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Maps Preserving Commutativity up to a Factor on Standard Operator Algebras

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Abstract Let X, Y be real or complex Banach spaces with dimension greater than 2 and $\mathcal{A}$, $\mathcal{B}$ be standard operator algebras on X and Y, respectively. Let $\Phi:\mathcal{A}\to\mathcal{B}$ be a unital surjective map. In this paper, we characterize the map Φ on $\mathcal{A}$ which satisfies $(A-B)R=\xi R(A-B)\Leftrightarrow(\Phi(A)-\Phi(B))\Phi(R)=\xi\Phi(R)(\Phi(A)-\Phi(B))$ for $A,B,R\in\mathcal{A}$ and for some scalar ξ.

Keywords preservers; standard operator algebras; commutativity up to a factor.

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1. Introduction

Let x and y be two elements in an algebra. If x and y satisfy the algebraic relation $xy=\xi yx$ for a nonzero scalar ξ, we say that x and y are commutative up to a factor. If x and y are commutative up to some factor, they are also said to be of quasi-commutativity. The commutativity up to a factor has important applications in quantum mechanics. We refer the reader to [1] for more information.

Transformations on quantum structures which preserve some relation or operation are usually called symmetries in physics and have been studied by different authors [2]. From a mathematical point of view, maps preserving given algebraic property are called preservers and are extensively studied. Let $\mathcal{A}$ and $\mathcal{B}$ be algebras. Recall that a map Φ from $\mathcal{A}$ into $\mathcal{B}$ preserves commutativity up to a factor ξ in both directions if $AB=\xi BA\Leftrightarrow\Phi(A)\Phi(B)=\xi\Phi(B)\Phi(A)$. The assumption of preserving commutativity up to a factor can be reformulated as preserving ξ-Lie zero products. The concept of ξ-Lie products is introduced and discussed in the recent papers by Qi and Hou (Ref. [3, 4] and the references therein).

Actually, when $\xi=0,\pm1$, motivated by theory and applications, the problem for maps preserving commutativity up to a factor ξ has been studied in many literatures [5–11]. For $\xi\neq 0,\pm1$, the study of preserving commutativity up to a factor was initiated by Cui and Hou in [12]. Particularly, authors in [12] gave a characterization of unital additive surjections which preserve commutativity up to a factor in both directions between standard operator algebras on real or

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complex infinite dimensional Banach spaces. Molnár in [13] characterized bijective linear maps on the $n \times n$ complex matrices or on the $n \times n$ self-adjoint matrices preserving commutativity up to a factor in both directions. Recently, we in [14] characterized surjective weakly continuous linear map on nest subalgebras with non-trivial nests of any factor von Neumann algebra preserving commutativity up to a factor in both directions.

The purpose of this paper is to improve the result of [12] by replacing the assumptions of "additive" and "preserving commutativity up to a factor" by the assumption that $(A-B)R = \xi R(A-B) \Leftrightarrow (\Phi(A)-\Phi(B))\Phi(R) = \xi\Phi(R)(\Phi(A)-\Phi(B))$ for $A, B, R \in \mathcal{A}$ and for some $\xi \in \mathbb{F}$ with $\xi \neq 0, 1$. Here we mention that the proof of the main theorem is different from the one in [12] and our result holds for finite dimensional case, too.

Let X be a Banach space over the field $\mathbb{F}(= \mathbb{R}$ or $\mathbb{C}$, the field of real numbers or the field of complex numbers), and $\mathcal{B}(X)$ denote the algebra of all bounded linear operators on X. As usual, denote by $\mathcal{F}(X)$ the set of all finite rank operators and $\mathcal{I}_1(X)$ the set of all rank one idempotent operators in $\mathcal{B}(X)$. A standard operator algebra $\mathcal{A}$ on X is a subalgebra (not necessarily closed) of $\mathcal{B}(X)$ which contains the identity I and $\mathcal{F}(X)$. The dual of X will be denoted by X' and the conjugate of $T \in \mathcal{B}(X)$ by T' throughout.

