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The Existence and Stability of Positive Quasi-Periodic Solutions for the 3-Dimensional Lotka-Volterra System

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Abstract This work focuses on the existence and stability of positive quasi-periodic solutions for the 3-dimensional Lotka-Volterra system. Using KAM (Kolmogorov-Arnold-Moser) theory and Newton iteration, it is shown that there exists a positive quasi-periodic solution in a Cantor family for the 3-dimensional Lotka-Volterra system. On the above basis, we can show the stability of the solution with the help of Lyapunov function.

Keywords Lotka-Volterra system; positive quasi-periodic solution; KAM theory; Newton iteration; Lyaponov function.

MR(2010) Subject Classification 34B18; 39A24

1. Introduction and main results

In recent years, various mathematical models have been applied in the study of population dynamics, and one of the most famous models is the Lotka-Volterra model. Owing to its theoretical and practical significance, the Lotka-Volterra system has been studied extensively in [1, 2, 4, 7–9]. In this paper, we consider the 3-dimensional Lotka-Volterra system

$$\begin{cases} \mathrm{d}y_1/\mathrm{d}t = y_1((\lambda_1+\widetilde{h}_1(t)) - (\lambda_{11}+\widetilde{h}_{11}(t))y_1 - \sum_{i=2}^{3}(\lambda_{1i}+\widetilde{h}_{1i}(t))y_i), \\ \mathrm{d}y_2/\mathrm{d}t = y_2((\lambda_2+\widetilde{h}_2(t)) + (\lambda_{21}+\widetilde{h}_{21}(t))y_1 - \sum_{i=2}^{3}(\lambda_{2i}+\widetilde{h}_{2i}(t))y_i), \\ \mathrm{d}y_3/\mathrm{d}t = y_3((\lambda_3+\widetilde{h}_3(t)) + (\lambda_{31}+\widetilde{h}_{31}(t))y_1 - \sum_{i=2}^{3}(\lambda_{3i}+\widetilde{h}_{3i}(t))y_i). \end{cases} \tag{1}$$

This system describes a situation that three populations have the relations of predator-prey and competition, where population y_1 is prey, y_2 and y_3 prey on y_1, λ_i, λ_{ij} $(i,j=1,2,3)$ are positive constants, and $\widetilde{h}_i(t), \widetilde{h}_{ij}(t)$ $(i,j=1,2,3)$ are the quasi-periodic functions in time t.

In [6], the authors proved the existence of positive quasi-periodic solutions for the 2-dimensional Lotka-Volterra system. In fact, when dealing with such problem in the above paper, it is inevitable to encounter the so-called small divisor problem. As we know, KAM theory is a very powerful tool to cope with this problem; hence, the authors have perfectly utilized the KAM technique and Newton iteration to structure a positive quasi-periodic solution in a Cantor family. A natural question is: can we get the analogous conclusion for the 3-dimensional situation? After

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a further study, we find that the "twist" condition is hard to be proved, which is essential to apply KAM theory, because of a more complex coefficient matrix in the 3-dimensional system. However, we can overcome the difficulty by the creative skills in [11] where a finite number of terms in the Fourier expansion of the perturbation are killed in each iteration, and the remainder is included in the time-independent term.

Since error is inevitable in the observation, the biological significance of the stability for solutions is self-evident. So it is important for us to know whether the solution of the Lotka-Volterra system is stable. Luckily, we can prove solution's stability of (1) by Lyaponov function, which is not discussed in [6].

Before giving our results, we need the following definitions that have been concerned in [6].

Definition 1.1 *A function F is called quasi-periodic function in time t with the basic frequencies $\omega=(\omega_1,\dots,\omega_n)$, if there exists a function $\mathcal{F}(\theta_1,\dots,\theta_n)$ that is 2π-periodic in all its arguments θ_j $(j=1,\dots,n)$ and satisfies $F(t)=\mathcal{F}(\omega_1 t,\dots,\omega_n t)$. We call $\mathcal{F}$ the hull of $F(t)$.*

If $\mathcal{F}$ is analytic in the complex strip

$$D_\sigma=\{\theta\in\mathbb{T}^n=(\mathbb{C}/2\pi\mathbb{Z})^n:|\mathrm{Im}\,\theta|=\max_i|\mathrm{Im}\,\theta_i|<\sigma,\ i=1,\dots,n\},$$

then we say that F is analytic quasi-periodic in D_σ.

It is also known that the above analytical quasi-periodic function $\mathcal{F}(\theta)$ has Fourier expansion defined by

$$\mathcal{F}(\theta)=\sum_{k\in\mathbb{Z}^n}\widehat{\mathcal{F}}(k)e^{\sqrt{-1}\langle k,\theta\rangle},$$

where

$$\widehat{\mathcal{F}}(k)=\frac{1}{(2\pi)^n}\int_{\hat{\mathbb{T}}^n}\mathcal{F}(\theta)e^{-\sqrt{-1}\langle k,\theta\rangle}\mathrm{d}\theta$$

is called Fourier coefficients, $\hat{\mathbb{T}}^n=(\mathbb{R}/2\pi\mathbb{Z})^n$, and $\langle k,\theta\rangle=k_1\theta_1+k_2\theta_2+\dots+k_n\theta_n$ is the usual inner product. In D_σ we define

$$\|F\|^{D_\sigma}:=\|\mathcal{F}(\theta)\|^{D_\sigma}:=\sum_{k\in\mathbb{Z}^n}|\widehat{\mathcal{F}}(k)|e^{|k|\sigma}.$$

It is easy to see that $\|\cdot\|^{D_\sigma}$ is a norm.

Analogously, if the frequency $\omega\in\mathcal{O}$ is seen as a parameter, where

$$\mathcal{O}=\{\omega\in\mathbb{C}^n:|\mathrm{Im}\,\omega|=\max_i|\mathrm{Im}\,\omega_i|<q,\ i=1,\dots,n\},$$

we define

$$\|F\|^{D_\sigma\times\mathcal{O}}:=\|\mathcal{F}(\theta;\omega)\|^{D_\sigma\times\mathcal{O}}:=\sum_{k\in\mathbb{Z}^n}\|\widehat{\mathcal{F}}(k,\omega)\|^{\mathcal{O}}e^{|k|\sigma},$$

where

$$\|\widehat{\mathcal{F}}(k,\omega)\|^{\mathcal{O}}=\sup_{\omega\in\mathcal{O}}\|\widehat{\mathcal{F}}(k,\omega)\|.$$

When $F=(F_{ij})$ is a matrix-valued function in $D_\sigma\times\mathcal{O}$, we define

$$\|F\|^{D_\sigma\times\mathcal{O}}:=\sum_{k\in\mathbb{Z}^n}\|\widehat{\mathcal{F}}(k,\omega)\|^{\mathcal{O}}e^{|k|\sigma},$$

where

$$\|\widehat{\mathcal{F}}(k,\omega)\|^{\mathcal{O}} = \sup_{\omega\in\mathcal{O}} \|\widehat{\mathcal{F}}(k,\omega)\|,$$

with $\|\widehat{\mathcal{F}}(k,\omega)\|$ being the sup-norm of the matrix $\widehat{\mathcal{F}}(k,\omega) = (\widehat{\mathcal{F}}_{ij}(k,\omega))$, i.e.,

$$\|\widehat{\mathcal{F}}(k,\omega)\| = \max_{ij} |\widehat{\mathcal{F}}_{ij}(k,\omega)|.$$

Remark For convenience, we use the decorated letter of a function or itself to express its hull in the following paper.

