|Q^{-1}\|^{\frac{t+1-k}{s}}]\|\Delta X\| \equiv \gamma\|\Delta X\|. \end{aligned}\tag{3.5}$$

If $\gamma > 0$, then by (3.4), (3.5) and the fact that $\tilde{X} \ge Q^{\frac{1}{s}}$, we obtain

$$\begin{aligned}\|\Delta X\| &\le \frac{1}{\gamma}\|\Delta Q + \Delta A^*\tilde{X}^{-t}\Delta A + \Delta A^*\tilde{X}^{-t}A + A^*\tilde{X}^{-t}\Delta A\|\\ &\le \frac{1}{\gamma}(\|\Delta Q\| + \|\Delta A\|^2\|\tilde{X}^{-1}\|^t + 2\|\tilde{X}^{-1}\|\|A\|\|\Delta A\|)\\ &\le \frac{1}{\gamma}[\|\Delta Q\| + \|\Delta A\|\|\tilde{Q}^{-1}\|^{\frac{t}{s}}(\|\Delta A\| + 2\|A\|)].\end{aligned}$$

Moreover, since $\|X\| \ge \|Q\|^{\frac{1}{s}}$, we have

$$\begin{aligned}\frac{\|\Delta X\|}{\|X\|} &\le \frac{1}{\gamma}[\frac{\|\Delta Q\|}{\|Q\|}\frac{\|Q\|}{\|X\|} + \frac{\|\Delta A\|}{\|A\|}\frac{\|\tilde{Q}^{-1}\|^{\frac{t}{s}}\|A\|^2}{\|X\|}(\frac{\|\Delta A\|}{\|A\|}+2)]\\ &\le \frac{1}{\gamma}[\frac{\|\Delta Q\|}{\|Q\|}\|Q\|^{1-\frac{1}{s}} + \frac{\|\Delta A\|}{\|A\|}\frac{\|\tilde{Q}^{-1}\|^{\frac{t}{s}}\|A\|^2}{\|Q\|^{\frac{1}{s}}}(\frac{\|\Delta A\|}{\|A\|}+2)].\end{aligned}$$

The proof is completed. □

Lemma 3.1 ([18]) *Let $B \in \mathscr{C}^{n\times n}$ be nonsingular and $E \in \mathscr{C}^{n\times n}$. If $\|B^{-1}\|\|E\| < 1$, then*

$C = B + E$ is also nonsingular and

$$\|C^{-1}\| \leq \frac{\|B^{-1}\|}{1 - \|B^{-1}\|\|E\|}. \tag{3.6}$$

Theorem 3.2 *If $\{[\frac{t}{s}(\|A\| + \|\Delta A\|)^2]^{\frac{s}{s+t}} + \|\Delta Q\|\}\|Q^{-1}\| < 1$, then both Eqs. (1.1) and (3.1) have unique solutions X and $\tilde{X}$, respectively. And the two solutions satisfy the estimates (3.2) and (3.3).*

Proof From the assumption, we see that $\|\Delta Q\|\|Q^{-1}\| < 1$. Then it follows from Lemma 3.1 that

$$\|\tilde{Q}^{-1}\| \leq \frac{\|Q^{-1}\|}{1 - \|Q^{-1}\|\|\Delta Q\|}. \tag{3.7}$$

If

$$\{[\frac{t}{s}(\|A\| + \|\Delta A\|)^2)]^{\frac{s}{s+t}} + \|\Delta Q\|\}\|Q^{-1}\| < 1, \tag{3.8}$$

then

$$[\frac{t}{s}(\|A\| + \|\Delta A\|)^2]^{\frac{s}{s+t}}\|Q^{-1}\| < 1 - \|Q^{-1}\|\|\Delta Q\|,$$

which yields

$$\frac{t}{s}(\|A\| + \|\Delta A\|)^2\|Q^{-1}\|^{\frac{s+t}{s}} < (1 - \|Q^{-1}\|\|\Delta Q\|)^{\frac{s+t}{s}},$$

i.e.,

$$\frac{t}{s}(\|A\| + \|\Delta A\|)^2(\frac{\|Q^{-1}\|}{1 - \|Q^{-1}\|\|\Delta Q\|})^{\frac{s+t}{s}} < 1. \tag{3.9}$$