2. Main result and proof

Theorem 2.1 *Let X, Y be infinite dimensional Banach spaces over the real or complex field $\mathbb{F}$ and $\mathcal{A}$, $\mathcal{B}$ be standard operator algebras on X and Y, respectively. Let ξ be a scalar with $\xi \neq 0, 1$ and $\Phi : \mathcal{A} \to \mathcal{B}$ be a unital surjective map with the property that $(A-B)R = \xi R(A-B) \Leftrightarrow (\Phi(A)-\Phi(B))\Phi(R) = \xi\Phi(R)(\Phi(A)-\Phi(B))$ for any $A, B, R \in \mathcal{A}$. Then one of the followings holds.*

(1) X is real, there exists an invertible bounded linear operator $T : X \to Y$ such that $\Phi(A) = TAT^{-1}$, $\forall A \in \mathcal{A}$.

(2) X is complex , then one of the (a)–(c) holds.

(a) $\xi \in \mathbb{R}$, then there exists an invertible bounded linear or conjugate linear operator $T : X \to Y$ such that $\Phi(A) = TAT^{-1}$, $\forall A \in \mathcal{A}$;

(b) $\xi \in \mathbb{C} \setminus \mathbb{R}$ and $|\xi| \neq 1$, there exists an invertible bounded linear operator $T : X \to Y$ such that $\Phi(A) = TAT^{-1}$, $\forall A \in \mathcal{A}$;

(c) $|\xi| = 1$, either there exists an invertible bounded linear operator $T : X \to Y$ such that $\Phi(A) = TAT^{-1}$, $\forall A \in \mathcal{A}$ or there exists an invertible bounded conjugate linear operator $T : X' \to Y$ such that $\Phi(A) = TA'T^{-1}$, $\forall A \in \mathcal{A}$.

For X is finite-dimensional case, we can suppose Φ acts on the space of $n \times n$ complex matrices ($n = \dim X$).

Theorem 2.2 *Let ξ be a scalar with $\xi \neq 0, 1$ and $\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ $(n > 2)$ be a unital surjective map. Let Φ satisfy that $(A-B)R = \xi R(A-B) \Leftrightarrow (\Phi(A)-\Phi(B))\Phi(R) = \xi\Phi(R)(\Phi(A)-\Phi(B))$ for any $A, B, R \in M_n(\mathbb{F})$. Then one of the followings holds.*

(1) If $\mathbb{F}$ is the real field $\mathbb{R}$, then there exists a nonsingular matrix $T \in M_n(\mathbb{R})$ such that

$\Phi(A)=TAT^{-1}$, $\forall A\in M_n(\mathbb{F})$.

(2) If $\mathbb{F}$ is complex field $\mathbb{C}$, then there exist a nonsingular matrix $T\in M_n(\mathbb{C})$ and a ring automorphism τ of $\mathbb{C}$ with $\tau(\xi)=\xi$ such that either $\Phi(A)=T\tau(A)T^{-1}$, $\forall A\in M_n(\mathbb{F})$ or there exist a nonsingular matrix $T\in M_n(\mathbb{C})$ and a ring automorphism τ of $\mathbb{C}$ with $\tau(\xi)=\frac{1}{\xi}$ such that $\Phi(A)=T\tau(A)^{tr}T^{-1}$, $\forall A\in M_n(\mathbb{F})$.

Here $\tau(A)$ denotes the matrix obtained from A by applying τ to every entry of it and $\tau(A)^{tr}$ is the transpose of $\tau(A)$.

To prove Theorems 2.1 and 2.2, we need the following lemmas.

Lemma 2.3 *Let X, Y be infinite dimensional Banach spaces over the real or complex field $\mathbb{F}$ and $\Phi:\mathcal{I}_1(X)\to\mathcal{I}_1(Y)$ be a bijective map with the property*

$$PQ=QP=0\Leftrightarrow\Phi(P)\Phi(Q)=\Phi(Q)\Phi(P)=0$$

for all $P,Q\in\mathcal{I}_1(X)$. Then one of the followings holds.

(1) If X is real, then either there exists an invertible bounded linear operator $T:X\to Y$ such that

$$\Phi(P)=TPT^{-1},\quad\forall P\in\mathcal{I}_1(X),$$

or there exists an invertible bounded linear operator $T:X'\to Y$ such that

$$\Phi(P)=TP'T^{-1},\quad\forall P\in\mathcal{I}_1(X).$$

(2) If X is complex, then either there exists an invertible bounded linear or conjugate linear operator $T:X\to Y$ such that

$$\Phi(P)=TPT^{-1},\quad\forall P\in\mathcal{I}_1(X),$$

or there exists an invertible bounded linear or conjugate linear operator $T:X'\to Y$ such that

$$\Phi(P)=TP'T^{-1},\quad\forall P\in\mathcal{I}_1(X).$$

Denote by $\mathcal{I}_1(\mathbb{F}^n)$ the set of all rank one idempotent matrices in $M_n(\mathbb{F})$ $(n>2)$.