After giving the above definitions we present the main results of this paper:

Theorem 1.1 *Let positive constants λ_i, λ_{ij} $(i,j=1,2,3)$ and a compact set of positive Lebesgue measure $\Pi \subset \mathbb{R}^n_+$ be given. Assume the functions $\widetilde{h}_i(t), \widetilde{h}_{ij}(t)$ $(i,j=1,2,3)$ are real analytic quasi-periodic functions in D_σ $(\sigma>0)$ with basic frequency $\omega=(\omega_1,\ldots,\omega_n)\in\Pi$. Then, for "most" $\omega\in\Pi$ (in the sense of Lebesgue measure), the equations (1) possesses a positive quasi-periodic solution, provided the norms $\|\widetilde{h}_i(t)\|^{D_\sigma\times\mathcal{O}}, \|\widetilde{h}_{ij}(t)\|^{D_\sigma\times\mathcal{O}}$ $(i,j=1,2,3)$ are bounded by a sufficiently small constant $\epsilon>0$ and $\mathcal{O}$ is a complex q-neighborhood of Π.*

Theorem 1.2 *If (1) satisfies the conditions in Theorem 1.1 and $\sum_{j\neq i}\lambda_{ji}+1<\lambda_{ii}$ $(i,j=1,2,3)$, the positive quasi-periodic solution $y^*(t)=(y_1^*(t),y_2^*(t),y_3^*(t))^T$ that we have structured is stable. That is, if $y(t)=(y_1(t),y_2(t),y_3(t))^T$ is any positive solution, the following equations are established.*

$$\lim_{t\to\infty}|y_i(t)-y_i^*(t)|=0, \quad i=1,2,3.$$

We will make the following arrangements in the rest paper: in the second part, we do coordinate changes to transform (1) into the ordinary vector form that is helpful for us to solve the problems. In the third part, some significant iterative lemmas that would be cited in the following proof will be given. In the fourth part, we plan to structure the positive quasi-periodic solution by Newton iteration. In the fifth part, we will prove the stability of the solution. In the end some technical lemmas will be presented.

2. Coordinate changes

Since we assume the norms $\|\widetilde{h}_i(t)\|^{D_\sigma\times\mathcal{O}}, \|\widetilde{h}_{ij}(t)\|^{D_\sigma\times\mathcal{O}}$ are bounded by a sufficiently small constant $\epsilon>0$, we can write

$$\widetilde{h}_i(t)=\epsilon h_i(t),\ \widetilde{h}_{ij}(t)=\epsilon h_{ij}(t), \quad i,j=1,2,3$$

with $\|h_i(t)\|^{D_\sigma\times\mathcal{O}}\le 1$, $\|h_{ij}(t)\|^{D_\sigma\times\mathcal{O}}\le 1$. Suppose the frequency $\omega\in\Pi$ satisfies the Diophantine conditions:

$$|\langle k,\omega\rangle| \ge \frac{\gamma}{|k|^{n+1}}, \quad \forall\ 0\neq k\in\mathbb{Z}^n,$$

otherwise, we can remove the set containing those ω that do not satisfy the Diophantine conditions, whose measure is less than or equal to $\gamma\sum_{k\in\mathbb{Z}^n}|k|^{-(n+1)}\le C\gamma$.

Without loss of generality, we assume that $\widehat{h}_i(0)=0$ $(i=1,2,3)$, otherwise, we can replace λ_i with $\widehat{h}_i(0)+\lambda_i$. Thus we can expand $h_i(t)$ into Fourier series

$$h_i(t)=\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\widehat{h}_i(k)e^{\sqrt{-1}\langle k,\omega\rangle t},\quad i=1,2,3.$$

Since ω satisfies the Diophantine conditions and $h_i(t)$ are analytic and quasi-periodic in D_σ with $\sigma>0$, we have

$$\int_0^t h_i(s)\mathrm{d}s=\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\frac{\widehat{h}_i(k)}{\sqrt{-1}\langle k,\omega\rangle}(e^{\sqrt{-1}\langle k,\omega\rangle t}-1),\quad i=1,2,3$$

are analytic and quasi-periodic too. We set the following linear quasi-periodic coordinate changes

$$y_i(t)=e^{\int_0^t\epsilon h_i(s)ds}\widetilde{y}_i(t),\quad i=1,2,3,$$

then equations (1) become

$$\begin{cases}\mathrm{d}\widetilde{y}_1/\mathrm{d}t=\lambda_1\widetilde{y}_1-(\lambda_{11}+\epsilon H_{11}(t))\widetilde{y}_1^2-\sum_{i=2}^3(\lambda_{1i}+\epsilon H_{1i})\widetilde{y}_i\widetilde{y}_1\\ \mathrm{d}\widetilde{y}_2/\mathrm{d}t=\lambda_2\widetilde{y}_2+(\lambda_{21}+\epsilon H_{21}(t))\widetilde{y}_2^2-\sum_{i=2}^3(\lambda_{2i}+\epsilon H_{2i})\widetilde{y}_i\widetilde{y}_2\\ \mathrm{d}\widetilde{y}_3/\mathrm{d}t=\lambda_3\widetilde{y}_3+(\lambda_{31}+\epsilon H_{31}(t))\widetilde{y}_3^2-\sum_{i=2}^3(\lambda_{3i}+\epsilon H_{3i})\widetilde{y}_i\widetilde{y}_3\end{cases}\tag{2}$$

where

$$H_{ij}(t)=h_{ij}(t)+\epsilon^{-1}(\lambda_{ij}+\epsilon h_{ij}(t))(e^{\int_0^t\epsilon h_i(s)\mathrm{d}s}-1),\quad i,j=1,2,3.$$

Let

$$\widetilde{y}_i(t)=\alpha_i e^{\epsilon x_i(t)},\quad i=1,2,3,$$

where α_i are constants to be determined. Then the equations (2) become

$$\begin{cases}\epsilon\mathrm{d}x_1/\mathrm{d}t=\lambda_1-\sum_{i=1}^3\alpha_i(\lambda_{1i}+\epsilon H_{1i})e^{\epsilon x_i(t)}\\ \epsilon\mathrm{d}x_2/\mathrm{d}t=\lambda_2+\alpha_1(\lambda_{21}+\epsilon H_{21})e^{\epsilon x_1(t)}-\sum_{i=2}^3\alpha_i(\lambda_{2i}+\epsilon H_{2i})e^{\epsilon x_i(t)}\\ \epsilon\mathrm{d}x_3/\mathrm{d}t=\lambda_3+\alpha_1(\lambda_{31}+\epsilon H_{31})e^{\epsilon x_1(t)}-\sum_{i=2}^3\alpha_i(\lambda_{3i}+\epsilon H_{3i})e^{\epsilon x_i(t)}.\end{cases}\tag{3}$$