Then by (3.7) and (3.9), we have

$$\frac{t}{s}\|\tilde{A}\|^2\|\tilde{Q}^{-1}\|^{\frac{s+t}{s}} \leq \frac{t}{s}(\|A\| + \|\Delta A\|)^2(\frac{\|Q^{-1}\|}{1 - \|Q^{-1}\|\|\Delta Q\|})^{\frac{s+t}{s}} < 1. \tag{3.10}$$

According to Theorem 2.2, Eq. (3.1) has a unique solution $\tilde{X}$. Moreover, from (3.8), we see that

$$\frac{t}{s}\|A\|^2\|Q^{-1}\|^{\frac{s+t}{s}} < 1, \tag{3.11}$$

which implies Eq. (1.1) also has a unique solution X.

It follows from (3.7) and (3.9) that

$$\frac{t}{s}\|A\|^2\|\tilde{Q}^{-1}\|^{\frac{s+t}{s}} \leq \frac{t}{s}(\|A\| + \|\Delta A\|)^2(\frac{\|Q^{-1}\|}{1 - \|Q^{-1}\|\|\Delta Q\|})^{\frac{s+t}{s}} < 1,$$

which yields

$$\|\tilde{Q}^{-1}\| < (\frac{s}{t\|A\|^2})^{\frac{s}{s+t}}.$$

On the other hand, by (3.11), we have

$$\|Q^{-1}\| < (\frac{s}{t\|A\|^2})^{\frac{s}{s+t}}.$$

From the above two inequalities, we obtain

$$\|\tilde{Q}^{-1}\|, \|Q^{-1}\| < (\frac{s}{t\|A\|^2})^{\frac{s}{s+t}}, \tag{3.12}$$

which implies

$$\|A\|^2 < \frac{s}{t}\min\{\lambda_n(Q), \lambda_n(\tilde{Q})\}^{\frac{s+t}{s}}. \tag{3.13}$$

Then we have

$$\begin{aligned}
&\|A\|^2 \sum_{k=1}^{t} \|\tilde{Q}^{-1}\|^{\frac{k}{s}} \|Q^{-1}\|^{\frac{t+1-k}{s}} \\
&< t\|A\|^2 \Big(\frac{s}{t\|A\|^2}\Big)^{\frac{s}{s+t}\cdot\frac{t+1}{s}} = t\Big(\frac{s}{t}\Big)^{\frac{t+1}{s+t}} \|A\|^{2(1-\frac{t+1}{s+t})} \\
&< t\Big(\frac{s}{t}\Big)^{\frac{t+1}{s+t}} \Big(\frac{s}{t}\min\{\lambda_n(Q), \lambda_n(\tilde{Q})\}^{\frac{s+t}{s}}\Big)^{\frac{s-1}{s+t}} = s\cdot\min\{\lambda_n(Q), \lambda_n(\tilde{Q})\}^{\frac{s-1}{s}},
\end{aligned} \tag{3.14}$$

which implies

$$\gamma = s\cdot\min\{\lambda_n(Q), \lambda_n(\tilde{Q})\}^{\frac{s-1}{s}} - \|A\|^2 \sum_{k=1}^{t} \|\tilde{Q}^{-1}\|^{\frac{k}{s}} \|Q^{-1}\|^{\frac{t+1-k}{s}} > 0.$$

Hence by Theorem 3.1, for the solutions X and $\tilde{X}$, we have the estimates (3.2) and (3.3). □

4. Numerical examples

In this section, we give a numerical example to show that our perturbation estimates (3.2) and (3.3) are close to the actual perturbation for Eq. (1.1). Let I_n denote the $n\times n$ unit matrix and ones(n) denote the $n\times n$ matrix whose all elements are one. All the tests are performed by MATLAB 7.4 with machine precision around 10^{-16}.