Lemma 2.4 *Let $\Phi:\mathcal{I}_1(\mathbb{F}^n)\to\mathcal{I}_1(\mathbb{F}^n)$ be a bijective map with the property*

$$PQ=QP=0\Leftrightarrow\Phi(P)\Phi(Q)=\Phi(Q)\Phi(P)=0$$

for all $P,Q\in\mathcal{I}_1(\mathbb{F}^n)$. Then one of the followings holds.

(1) If $\mathbb{F}$ is the real field $\mathbb{R}$, then either there exists a nonsingular matrix $T\in M_n(\mathbb{R})$ such that

$$\Phi(P)=TPT^{-1},\quad\forall P\in\mathcal{I}_1(\mathbb{R}^n),$$

or

$$\Phi(P)=TP^{tr}T^{-1},\quad\forall P\in\mathcal{I}_1(\mathbb{R}^n).$$

(2) If $\mathbb{F}$ is the complex field $\mathbb{C}$, then either there exist a nonsingular matrix $T \in M_n(\mathbb{C})$ and a ring automorphism τ of $\mathbb{C}$ such that either Φ is of the form

$$\Phi(P) = T\tau(P)T^{-1}, \quad \forall P \in \mathcal{I}_1(\mathbb{C}^n),$$

or

$$\Phi(P) = T\tau(P)^{tr}T^{-1}, \quad \forall P \in \mathcal{I}_1(\mathbb{C}^n).$$

The proofs of Lemmas 2.3 and 2.4 are similar to that of [15, Main Theorem] and we omit them here.

The Proof of Theorem 2.1 We complete the proof by checking several claims.

Claim 1 $\Phi(0) = 0$ and Φ is injective.

It is clear to see that $(A - B)0 = \xi 0(A - B)$ for arbitrary $A, B \in \mathcal{A}$, and this implies that $(\Phi(A) - \Phi(B))\Phi(0) = \xi\Phi(0)(\Phi(A) - \Phi(B))$. Noticing that Φ is surjective, take $\Phi(A) = I$ and $\Phi(B) = 0$, thus we have $\Phi(0) = 0$.

Suppose that $\Phi(A) = \Phi(B)$ for some $A, B \in \mathcal{A}$. Then for every $R \in \mathcal{A}$, $(\Phi(A)-\Phi(B))\Phi(R) = \xi\Phi(R)(\Phi(A) - \Phi(B))$ which implies that $(A - B)R = \xi R(A - B)$. Taking $R = I$, one arrives at $A = B$. Hence, Φ is injective.

Claim 2 Φ preserves idempotents in both directions and preserves square-zero. Moreover, for every idempotent $P \in \mathcal{A}$ and every scalar $\alpha \in \mathbb{F}$, there exists a bijective function $h : \mathbb{F} \to \mathbb{F}$ such that $\Phi(\alpha P) = h(\alpha)\Phi(P)$.

It follows from $(I - P)P = \xi P(I - P)$ that $\Phi(P) - \Phi(P)^2 = \xi\Phi(P) - \xi\Phi(P)^2$ for every idempotent $P \in \mathcal{A}$, which yields that $\Phi(P) = \Phi(P)^2$, i.e., Φ preserves idempotents. Let $D \in \mathcal{A}$ with $D^2 = 0$. It follows from $(0-D)D = \xi D(0-D)$ that $\Phi(D)^2 = 0$, i.e., Φ preserves square-zero. Considering Φ^{-1}, we have that Φ preserves idempotents in both directions. For arbitrary scalar $\alpha \in \mathbb{F}$, $(I - P)(\alpha P) = \xi(\alpha P)(I - P)$. Hence,

$$(1 - \xi)\Phi(\alpha P) = \Phi(P)\Phi(\alpha P) - \xi\Phi(\alpha P)\Phi(P). \tag{2.1}$$

Multiplying Eq.(2.1) from the left and the right by $\Phi(P)$, respectively, leads to

$$\Phi(P)\Phi(\alpha P) = \Phi(\alpha P)\Phi(P) = \Phi(P)\Phi(\alpha P)\Phi(P). \tag{2.2}$$