Expand $e^{\epsilon x_i(t)}$ with Taylor expansion

$$e^{\epsilon x_i}=1+\epsilon x_i(t)+\frac{1}{2}e^{\theta\epsilon x_i}\epsilon^2x_i^2(t),\quad i=1,2,3,\ 0<\theta<1,$$

then bringing back (3), we obtain

$$\mathrm{d}x/\mathrm{d}t=(A+\epsilon B(t))x(t)+\xi(t)+F(t,x(t))\tag{4}$$

with

$$A=\begin{pmatrix}-\alpha_1\lambda_{11}&-\alpha_2\lambda_{12}&-\alpha_3\lambda_{13}\\ \alpha_1\lambda_{21}&-\alpha_2\lambda_{22}&-\alpha_3\lambda_{23}\\ \alpha_1\lambda_{31}&-\alpha_2\lambda_{32}&-\alpha_3\lambda_{33}\end{pmatrix},$$

$$B(t)=\begin{pmatrix}-\alpha_1C_{11}(t)&-\alpha_2C_{12}(t)&-\alpha_3C_{13}(t)\\ \alpha_1C_{21}(t)&-\alpha_2C_{22}(t)&-\alpha_3C_{23}(t)\\ \alpha_1C_{31}(t)&-\alpha_2C_{32}(t)&-\alpha_3C_{33}(t)\end{pmatrix},$$

$$\xi(t)=\begin{pmatrix} -\alpha_1C_{11}(t)-\alpha_2C_{12}(t)-\alpha_3C_{13}(t)\\ \alpha_1C_{21}(t)-\alpha_2C_{22}(t)-\alpha_3C_{23}(t)\\ \alpha_1C_{31}(t)-\alpha_2C_{32}(t)-\alpha_3C_{33}(t)\end{pmatrix},$$

$$F(t,x(t))=\begin{pmatrix} -\alpha_1(\lambda_{11}+\epsilon C_{11}(t))\frac{1}{2}e^{\theta\epsilon x_1}x_1^2(t)\epsilon-G_1\\ \alpha_1(\lambda_{21}+\epsilon C_{21}(t))\frac{1}{2}e^{\theta\epsilon x_1}x_1^2(t)\epsilon-G_2\\ \alpha_1(\lambda_{31}+\epsilon C_{31}(t))\frac{1}{2}e^{\theta\epsilon x_1}x_1^2(t)\epsilon-G_3\end{pmatrix}, \tag{5}$$

$$G_j=\sum_{i=2}^{3}\alpha_i(\lambda_{ji}+\epsilon C_{ji}(t))\frac{1}{2}e^{\theta\epsilon x_1}x_i^2(t)\epsilon,\quad i,j=1,2,3,$$

$$C_{ij}(t)=(\lambda_{ij}+\epsilon h_{ij}(t))e^{\int_0^t\epsilon h_i(s)\mathrm{d}s},\quad i,j=1,2,3,$$

while

$$\alpha_1=\frac{\lambda_1\lambda_{22}\lambda_{33}-\lambda_1\lambda_{23}\lambda_{32}+\lambda_2\lambda_{13}\lambda_{32}-\lambda_2\lambda_{12}\lambda_{33}+\lambda_3\lambda_{12}\lambda_{23}-\lambda_3\lambda_{13}\lambda_{22}}{\lambda_{11}\lambda_{22}\lambda_{33}-\lambda_{11}\lambda_{23}\lambda_{32}+\lambda_{12}\lambda_{21}\lambda_{33}-\lambda_{12}\lambda_{23}\lambda_{31}+\lambda_{13}\lambda_{31}\lambda_{22}-\lambda_{13}\lambda_{32}\lambda_{21}},$$

$$\alpha_2=\frac{\lambda_1\lambda_{21}\lambda_{33}-\lambda_1\lambda_{23}\lambda_{31}+\lambda_2\lambda_{11}\lambda_{33}+\lambda_2\lambda_{13}\lambda_{31}-\lambda_3\lambda_{11}\lambda_{23}-\lambda_3\lambda_{13}\lambda_{21}}{\lambda_{11}\lambda_{22}\lambda_{33}-\lambda_{11}\lambda_{23}\lambda_{32}+\lambda_{12}\lambda_{21}\lambda_{33}-\lambda_{12}\lambda_{23}\lambda_{31}+\lambda_{13}\lambda_{31}\lambda_{22}-\lambda_{13}\lambda_{32}\lambda_{21}},$$

$$\alpha_3=\frac{-\lambda_1\lambda_{21}\lambda_{32}+\lambda_1\lambda_{22}\lambda_{31}-\lambda_2\lambda_{11}\lambda_{32}-\lambda_2\lambda_{12}\lambda_{31}+\lambda_3\lambda_{11}\lambda_{22}+\lambda_3\lambda_{12}\lambda_{21}}{\lambda_{11}\lambda_{22}\lambda_{33}-\lambda_{11}\lambda_{23}\lambda_{32}+\lambda_{12}\lambda_{21}\lambda_{33}-\lambda_{12}\lambda_{23}\lambda_{31}+\lambda_{13}\lambda_{31}\lambda_{22}-\lambda_{13}\lambda_{32}\lambda_{21}}.$$

From (5), it is easy to get

$$\|F(t,x(t))\|^{D_\sigma\times\mathcal{O}}\le C\epsilon, \tag{6}$$

where C is a constant.

3. Iterative lemmas

The proof of Theorem 1.1 is based on some iterative lemmas. Before we state the main iterative lemmas, we need introduce some notations.

In the following we denote by $C, C_1, C_2, \cdots$ positive constants that will arise in the estimates. Denote by m the number of the iterative step, and let

(i) $\epsilon_m=\epsilon^{(\frac{4}{3})^{m-1}}$, $\epsilon_1=\epsilon$, $m=1,2,3,\ldots$; especially, set $\epsilon_0=1$.

(ii) $\sigma_m=\sigma-\sigma\frac{\sum_{j=1}^m 3^{-j}}{2\sum_{j=1}^\infty 3^{-j}}$, $\sigma_0=\sigma$.

(iii) $\beta_m=\beta-\beta\frac{\sum_{j=1}^m 3^{-j}}{2\sum_{j=1}^\infty 3^{-j}}$, $\beta_0=\beta$.

(iv) $D_m=\{\theta\in\mathbb{T}^n:|\mathrm{Im}\theta|=\max_i|\mathrm{Im}\,\theta_i|<\sigma_m,\ i=1,\ldots,n\}$, $U_m=U(\sigma_m,\beta_m)=\{(\theta,x)\in\mathbb{T}^n\times\mathbb{K}^3:|\mathrm{Im}\,\theta|<\sigma_m,|x|<\beta_m\}$. $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$.

(v) $q_m=\epsilon_{m+1}^{1/36}$.

(vi) $\gamma_m=\gamma/m^{2\times 3^2}$.

(vii) $C(m)$ be a constant of the form $a_1m^{a_2}$, where a_1,a_2 are constants independent of m.

(viii) Let $\mathcal{D}\subset\mathbb{C}^n$. If a matrix-valued function $F:U_m\times\mathcal{D}\to\mathbb{K}^3$ is real analytic in variables $(\theta,x)\in U_m$, $\omega\in\mathcal{D}$, and there are a series of functions $\{F_i(\theta;\omega)\}$ for $l\in\{0,1,2,\ldots\}$ satisfying

$$F(\theta,x;\omega)=\sum_{i=l}^{\infty}F_i(\theta;\omega)x^i,$$

then we write

$$F = O_{\sigma_m,\beta_m,\mathcal{D}}(x^l).$$

And by (5), we have

$$F(\theta, x; \omega) = O_{\sigma_0,\beta_0,\mathcal{O}}(x^2). \tag{7}$$

Let $\Pi = \Pi_0 \supset \Pi_1 \supset \cdots \supset \Pi_{m-1}$ be the compact sets in $\mathbb{R}^n_+$ and $\Pi_m \subset \Pi_{m-1}$ be defined in the proof of Lemma 3.2. Let $\mathcal{O}_l$ be the complex q_l-neighborhood of Π_l for $l = 0, 1, \ldots, m$ and

$$\widetilde{\mathcal{R}}_k(m) := \{\omega \in \Pi_{m-1} : \big||\sqrt{-1}\langle k, \omega\rangle E_{3^2} - E_3 \otimes A_m + A_m^T \otimes E_3|_d\big| \le \gamma_m/|k|^{\tau_1}\},$$

$$\widetilde{\Pi}_0 = \Pi_0, \quad \widetilde{\Pi}_m = \widetilde{\Pi}_{m-1} \backslash \bigcup_{0<|k|\le M_m} \widetilde{\mathcal{R}}_k(m),$$

where $\otimes$ is kronecker product, E_m is the $m \times m$ identity matrix, A_m is a 3×3 constant matrix and satisfies $(H1)_{m+1}$ in Lemma 3.1, $\tau_1 = 9(n+1)$ and $M_m = |\ln \epsilon_m|/(\sigma_{m-1} - \sigma_m)$. Assume $\widetilde{\mathcal{O}}_l$ is the complex q_l-neighborhood of $\widetilde{\Pi}_l$.