Example 4.1 Consider the matrix equation

$$X - A^*X^{-2}A = Q, \tag{4.1}$$

where $A = \frac{\hat{A}}{2\|\hat{A}\|}$ and

$$\hat{A} = \begin{bmatrix} 2 & 1+i & & & & \\ 1-i & 2 & 1+i & & & \\ & 1-i & 2 & 1+i & & \\ & & \ddots & & & \\ & & 1-i & 2 & 1+i \\ & & & 1-i & 2 \end{bmatrix} \in C^{n\times n}$$

with the solution

$$X = 2I_n + \mathrm{ones}(n) \text{ and } Q = X - A^*X^{-2}A.$$

Suppose that Eq. (4.1) is perturbed to the equation $X_j - A_j^*X_j^{-2}A_j = Q_j$, where $A_j = A + 10^{-j}A_0$ with $A_0 = \frac{1}{\|C+C^*\|}(C+C^*)$ and C is a random matrix generated by MATLAB function randn. The exact solution $X_j = X + 10^{-j}I_n$ and $Q_j = X_j - A_j^*X_j^{-2}A_j$.

We denote the left hand of the inequality proposed in the assumption of Theorem 3.2 by

$$\eta = \{[\frac{t}{s}(\|A\| + \|\Delta A\|)^2]^{\frac{s}{s+t}} + \|\Delta Q\|\}\|Q^{-1}\|.$$

The perturbation bounds given by (3.2) and (3.3) are denoted by err_A and err_R, respectively, i.e.,

$$\mathrm{err}_A = \frac{1}{\gamma}[\|\Delta Q\| + \|\tilde{Q}^{-1}\|^{\frac{t}{s}}\|\Delta A\|(2\|A\| + \|\Delta A\|)],$$

and

$$\mathrm{err}_R = \frac{1}{\gamma}[\frac{\|\Delta Q\|}{\|Q\|}\|Q\|^{1-\frac{1}{s}} + \frac{\|\Delta A\|}{\|A\|}\frac{\|\tilde{Q}^{-1}\|^{\frac{t}{s}}\|A\|^2}{\|Q\|^{\frac{1}{s}}}(\frac{\|\Delta A\|}{\|A\|} + 2)].$$

For the cases $n = 5, 10$ and 20, set $j = 1, 2, 3, \ldots$. Some numerical results are listed in Tables 1–3.

j	$\|\Delta X\|$	err_A	$\frac{\|\Delta X\|}{\|X\|}$	err_R	$\eta < 1$
1	0.1000	0.1428	0.0143	0.0204	Yes
2	0.0100	0.0146	0.0014	0.0021	Yes
3	1.0000e-003	0.0015	1.4286e-004	2.0942e-004	Yes
4	1.0000e-004	1.4703e-004	1.4286e-005	2.1020e-005	Yes

Table 1 Perturbation estimates of the solution of Eq. (4.1) for $n = 5$

j	$\|\Delta X\|$	err_A	$\frac{\|\Delta X\|}{\|X\|}$	err_R	$\eta < 1$
1	0.1000	0.1477	0.0083	0.0123	Yes
2	0.0100	0.0149	8.3333e-004	0.0012	Yes
3	1.0000e-003	0.0016	8.3333e-005	1.2968e-004	Yes
4	1.0000e-004	1.5279e-004	8.3333e-006	1.2735e-005	Yes

Table 2 Perturbation estimates of the solution of Eq. (4.1) for $n = 10$

j	$\|\Delta X\|$	err_A	$\frac{\|\Delta X\|}{\|X\|}$	err_R	$\eta < 1$
1	0.1000	0.1519	0.0045	0.0069	Yes
2	0.0100	0.0154	4.5455e-004	6.9815e-004	Yes
3	1.0000e-003	0.0015	4.5455e-005	6.9839e-005	Yes
4	1.0000e-004	1.5368e-004	4.5455e-006	6.9858e-006	Yes

Table 3 Perturbation estimates of the solution of Eq. (4.1) for $n = 20$

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