Substituting Eq.(2.2) into Eq.(2.1), we get

$$\Phi(\alpha P) = \Phi(P)\Phi(\alpha P). \tag{2.3}$$

On the other hand, it follows from $(\alpha I - \alpha P)P = \xi P(\alpha I - \alpha P)$ that

$$\Phi(\alpha I)\Phi(P) - \Phi(\alpha P)\Phi(P) = \xi\Phi(P)\Phi(\alpha I) - \xi\Phi(P)\Phi(\alpha P).$$

By Eq.(2.2), one obtains that

$$\Phi(\alpha I)\Phi(P) - \xi\Phi(P)\Phi(\alpha I) = (1 - \xi)\Phi(P)\Phi(\alpha P). \tag{2.4}$$

Multiplying the both sides of Eq.(2.4) by $\Phi(P)$, respectively, leads to

$$(1+\xi)\Phi(P)\Phi(\alpha I)\Phi(P) = \xi\Phi(P)\Phi(\alpha I) + \Phi(\alpha I)\Phi(P). \tag{2.5}$$

Multiplying the both sides of Eq.(2.5) by $\Phi(P)$, respectively, leads to

$$(1-\xi)\Phi(P)\Phi(\alpha I)\Phi(P) = \Phi(\alpha I)\Phi(P) - \xi\Phi(P)\Phi(\alpha I). \tag{2.6}$$

Comparing Eqs.(2.5) and (2.6), we see that

$$\Phi(P)\Phi(\alpha I)\Phi(P) = \Phi(\alpha I)\Phi(P) = \Phi(P)\Phi(\alpha I). \tag{2.7}$$

Since Φ is surjective and preserves idempotents in both directions, it follows from Eq.(2.7) and the fact that every operator in $\mathcal{F}(X)$ can be written as the linear combination of finite many idempotents of finite rank in $\mathcal{B}(X)$ that

$$\Phi(\alpha I) = h(\alpha)I \tag{2.8}$$

for some scalar $h(\alpha) \in \mathbb{F}$. It is clear that $h : \mathbb{F} \to \mathbb{F}$ is bijective as Φ is. Substituting Eq.(2.8) into Eq.(2.4) and by Eq.(2.3), we see that $\Phi(\alpha P) = h(\alpha)\Phi(P)$.

Claim 3 The theorem holds true for all rank one idempotents in $\mathcal{A}$.

Firstly, we show that Φ preserves the orthogonality of idempotents. Indeed, let idempotents $P_1, P_2 \in \mathcal{A}$ satisfy $P_1P_2 = P_2P_1 = 0$. It follows from $P_1P_2 = \xi P_2P_1$ that

$$\Phi(P_1)\Phi(P_2) = \xi\Phi(P_2)\Phi(P_1). \tag{2.9}$$

Multiplying the both sides of Eq.(2.9) by $\Phi(P_1)$, respectively, leads to

$$\Phi(P_1)\Phi(P_2) = \xi^2\Phi(P_2)\Phi(P_1) = \xi\Phi(P_2)\Phi(P_1).$$

It is clear that $\Phi(P_1)\Phi(P_2) = \Phi(P_2)\Phi(P_1) = 0$. Next we show that Φ preserves the order of idempotents. Let $P_1, P_2 \in \mathcal{A}$ be two idempotents with $P_1 \leq P_2$. It follows from $(I - \Phi(P_2))\Phi(P_1) = \xi\Phi(P_1)(I - \Phi(P_2))$ that

$$(1-\xi)\Phi(P_1) = \Phi(P_2)\Phi(P_1) - \xi\Phi(P_1)\Phi(P_2).$$

Multiplying the above equation by $\Phi(P_1)$, respectively, we get $\Phi(P_1) = \Phi(P_2)\Phi(P_1) = \Phi(P_1)\Phi(P_2)$. Therefore, $\Phi(P_1) \leq \Phi(P_2)$. Consequently, Φ preserves rank one idempotents.

Since Φ^{-1} has the same property as Φ, Φ maps $\mathcal{I}_1(X)$ onto $\mathcal{I}_1(Y)$ and preserves the orthogonality of rank one idempotents. By Lemma 2.3, the followings hold.