What we would like to remind readers is there is a relation $\Pi_m \supset \widetilde{\Pi}_m \supset \Pi_{m-1}$ established, if we carefully observe the definition of Π_m in the proof of Lemma 3.2.

Lemma 3.1 *Consider a quasi-periodic differential equation*

$$\mathrm{d}x/\mathrm{d}t = (A_{m-1} + \epsilon_m Q_m(t; \omega))x, \tag{8}$$

where the following conditions are satisfied

$(H1)_m$ $A_{m-1} = A + \epsilon_1\widetilde{\mathcal{L}}_1(\omega) + \cdots + \epsilon_{m-1}\widetilde{\mathcal{L}}_{m-1}(\omega), m \ge 2, A_0 = A$ *with* $\widetilde{\mathcal{L}}_l(\omega)$ *analytic in* $\widetilde{\mathcal{O}}_l$ *and* $\|\widetilde{\mathcal{L}}_l\|^{\widetilde{\mathcal{O}}_l} \le C$ *for* $l = 1, \ldots, m-1$;

$(H2)_m$ *The hull* $\mathcal{Q}_m$ *of* $Q_m(t; \omega)$ *is analytic in* $D_{m-1} \times \widetilde{\mathcal{O}}_{m-1}$ *and*

$$\|\mathcal{Q}_m(\theta; \omega)\|^{D_{m-1}\times\widetilde{\mathcal{O}}_{m-1}} \le C, \tag{9}$$

where C is constant.

Then there is a quasi-periodic transformation

$$x = (E + \epsilon_m P_m(t))y, \tag{10}$$

where $P_m(t)$ is quasi-periodic with frequency ω. Its hull $\mathcal{P}_m(\theta; \omega)$ is analytic in $D_m \times \widetilde{\mathcal{O}}_m$ and

$$\|\mathcal{P}_m(\theta; \omega)\|^{D_m\times\widetilde{\mathcal{O}}_m} \le C(m)$$

such that (8) is changed into

$$\mathrm{d}y/\mathrm{d}t = (A_m + \epsilon_{m+1}Q_{m+1}(t))y, \tag{11}$$

where A_m and Q_{m+1} satisfy the conditions $(H1)_{m+1}$ and $(H2)_{m+1}$.

Proof This proof can be found in [12, p. 4074].

Lemma 3.2 *Consider the following differential equation*

$$\mathrm{d}x/\mathrm{d}t = A_m(\omega)x + \xi(\theta; \omega), \tag{12}$$

where $\theta = \omega t$, $A_m(\omega)$ is a 3×3 constant matrix and satisfies $(H1)_{m+1}$, and the function $\xi(\theta;\omega)$ is analytic in $D_m \times \widetilde{\mathcal{O}}_m$. Then equations (12) has an analytic quasi-periodic solution in one of its subsets $D_m \times \mathcal{O}_m$ and we have the following estimation of the norm for the solution

$$\|x(t)\|^{D_m\times\mathcal{O}_m} \le C\gamma_m^{-1}\|\xi\|^{D_m\times\mathcal{O}_m}/\beta_m^{\tau+n}, \tag{13}$$

where $\tau = 9n+11$.

Proof We can expand $x(\theta), \xi(\theta)$ into Fourier series

$$x(t) = \sum_{k\in\mathbb{Z}^n} \widehat{x}(k)e^{\sqrt{-1}\langle k,\omega\rangle t}, \quad \xi(t) = \sum_{k\in\mathbb{Z}^n} \widehat{\xi}(k)e^{\sqrt{-1}\langle k,\omega\rangle t}.$$

Contrasting the corresponding Fourier coefficients on both sides of (12), we get

$$\mathrm{diag}(\sqrt{-1}\langle k,\omega\rangle E - A_m)X = \Xi, \tag{14}$$

where diag$(*)$ is the diagonal matrix whose diagonal entry is $*$, E is the 3×3 identity matrix, $X = (\ldots\widehat{x}(k)\ldots)^T$, and $\Xi = (\ldots\widehat{\xi}(k)\ldots)^T$.

Set

$$G(\omega) = \sqrt{-1}\langle k,\omega\rangle E - A_m, \quad \mathcal{M}_k(\omega) = |G(\omega)|_d$$

and

$$\mathcal{R}_k(m) = \{\omega \in \Pi_{m-1} : |\mathcal{M}_k| < \gamma_m/|k|^{\tau_1}\}, \tag{15}$$

where $|\cdot|_d$ is the determinant of a matrix and

$$\Pi_m = \widetilde{\Pi}_m \backslash \bigcup_{0<|k|\le M_m} \mathcal{R}_k(m) \tag{16}$$

with $M_m = |\ln\epsilon_m|/(\sigma_{m-1}-\sigma_m)$ being the number of Fourier coefficients we must consider at the mth step of the iteration. Denote by $\mathcal{O}_m$ the complex q_m-neighborhood of Π_m. By the definition of Π_m, we can get

$$|\mathcal{M}_k| \ge \gamma_m/|k|^{\tau_1}$$

for $0<|k|\le M_m$ and $\omega\in\Pi_m$.

It is easy to know that

$$\|G(\omega)\|^{\mathcal{O}_m} \le C_1|k|, \quad k\ne 0,$$

where $C_1 = 2(\max\{|\omega| : \omega\in\Pi\} + \|A\| + 1)$. Since $|G(\omega)|_d = \mathcal{M}_k(\omega)$, $G^{-1}(\omega)$ exists for $\omega\in\Pi_m$ and

$$G^{-1}(\omega) = \frac{\mathrm{adj}\, G(\omega)}{\mathcal{M}_k(\omega)},$$

where adj is the adjoint of a matrix. Thus, for $0<|k|\le M_m$, there exists a constant C_2 satisfying

$$\|G^{-1}(\omega)\|^{\Pi_m} \le C_2\frac{|k|^2}{\gamma_m/|k|^{\tau_1}} = C_2\gamma_m^{-1}|k|^{\tau},$$

where $\tau = \tau_1 + 2 = 9n+11$.

Now, we assume that $\omega\in\mathcal{O}_m$. Then there is an $\omega_0\in\Pi_m$ such that $|\omega-\omega_0|\le q_m$. Thus,

$$\|G^{-1}(\omega_0)\|\|G(\omega)-G(\omega_0)\| \le \|G^{-1}(\omega)\|^{\Pi_m}\|\nabla_\omega G(\omega)\|^{\mathcal{O}_m}\cdot|\omega-\omega_0|$$

$$
\begin{aligned}
&\leq C_2\gamma_m^{-1}|k|^{\tau}\|\ G(\omega)\|^{\ \mathcal{O}_{m-1}}\frac{q_m}{q_{m-1}-q_m}\\
&\leq C_2\gamma_m^{-1}|k|^{\tau+1}\frac{q_m}{q_{m-1}-q_m}\\
&\leq C_2M_m^{\tau+1}\gamma_m^{-1}\frac{q_m}{q_{m-1}-q_m}\\
&\leq \frac{C_2m^{9(6+2n)}|\ln\epsilon_m|^{\tau+1}\epsilon_{m+1}^{1/36}(\gamma\sigma_0)^{-1}}{\epsilon_m^{1/36}-\epsilon_{m+1}^{1/36}}\\
&<\frac{1}{2},
\end{aligned}
$$

where ∇ is the differential operator of vector. Therefore, $E+G^{-1}(\omega_0)(G(\omega)-G(\omega_0))$ has its inverse which is analytic in $\mathcal{O}_m$ since

$$(E+G^{-1}(\omega_0)(G(\omega)-G(\omega_0)))^{-1}=\sum_{j=0}^{\infty}(-G^{-1}(\omega_0)(G(\omega)-G(\omega_0)))^j.$$

So, $G(\omega)$ has its inverse for $\omega\in\mathcal{O}_m$ and

$$
\begin{aligned}
\|G^{-1}(\omega)\|&=\|(E+G^{-1}(\omega_0)(G(\omega)-G(\omega_0)))^{-1}\cdot G^{-1}(\omega_0)\|\\
&\leq\|(E+G^{-1}(\omega_0)(G(\omega)-G(\omega_0)))^{-1}\|\cdot\|G^{-1}(\omega_0)\|\\
&\leq C\gamma_m^{-1}|k|^{\tau}.
\end{aligned}
$$

In the end, we get

$$
\begin{aligned}
\|x(t)\|^{D_m\times\mathcal{O}_m}&=\sum_{k\in\mathbb{Z}^n}\|\widehat{x}(k)\|^{\mathcal{O}_m}e^{|k|\sigma_m}\leq\sum_{k\in\mathbb{Z}^n}\gamma_m^{-1}|k|^{\tau}\|\widehat{\xi}(k)\|^{\mathcal{O}_m}e^{|k|\sigma_m}\\
&\leq C\gamma_m^{-1}|k|^{\tau}e^{-|k|(\sigma_{m-1}-\sigma_m)}\|\xi\|^{D_m\times\mathcal{O}_m}\leq C\gamma_m^{-1}\|\xi\|^{D_m\times\mathcal{O}_m}/\beta_m^{\tau+n},
\end{aligned}
$$

where the last inequality follows from Lemma 6.1. So (12) has a bounded analytic quasi-periodic solution in the set $D_m\times\mathcal{O}_m$.

Lemma 3.3 *If $\xi(t)$ is an analytic quasi-periodic function and its hull*

$$\|\xi(\theta;\omega)\|^{D_{m+1}\times\widetilde{\mathcal{O}}_{m+1}}\leq\epsilon_m.$$

Meanwhile,

$$\xi(\theta;\omega)=\xi_1(\theta;\omega)+\xi_2(\theta;\omega),$$

where

$$\xi_1(\theta;\omega)=\sum_{0<|k|\leq M_m}\widehat{\xi}(k)e^{\sqrt{-1}\langle k,\theta\rangle}$$

and

$$\xi_2(\theta;\omega)=\sum_{|k|>M_m}\widehat{\xi}(k)e^{\sqrt{-1}\langle k,\theta\rangle}.$$

Then the following inequality holds

$$\|\xi_2(\theta;\omega)\|^{D_{m+1}\times\mathcal{O}_{m+1}}\leq\epsilon_{m+1}.$$

Proof In view of

$$\xi_2(\theta;\omega)=\sum_{|k|>M_m}\widehat{\xi}(k)e^{\sqrt{-1}\langle k,\theta\rangle}$$

and the definition of $\|\xi_2(\theta;\omega)\|^{D_{m+1}\times\mathcal{O}_{m+1}}$, we get

$$\begin{aligned}\|\xi_2(\theta;\omega)\|^{D_{m+1}\times\mathcal{O}_{m+1}}&=\sum_{|k|>M_m}\|\widehat{\xi}(k;\omega)\|^{\mathcal{O}_{m+1}}e^{|k|\sigma_{m+1}}\\&=\sum_{|k|>M_m}\|\widehat{\xi}(k;\omega)\|^{\mathcal{O}_{m+1}}e^{|k|\sigma_m}e^{-|k|\sigma_m}e^{|k|\sigma_{m+1}}\\&\le\sum_{|k|>M_m}\|\xi(\theta;\omega)\|^{D_{m+1}\times\tilde{\mathcal{O}}_{m+1}}e^{-|k|\sigma_m}e^{|k|\sigma_{m+1}}\\&\le\epsilon_m\sum_{|k|>M_m}e^{-|k|(\sigma_m-\sigma_{m+1})}\\&=\epsilon_m^2\sum_{|k|>0}e^{-|k|(\sigma_m-\sigma_{m+1})}\\&\le C(m)\epsilon_m^2\le\epsilon_{m+1}(\epsilon\ll 1).\end{aligned}$$

4. Newton iteration

When we use the Newton iteration to structure the solution of (4), we need the following definition to describe the approximation.

Definition 4.1 *Consider the M-dimensional system $\mathrm{d}x/\mathrm{d}t=f(t,x)$, where $M\in\mathbb{Z}_+$. If there exists $x=x(t)$ defined in some set Ω such that*

$$\sup_{t\in\Omega}|\mathrm{d}x/\mathrm{d}t-f(t,x(t))|<C\epsilon,$$

where C is a constant, then we call $x(t)$ an ϵ-approximate solution of the system $\mathrm{d}x/\mathrm{d}t=f(t,x)$.

Lemma 4.1 *Assume the equation*

$$\mathrm{d}x/\mathrm{d}t=A(t)x+\xi(t) \tag{17}$$

is defined in Ω. $A(t)$ and $\xi(t)$ are real analytic. For any analytic linear transformation $x=P(t)y$, where $P(t)$ is bounded and has a bounded inverse, (17) can be turned into

$$\mathrm{d}y/\mathrm{d}t=P^{-1}APy-P^{-1}\dot{P}y+P^{-1}\xi, \tag{18}$$

where $\cdot$ means the derivative with respect to time t. If $y_0(t)$ is an ϵ-approximate solution of (18), $P(t)y_0(t)$ is an ϵ-approximate solution of (17).

Proof We calculate derivative on the both sides of $x=P(t)y$ with respect to t

$$\dot{x}=\dot{P}y+P\dot{y},$$

bringing it back to (17), then we can get (18) directly.

If $y_0(t)$ is an ϵ-approximate solution of (18), then

$$|\dot{y}_0 - P^{-1}APy_0 + P^{-1}\dot{P}y_0 - P^{-1}\xi| \le C\epsilon.$$

Due to $\|P\|, \|P^{-1}\| \le C$,

$$\begin{aligned}|\mathrm{d}(Py_0)/\mathrm{d}t - APy_0 - \xi| =& |P\dot{y}_0 + \dot{P}y_0 - APy_0 - \xi| \\ =& |P(\dot{y}_0 - P^{-1}APy_0 + P^{-1}\dot{P}y_0 - P^{-1}\xi)| \\ \le& \|P\| \cdot |\dot{y}_0 - P^{-1}APy_0 + P^{-1}\dot{P}y_0 - P^{-1}\xi| \\ \le& C\epsilon.\end{aligned}$$

Thus $P(t)y_0(t)$ is an ϵ-approximate solution of (17).