If $\Phi(P) = TPT^{-1}$ holds for all $P \in \mathcal{I}_1(X)$, we define $\Psi : \mathcal{A} \to \mathcal{C}$ by

$$\Psi(A) = T^{-1}\Phi(A)T, \quad \forall A \in \mathcal{A},$$

where $\mathcal{C}$ is a standard operator algebra on Y. If $\Phi(P) = TP'T^{-1}$ for all $P \in \mathcal{I}_1(X)$, then $(T^{-1})'\pi$ is bijective, here π is the natural embedding of X into X''. Therefore, X is reflective. We define $\Psi : \mathcal{A} \to \mathcal{D}$ by

$$\Psi(A) = \pi^{-1}T'\Phi(A)'(T^{-1})'\pi, \quad \forall A \in \mathcal{A},$$

where $\mathcal{D}$ is a standard operator algebra on Y.

Obviously, in any case, Ψ has the same properties as Φ, that is, Ψ is unital and satisfies $(A-B)R=\xi R(A-B) \Leftrightarrow (\Psi(A)-\Psi(B))\Psi(R)=\xi\Psi(R)(\Psi(A)-\Psi(B))$. Moreover, $\Psi(P)=P$ for all $P\in\mathcal{I}_1(X)$. Next we show that $\Psi(A)=A$ for all $A\in\mathcal{A}$.

Claim 4 For every rank one nilpotent operator $x\otimes f\in\mathcal{A}$, $\Psi(x\otimes f)=x\otimes f$.

Choose $y\in X$ and $g\in X'$ with $f(y)=1$, $g(x)=1$ and $g(y)=0$. It is easy to check that $(-\xi x\otimes g-y\otimes f)(x\otimes f)=\xi(x\otimes f)(-\xi x\otimes g-y\otimes f)$. From Claims 2 and 3, we know that $\Psi(-\xi x\otimes g)=h(-\xi)x\otimes g$. Then

$$(h(-\xi)x\otimes g-y\otimes f)\Psi(x\otimes f)=\xi\Psi(x\otimes f)(h(-\xi)x\otimes g-y\otimes f). \tag{2.10}$$

Letting the both sides of Eq.(2.10) act at x, one gets that

$$h(-\xi)\langle\Psi(x\otimes f)x,g\rangle x-\langle\Psi(x\otimes f)x,f\rangle y=h(-\xi)\xi\Psi(x\otimes f)x.$$

Since $h(-\xi)\neq 0$ by the injectivity of h, $\langle\Psi(x\otimes f)x,g\rangle=\xi\langle\Psi(x\otimes f)x,g\rangle$. Therefore, we have $\langle\Psi(x\otimes f)x,g\rangle=0$. It follows that

$$\Psi(x\otimes f)x=-\frac{1}{h(-\xi)\xi}\langle\Psi(x\otimes f)x,f\rangle y. \tag{2.11}$$

Letting the both sides of (2.10) act at y, one sees that

$$h(-\xi)\langle\Psi(x\otimes f)y,g\rangle x-\langle\Psi(x\otimes f)y,f\rangle y=-\xi\Psi(x\otimes f)y.$$

So

$$\langle\Psi(x\otimes f)y,f\rangle=\xi\langle\Psi(x\otimes f)y,f\rangle.$$

Hence, we have $\langle\Psi(x\otimes f)y,f\rangle=0$. It follows that

$$\Psi(x\otimes f)y=-\frac{h(-\xi)}{\xi}\langle\Psi(x\otimes f)y,g\rangle x. \tag{2.12}$$

For $\forall z\in\ker f\cap\ker g$, choose $k\in X'$ such that $k(z)=1$ and $k(x)=0$. Obviously, $(x\otimes f)(z\otimes k)=\xi(z\otimes k)(x\otimes f)$, which implies that $\Psi(x\otimes f)(z\otimes k)=\xi(z\otimes k)\Psi(x\otimes f)$. Multiplying this equation by $z\otimes k$, respectively, we easily check that $\Psi(x\otimes f)z\otimes k=0$ and $\Psi(x\otimes f)z=0$. This, together with Eqs(2.11) and (2.12) imply that there exist some scalars $\gamma,\delta\in\mathbb{F}$ such that $\Psi(x\otimes f)=\gamma x\otimes f+\delta y\otimes g$. Because Ψ preserves square-zero by Claim 2, it follows from $(x\otimes f)^2=0$ that $\Psi(x\otimes f)^2=(\gamma x\otimes f+\delta y\otimes g)^2=0$. Therefore, $\gamma\delta=0$ and consequently, we have $\Psi(x\otimes f)=\gamma x\otimes f$ or $\Psi(x\otimes f)=\delta y\otimes g$. Since $(x\otimes g-x\otimes f)(x+y)\otimes f=\xi(x+y)\otimes f(x\otimes g-x\otimes f)$,

$$(x\otimes g-\Psi(x\otimes f))(x\otimes f+y\otimes f)=\xi(x\otimes f+y\otimes f)(x\otimes g-\Psi(x\otimes f)). \tag{2.13}$$