We will use mathematical induction to structure the ϵ_m-approximate solution of (4). In the first place, let us try to find an ϵ-approximate solution of the equations (4). Consider a part of (4)

$$\mathrm{d}x/\mathrm{d}t = Ax + \xi_1(\theta;\omega), \tag{19}$$

where

$$\begin{aligned}\xi(\theta;\omega) &= \xi_1(\theta;\omega) + \xi_2(\theta;\omega), \\ \xi_1(\theta;\omega) &= \sum_{0<|k|\le M_1} \widehat{\xi}(k) e^{\sqrt{-1}\langle k,\theta\rangle}, \\ \xi_2(\theta;\omega) &= \sum_{|k|>M_1} \widehat{\xi}(k) e^{\sqrt{-1}\langle k,\theta\rangle}.\end{aligned}$$

Since $D_1 \times \mathcal{O}_1 \subset D_0 \times \mathcal{O}_0$, we think over the solution of (19) in $D_1 \times \mathcal{O}_1$. When we notice the specific form of ξ, it is easy to know $\|\xi\|^{D_0\times\mathcal{O}_0} \le C$. As a subset of $D_0 \times \mathcal{O}_0$, the same inequality is right in $D_1 \times \widetilde{\mathcal{O}}_1$.

$$\|\xi\|^{D_1\times\widetilde{\mathcal{O}}_1} \le C.$$

By Lemma 3.3, it is easy to know

$$\|\xi_2\|^{D_1\times\mathcal{O}_1} \le \epsilon.$$

In fact, the present conditions do not completely fit the conditions in Lemma 3.3 if readers observe carefully. However, we can directly get the above inequality by the proof process of Lemma 3.3.

According to (13) in Lemma 3.2, we know (19) has a solution x_1 in $D_1 \times \mathcal{O}_1$ whose norm satisfies

$$\|x_1\|^{D_1\times\mathcal{O}_1} \le C\gamma_1^{-1}/\beta_1^{\tau+n}\|\xi_1\|^{D_1\times\mathcal{O}_1} \le C\gamma_1^{-1}/\beta_1^{\tau+n}.$$

Let us confirm x_1 is an ϵ-approximate solution of (4)

$$\begin{aligned}&\|\mathrm{d}x_1/\mathrm{d}t - (A+\epsilon B(t))x_1(t) - \xi(t) - F(t,x_1(t))\|^{D_1\times\mathcal{O}_1} \\ &= \|\epsilon B(t)x_1(t) + \xi_2(t) + F(t,x_1(t))\|^{D_1\times\mathcal{O}_1} \\ &\le \|\epsilon B(t)x_1(t)\|^{D_1\times\mathcal{O}_1} + \|\xi_2(t)\|^{D_1\times\mathcal{O}_1} + \|F(t,x_1(t))\|^{D_1\times\mathcal{O}_1}\end{aligned}$$

$\leq C\epsilon.$

In fact $\|F(t,x_1(t)\|^{D_1\times\mathcal{O}_1} \leq C\epsilon$ can be inferred from its special form (5). So x_1 is an ϵ-approximate solution of (4).

In the second place, we assume that $x^m = \sum_{i=1}^m x_i$ is an analytic quasi-periodic ϵ_m-approximate solution and $\|x_i\|^{D_i\times\mathcal{O}_i} \leq C\gamma_i^{-1}\epsilon_{i-1}/\beta_i^{\tau+n}$. It is easy to know that when $i=1$ the assumption holds true.

Assume $x = x^m + \tilde{x}$ is the solution of (4). Then

$$\begin{aligned}\mathrm{d}x^m/\mathrm{d}t + \mathrm{d}\tilde{x}/\mathrm{d}t =&(A+\epsilon B(t))(x^m+\tilde{x}) + \xi(t) + F(t,x^m+\tilde{x})\\ =&(A+\epsilon B(t))x^m + \xi(t) + F(t,x^m) + (A+\epsilon B(t))\tilde{x} + \partial_x F(t,x^m)\tilde{x}+\\ &\int_0^1\int_0^1 \partial_x^2 F(t,\mu\nu\tilde{x}+x^m)\mu\tilde{x}^2\mathrm{d}\mu\mathrm{d}\nu.\end{aligned}$$

We get a new equation

$$\mathrm{d}\tilde{x}/\mathrm{d}t = (A+\epsilon B(t)+\partial_x F(t,x^m))\tilde{x} + \xi^{m+1}(t) + \Upsilon^{m+1}(t,\tilde{x}), \tag{20}$$

where

$$\begin{aligned}\xi^{m+1}(t) &= \mathrm{d}x^m/\mathrm{d}t - ((A+B(t))x^m + \xi(t) + F(t,x^m)),\\ \Upsilon^{m+1}(t,\tilde{x}) &= \int_0^1\int_0^1 \partial_x^2 F(t,\mu\nu\tilde{x}+x^m)\mu\tilde{x}^2\mathrm{d}\mu\mathrm{d}\nu = O_{\sigma_m,\beta_m,\mathcal{O}_m}(\tilde{x}^2).\end{aligned} \tag{21}$$

According to the assumption that x^m is the ϵ_m-approximate solution of (4), we obtain

$$\|\xi^{m+1}(\theta;\omega)\|^{D_m\times\mathcal{O}_m} \leq C\epsilon_m.$$

Similarly, we consider a part of (20)

$$\mathrm{d}x/\mathrm{d}t = (A+\epsilon B(t)+\partial_x F(t,x^m))x + \xi_1^{m+1}(\theta;\omega), \tag{22}$$

where

$$\begin{aligned}\xi^{m+1}(\theta;\omega) &= \xi_1^{m+1}(\theta;\omega) + \xi_2^{m+1}(\theta;\omega),\\ \xi_1^{m+1}(\theta;\omega) &= \sum_{0<|k|\leq M_m} \widehat{\xi}^{m+1}(k)e^{\sqrt{-1}\langle k,\theta\rangle},\\ \xi_2^{m+1}(\theta;\omega) &= \sum_{|k|>M_m} \widehat{\xi}^{m+1}(k)e^{\sqrt{-1}\langle k,\theta\rangle}.\end{aligned}$$

In fact, there exists a transformation $y = P^m x$ ($P^m = (E+\epsilon_m P_m)P^{m-1}$, that is, we just do once transformation at the mth step based on the former step) to change (22) into

$$\mathrm{d}y/\mathrm{d}t = (A_m + \epsilon_{m+1}Q_{m+1}(t,\omega))y + (P^m)^{-1}\xi_1^{m+1}, \tag{23}$$

where A_m, Q_{m+1} satisfy the conditions $(H1)_{m+1}$ and $(H2)_{m+1}$. What's more, the norms of P^m and $(P^m)^{-1}$ satisfy the following inequalities in $D_{m+1}\times\widetilde{\mathcal{O}}_{m+1}$

$$\|\mathcal{P}^m(\theta;\omega)\|^{D_{m+1}\times\widetilde{\mathcal{O}}_{m+1}} \leq 2, \quad \|(\mathcal{P}^m(\theta;\omega))^{-1}\|^{D_{m+1}\times\widetilde{\mathcal{O}}_{m+1}} \leq 2.$$

The proof can be found in the following remark.

Consider the main part of (23)

$$\mathrm{d}y/\mathrm{d}t = A_m y + (P^m)^{-1}\xi_1^{m+1}. \tag{24}$$

According to the Lemma 3.2, the above equation has an analytic quasi-periodic solution y_{m+1} that satisfies

$$\|y_{m+1}\|^{D_{m+1}\times\mathcal{O}_{m+1}} \le C\gamma_{m+1}^{-1}\|\xi_1^{m+1}\|/\beta_{m+1}^{\tau+n} \le C\gamma_{m+1}^{-1}\epsilon_m/\beta_{m+1}^{\tau+n}.$$

Similarly, y_{m+1} is an ϵ_{m+1}-approximate solution of (23) in $D_{m+1}\times\mathcal{O}_{m+1}$. Now, let us test this conclusion.

$$\begin{aligned}&\|\mathrm{d}y_{m+1}/\mathrm{d}t - (A_m + \epsilon_{m+1}Q_{m+1}(t,\omega))y - (P^m)^{-1}\xi_1^{m+1}\|^{D_{m+1}\times\mathcal{O}_{m+1}}\\ &= \|\epsilon_{m+1}Q_{m+1}(t,\omega)y_{m+1}\|^{D_{m+1}\times\mathcal{O}_{m+1}}\\ &\le C\epsilon_{m+1}.\end{aligned}$$

According to Lemma 4.1, $x_{m+1} = (P^m)^{-1}y_{m+1}$ is an ϵ_{m+1}-approximate solution of (22). In the meanwhile, according to Lemma 3.3 and (21), the corresponding abandon part

$$\|\xi_2^{m+1}\|^{D_{m+1}\times\mathcal{O}_{m+1}} \le C\epsilon_{m+1},$$

$$\|\Upsilon^{m+1}(t,x_{m+1})\|^{D_{m+1}\times\mathcal{O}_{m+1}} \le C(\gamma_{m+1}^{-1}\epsilon_m/\beta_{m+1})^2 \le C\epsilon_{m+1}.$$

To sum up, $x^{m+1} = x^m + x_{m+1}$ is an ϵ_{m+1}-approximate solution of (4).

Remark We will use mathematical induction to prove the existence of the transformation. First, for $k=1$, since $\|F(t,x^1(t)\|^{D_1\times\mathcal{O}_1} \le C\epsilon$, we know $\partial_x F(t,x^1) \le C\epsilon$ by Lemma 6.3. Meanwhile $\widetilde{\mathcal{O}}_0 = \mathcal{O}_0$, we can get the conclusion from Lemma 3.1 directly when $k=1$. On the other hand, we make such assumption that there exists a transformation $y = P^{m-1}x$ to change

$$\mathrm{d}x/\mathrm{d}t = (A + \epsilon B(t) + \partial_x F(t,x^{m-1}))x + \xi_1(t) \tag{25}$$

into

$$\mathrm{d}x/\mathrm{d}t = (A_{m-1} + \epsilon_m Q_m(t;\omega))x + (P^{m-1})^{-1}\xi_1(t),$$

where A_{m-1}, Q_m satisfy the conditions $(H1)_m$ and $(H2)_m$. In the following we will prove the corresponding conclusion is right when $k=m$. In fact, we can get the conclusion for any $1\le l<m$ by Taylor' expansion

$$\partial_x F(t,x^l) = \partial_x F(t,x^{l-1}) + \int_0^1 \partial_x^2 F(t, sx_l + x^{l-1})x_l\mathrm{d}s,$$

and

$$\int_0^1 \partial_x^2 F(t,sx_l + x^{l-1})x_l\mathrm{d}s = O(x_l) = O_{\sigma_l,\beta_l,\mathcal{O}_l}(\gamma_l^{-1}\epsilon_{l-1}/\beta_l^{\tau+n}). \tag{26}$$

So we can do two transformations for (25), in the first place $y' = P^{m-1}x$, then

$$\mathrm{d}y'/\mathrm{d}t = (A_{m-1} + \epsilon_m Q'_m(t,\omega))y' + (P^{m-1})^{-1}\xi_1^{m+1}(t),$$

where $Q'_m = Q_m(t,\omega) + (P^{m-1})^{-1}\int_0^1 \partial_x^2 F(t, sx_l + x^{l-1})x_l$. In the second place, from the Lemma 3.1, there exists a transformation $y = (E + \epsilon_m P_m)y'$ to take the above equation into

$$\mathrm{d}y/\mathrm{d}t = (A_m + \epsilon_{m+1}Q_{m+1}(t,\omega))y + (P^m)^{-1}\xi_1^{m+1}(t),$$

where $P^m = (E + \epsilon_m P_m)P^{m-1}$, A_m, Q_{m+1} satisfy the conditions $(H1)_{m+1}$ and $(H2)_{m+1}$. That is, there exists a transformation $y = P^m x$ to change (25) into the above equation. So the conclusion is proved by mathematical induction. What is more, from the induction process we know $P^m = \Pi_{i=1}^m (E + \epsilon_i P_i)$. Let us estimate the norms of P^m and $(P^m)^{-1}$ on $D_{m+1} \times \widetilde{\mathcal{O}}_{m+1}$, in fact, according to Lemma 3.1, $\|\mathcal{P}_i(\theta;\omega)\|^{D_i \times \widetilde{\mathcal{O}}_i} \le C(i)$. Hence,

$$\begin{aligned}\|\mathcal{P}^m(\theta;\omega)\|^{D_{m+1}\times\widetilde{\mathcal{O}}_{m+1}} \le& \Pi_{i=1}^m \|E + \epsilon_i \mathcal{P}_i(\theta;\omega)\|^{D_i\times\widetilde{\mathcal{O}}_i}\\ \le& \Pi_{i=1}^m (1 + C(i)\epsilon_i) \le 2,\end{aligned}$$

in the meanwhile

$$\|(E + \epsilon_i \mathcal{P}_i(\theta;\omega))^{-1}\| \le 1 + C\epsilon_i,$$

so $\|(\mathcal{P}^m(\theta;\omega))^{-1}\|^{D_{m+1}\times\widetilde{\mathcal{O}}_{m+1}} \le 2$, that is, P^m is bounded and has bounded inverse.

Proof of Theorem 1.1 Let $x^\infty(t) = \sum_{i\ge 1} x_i(t)$, $\Pi_\infty = \bigcap_{i=0}^\infty \Pi_i$ (See Lemma 4.2 for the estimate on the measure). In the meanwhile the sequence of coordinate changes we used in the proof is convergent

$$P^\infty := \lim_{m\to\infty} P^m := \lim_{m\to\infty} \Pi_{i=1}^m (E + \epsilon_i P_i(t)).$$

In fact, $\mathcal{O}_{m+1} \subset \widetilde{\mathcal{O}}_{m+1}$ $(m = 0, 1, \ldots)$, so

$$\begin{aligned}\|P^\infty\|^{D_\infty\times\mathcal{O}_\infty} =& \lim_{m\to\infty}\|P^m\|^{D_{m+1}\times\mathcal{O}_{m+1}} \le \Pi_{i=1}^\infty \|E + \epsilon_i P_i(t)\|^{D_{m+1}\times\widetilde{\mathcal{O}}_{m+1}}\\ \le& \Pi_{i=1}^\infty (1 + C(i)\epsilon_i) \le 2.\end{aligned}$$

Thus the infinite coordinate changes is convergent, and we have

$$\|x^\infty\|^{D_{\sigma_0/2}\times\mathcal{O}_\infty} \lessdot C\gamma^{-1}(1 + \sum_{i=1}^\infty \gamma_i^{-1}\epsilon_i) \le 2C\gamma^{-1}. \tag{27}$$

Then for $\omega \in \Pi_\infty$, $x^\infty(t) = (x_1^\infty(t), x_2^\infty(t), x_3^\infty(t))^T$ is a real analytic quasi-periodic solution of (4). Therefore, $(y_1, y_2, y_3)^T$ is a quasi-periodic solution of (1). $\square$

Remark In fact, we can get the positive quasi-periodic solution when α_i in the coordinate changes satisfy $\alpha_i > 0$, $i = 1, 2, 3$.

Lemma 4.2 (Estimates on the allowed frequencies set) *Let*

$$\Pi_\infty = \bigcap_{m=0}^\infty \Pi_m.$$

Then $\mathrm{Meas}\,\Pi_\infty = (\mathrm{Meas}\,\Pi_0)(1 - C\gamma)$.

Proof This lemma can be found in [12]. $\square$

In the foregoing paragraphs we have proved the existence of the solution of (1). Now let us prove the stability of the solution.