If $\Psi(x\otimes f)=\delta y\otimes g$, then $x\otimes(f+\delta\xi g)=\delta y\otimes(f-\xi g)$, which is impossible since f and g are linear independent. Thus $\Psi(x\otimes f)=\gamma x\otimes f$. Substituting this into Eq.(2.13), we get $\gamma=1$.

Claim 5 $\Psi(\alpha P)=\alpha P$ for every scalar $\alpha\in\mathbb{F}$ and every $P\in\mathcal{I}_1(X)$. Moreover, for every rank one operator $D\in\mathcal{A}$, $\Psi(D)=D$.

According to Claims 2 and 3, for every $P\in\mathcal{I}_1(X)$ and every scalar $\alpha\in\mathbb{F}$, there exists a bijective function $h:\mathbb{F}\to\mathbb{F}$ such that $\Psi(\alpha P)=h(\alpha)P$. What remains is to show that $h(\alpha)=\alpha$.

Pick $x, y \in X$ and $f, g \in X'$ such that $f(x) = g(y) = 1$ and $f(y) = g(x) = 0$. It follows from $(\alpha y \otimes g - \alpha^2 x \otimes g)x \otimes (f + \alpha g) = \xi x \otimes (f + \alpha g)(\alpha y \otimes g - \alpha^2 x \otimes g)$ that

$$\begin{aligned}0 &= (h(\alpha)y \otimes g - \alpha^2 x \otimes g)x \otimes (f + \alpha g) = \xi x \otimes (f + \alpha g)(h(\alpha)y \otimes g - \alpha^2 x \otimes g)\\ &= \xi(\alpha h(\alpha)x \otimes g - \alpha^2 x \otimes g).\end{aligned}$$

So $h(\alpha) = \alpha$. Moreover, by Claim 4, $\Psi(D) = D$ for every rank one operator $D \in \mathcal{A}$.

Claim 6 $\Psi(A) = A$ for every $A \in \mathcal{A}$.

For every $x \in X$ and $f \in X'$ with $f(x) = 1$, it is easy to see that $\mathrm{Rng}(A - (x \otimes f)A) \subseteq \mathrm{Rng}(I - x \otimes f)$. Thus there is a non-zero linear functional $k \in X'$ such that $k|_{\mathrm{Rng}(A-(x\otimes f)A)} = 0$. Denote $A - (x \otimes f)A$ by B. Let $y \in X$ be arbitrary, $g \in X'$ with $g(y) = 1$. Clearly, $(B - By \otimes g)(y \otimes k) = \xi(y \otimes k)(B - By \otimes g)$. So $(\Psi(B) - By \otimes g)(y \otimes k) = \xi(y \otimes k)(\Psi(B) - By \otimes g)$ and a simple computation leads to $\Psi(B) = B$.

To prove Claim 6, we may assume that $\mathrm{rank}(A) > 1$. For every nonzero $x \in X$ take nonzero $z \in X$ and $f \in X'$ such that $f(x) = 1$ and $f(Az) = 0$. Above paragraph shows that $\Psi(A - (x \otimes f)A) = A - (x \otimes f)A$. Note that, for every $k \in X'$ with $k(x) = 0$, we have

$$(A - (A - (x \otimes f)A))(z \otimes k) = \xi(z \otimes k)(A - (A - (x \otimes f)A)).$$

Thus

$$(\Psi(A) - (A - (x \otimes f)A))(z \otimes k) = \xi(z \otimes k)(\Psi(A) - (A - (x \otimes f)A)). \tag{2.14}$$

Letting Eq.(2.14) act at x yields $\langle(\Psi(A) - A)x, k\rangle = 0$. From the arbitrariness of k, it follows that $(\Psi(A) - A)x \in [x]$ for every $x \in X$. This implies that $\Psi(A) - A = \lambda I$ for some scalar $\lambda \in \mathbb{F}$. By Eq.(2.14), it is easily checked that $\lambda = 0$. So we have $\Psi(A) = A$, as desired. The remainder is to show that Φ has the form described in Theorem 2.1 for all elements in $\mathcal{A}$.