5. The proof of Theorem 1.2

Assume $y^*(t)=(y_1^*(t),y_2^*(t),y_3^*(t))^T$ is the special solution that we get from Newton iteration and $y(t)=(y_1(t),y_2(t),y_3(t))^T$ is any solution of (1).

It is easy to see from $\sum_{j\neq i}\lambda_{ji}+1<\lambda_{ii}$ $(i,j=1,2,3)$,

$$-(\lambda_{11}+\widetilde{h}_{11}(t))+(\lambda_{21}+\widetilde{h}_{21}(t))+(\lambda_{31}+\widetilde{h}_{31}(t))<-\mu,$$

$$(\lambda_{12}+\widetilde{h}_{12}(t))-(\lambda_{22}+\widetilde{h}_{22}(t))+(\lambda_{32}+\widetilde{h}_{32}(t))<-\mu,$$

$$(\lambda_{13}+\widetilde{h}_{13}(t))+(\lambda_{23}+\widetilde{h}_{23}(t))-(\lambda_{33}+\widetilde{h}_{33}(t))<-\mu,$$

where μ is a fixed positive constant. In order to prove the stability by Lyapunov function, we rewrite (1) with a simple vector form

$$\mathrm{d}y/\mathrm{d}t=g(t,y). \tag{28}$$

By the transformation $x=y-y^*$, (28) becomes

$$\mathrm{d}x/\mathrm{d}t=f(t,x), \tag{29}$$

where

$$f(t,x)=g(t,y)-\mathrm{d}y^*/\mathrm{d}t=g(t,x+y^*)-g(t,y^*).$$

It is obvious that $f(t,0)=0$. The V function is the key to proving the stability of solutions by Lyapunov function. Fortunately, we can define such Lyapunov function as follows

$$V(t,x)=\sum_{i=1}^{3}|\ln(x_i+y_i^*)-\ln y_i^*|=\sum_{i=1}^{3}|\ln y_i-\ln y_i^*|,$$

where $x=(x_1,x_2,x_3)^T$. It is easy to know $V(t,0)=0$, moreover, $V(t;x)>0$, for all $x\neq 0$. Thus the function V is positive definite.

Calculating the right upper derivative of $V(t,x)$ with respect to t gives

$$D^+V(t,x)=\sum_{i=1}^{3}\mathrm{sign}(y_i-y_i^*)(\frac{\dot{y}_i}{y_i}-\frac{\dot{y}_i^*}{y_i^*}).$$

That is,

$$\begin{aligned}D^+V(t,x)=&\mathrm{sign}(y_1-y_1^*)(\frac{\dot{y}_1}{y_1}-\frac{\dot{y}_1^*}{y_1^*})+\mathrm{sign}(y_2-y_2^*)(\frac{\dot{y}_2}{y_2}-\frac{\dot{y}_2^*}{y_2^*})+\mathrm{sign}(y_3-y_3^*)(\frac{\dot{y}_3}{y_3}-\frac{\dot{y}_3^*}{y_3^*})\\
=&\mathrm{sign}(y_1-y_1^*)[(\lambda_{11}+\widetilde{h}_{11}(t))(y_1^*-y_1)+\sum_{i=2}^{3}(\lambda_{1i}+\widetilde{h}_{1i}(t))(y_i^*-y_i)]+\\
&\mathrm{sign}(y_2-y_2^*)[-(\lambda_{21}+\widetilde{h}_{21}(t))(y_1^*-y_1)+\sum_{i=2}^{3}(\lambda_{2i}+\widetilde{h}_{2i}(t))(y_i^*-y_i)]+\end{aligned}$$

$$
\begin{aligned}
&\mathrm{sign}(y_3 - y_3^*)[-(\lambda_{31} + \widetilde{h}_{31}(t))(y_1^* - y_1) + \sum_{i=2}^{3}(\lambda_{3i} + \widetilde{h}_{3i}(t))(y_i^* - y_i)] \\
\leq& - (\lambda_{11} + \widetilde{h}_{11}(t))|y_1^* - y_1| + \sum_{i=2,3}(\lambda_{1i} + \widetilde{h}_{1i}(t))|y_i^* - y_i| - \\
&(\lambda_{22} + \widetilde{h}_{22}(t))|y_2^* - y_2| + \sum_{i=1,3}(\lambda_{2i} + \widetilde{h}_{2i}(t))|y_i^* - y_i| \\
&(\lambda_{33} + \widetilde{h}_{33}(t))|y_3^* - y_3| + \sum_{i=1,2}(\lambda_{3i} + \widetilde{h}_{3i}(t))|y_i^* - y_i| \\
=&(-(\lambda_{11} + \widetilde{h}_{11}(t)) + (\lambda_{21} + \widetilde{h}_{21}(t)) + (\lambda_{31} + \widetilde{h}_{31}(t)))|y_1^* - y_1| + \\
&(\lambda_{12} + \widetilde{h}_{12}(t)) - (\lambda_{22} + \widetilde{h}_{22}(t)) + (\lambda_{32} + \widetilde{h}_{32}(t)))|y_2^* - y_2| + \\
&((\lambda_{13} + \widetilde{h}_{13}(t)) + (\lambda_{23} + \widetilde{h}_{23}(t)) - (\lambda_{33} + \widetilde{h}_{33}(t)))|y_3^* - y_3| \\
\leq& - \mu\sum_{i=1}^{3}|y_i^*(t) - y_i(t)|,
\end{aligned}
$$

where *sign* is sign function and $\mu > 0$.

Thus, we know $D^+V(t,x) \leq 0$, $t \geq 0$. It follows from Lemma 6.2 that the zero solution of (28) is stable, that is,

$$\lim_{t\to\infty} |y_i^*(t) - y_i(t)| = 0, \quad i = 1, 2, 3.$$

Hence, Theorem 1.2 is proved.

6. Technical lemmas

Lemma 6.1 *For $\delta > 0, \nu > 0$, the following inequality holds:*

$$\sum_{k\in\mathbb{Z}^n} e^{-2|k|\delta}|k|^\nu \leq (\frac{\nu}{e})^\nu \frac{1}{\delta^{\nu+n}}(1+e)^n.$$

Proof The proof can be found in [5]. □

Lemma 6.2 *Given the system* $\mathrm{d}x/\mathrm{d}t = f(t, x(t), x(t-\sigma(t)))$ *where* $x \in \mathbb{R}^n$, $f \in \mathbb{C}[I \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$, $f(t,0,0) \equiv 0$, $0 \leq \sigma(t) < \infty$. *If there exists a positive definite function $V(t,x)$ in the set $G_H := \{(t,x), t \geq t_0, \|x\| < H\}$ so that $D^+V \leq 0$, then the zero solution of the system is stable.*

Proof The proof can be found in [11, p. 315]. □

Lemma 6.3 (Cauchy inequality) *Let E and F be two complex Banach spaces with norms $\|\cdot\|_E$ and $\|\cdot\|_F$, and let G be an analytic map from the open ball of radius r around v in E into F such that $\|G\|_F \leq M$ on the ball. The first derivative d_vG of G at v is a linear map from E into F, whose induced operator norm is*

$$\|d_vG\|_{F,E} = \max_{u\neq 0} \frac{\|d_vG(u)\|_F}{\|u\|_E}.$$

Then $\|d_vG\|_{F,E} \leq \frac{M}{r}$.

Proof Let $u \neq 0$ in E. Then $f(z) = F(v + zu)$ is an analytic map from the complex disc

$\|z\| < r/\|z\|_E$ in $\mathbb{C}$ into F that is uniformly bounded by M. Hence,

$$\|d_0 f\|_F = \|d_v F(u)\|_F \leq \frac{M}{r}\|u\|_E$$

by the usual Cauchy inequality. The above statement follows, since $u \neq 0$ was arbitrary.

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