Claim 7 The statements of the theorem hold true.

By the above several claims, now we know that the followings hold.

If X is real, then either there exists an invertible bounded linear operator $T : X \to Y$ such that

$$\Phi(A) = TAT^{-1}, \quad \forall A \in \mathcal{A},$$

or there exists an invertible bounded linear operator $T : X' \to Y$ such that

$$\Phi(A) = TA'T^{-1}, \quad \forall A \in \mathcal{A}.$$

If X is complex, then either there exists an invertible bounded linear or conjugate linear operator $T : X \to Y$ such that

$$\Phi(A) = TAT^{-1}, \quad \forall A \in \mathcal{A},$$

or there exists an invertible bounded linear or conjugate linear operator $T : X' \to Y$ such that

$$\Phi(A) = TA'T^{-1}, \quad \forall A \in \mathcal{A}.$$

Assume that Φ take the form $\Phi(A) = TAT^{-1}$ for all $A \in \mathcal{A}$, where $T : X \to Y$ is an invertible bounded linear or conjugate linear operator. We assert that T cannot be conjugate linear if $\xi \in \mathbb{C} \setminus \mathbb{R}$. Indeed, if T is conjugate linear, since Φ preserves commutativity up to the factor ξ in both directions, choosing $R, S \in \mathcal{A}$ with $RS = \xi SR \neq 0$, we get $(\xi - \bar{\xi})SR = 0$, and hence $\xi = \bar{\xi}$. This is a contradiction.

Assume that Φ take the form $\Phi(A) = TA'T^{-1}$ for all $A \in \mathcal{A}$. We assert that $T : X' \to Y$ cannot be an invertible bounded linear operator. Indeed, choosing $R, S \in \mathcal{A}$ with $RS = \xi SR \neq 0$, we get $TR'S'T^{-1} = \xi TS'R'T^{-1} = T\xi S'R'T^{-1}$. Thus $SR = \xi^2 SR$, contradicting $\xi \neq \pm 1$.

Only for the case $|\xi| = 1$, Φ may take the form $\Phi(A) = TA'T^{-1}$ for all $A \in \mathcal{A}$, where $T : X' \to Y$ is an invertible bounded conjugate linear operator. If T is conjugate linear, then a direct computation implies that $SR = |\xi| SR$ for $R, S \in \mathcal{A}$ satisfying $RS = \xi SR \neq 0$. Hence, $|\xi| = 1$. The proof is completed. $\square$

For the proof of Theorem 2.2, using Lemma 2.4, by a similar arguments as in the proof of Theorem 2.1, we can get the followings hold.

(I) If $\mathbb{F}$ is the real field $\mathbb{R}$, then either there exists a nonsingular matrix $T \in M_n(\mathbb{R})$ such that

$$\Phi(A) = TAT^{-1}, \quad \forall A \in M_n(\mathbb{R}),$$

or there exists a nonsingular matrix $T \in M_n(\mathbb{R})$ such that

$$\Phi(A) = TA^{tr}T^{-1}, \quad \forall A \in M_n(\mathbb{R}).$$

(II) If $\mathbb{F}$ is the complex field $\mathbb{C}$, then there exist a nonsingular matrix $T \in M_n(\mathbb{C})$ and a ring automorphism τ of $\mathbb{C}$ such that either Φ is of the form

$$\Phi(A) = T\tau(A)T^{-1}, \quad \forall A \in M_n(\mathbb{C}),$$

or

$$\Phi(A) = T\tau(A)^{tr}T^{-1}, \quad \forall A \in M_n(\mathbb{C}).$$

If Φ takes the form $\Phi(A) = T\tau(A)T^{-1}$ for all $A \in M_n(\mathbb{C})$, according to the property of map preserving commutativity up to a factor, then it is easy to see $\tau(\xi) = \xi$. Similarly, if Φ takes the form $\Phi(A) = T\tau(A)^{tr}T^{-1}$ for all $A \in M_n(\mathbb{C})$, then it is easy to see $\tau(\xi) = \frac{1}{\xi}$.

Remark 2.5 To get a characterization of surjective maps between standard operator algebras that preserve commutativity up to a factor in both directions, we conjecture that the assumption "Φ is unital" is not necessary. We are not able to solve this conjecture in the present paper. We pose this as an open problem.

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