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A Class of Projectively Flat Finsler Metrics

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Abstract In this paper, the authors study a class of Finsler metric defined by a Riemannian metric and a 1-form. We find a necessary and sufficient condition for the metric to be prejectively flat.

Keywords projectively flat; flag curvature; (α,β)-metrics.

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1. Introduction

It is an important problem in Finsler geometry to study and characterize projectively flat Finsler metric on an open domain in R^n. The flag curvature is an analogue of the sectional curvature in Riemannian geometry. Projectively flat Finsler metrics are of scalar flag curvature, but the flag curvature is not necessarily constant in contrast to the Riemannian case.

The main purpose of this paper is to study and characterize certain projectively flat Finsler metrics.

On every strongly convex domain $\mathcal{U}$ in R^n, Hilbert constructed a complete reversible projectively flat metric $H=H(x,y)$ with negative constant flag curvature $\mathbf{K}=-1$. Then Funk constructed a positively projectively flat metric $\Theta=\Theta(x,y)$ with $\mathbf{K}=-1/4$ on $\mathcal{U}$ so that its symmetrization is just the Hilbert metric, $H(x,y)=\frac{1}{2}(\Theta(x,y)+\Theta(x,-y))$. When $\mathcal{U}=\mathrm{B}^n$ is the unit ball in R^n, the Funk metric is given by

$$\Theta=\frac{\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}{1-|x|^2}+\frac{\langle x,y\rangle}{1-|x|^2},$$

where $y\in T_x\mathrm{B}^n\cong\mathrm{R}^n$. Here $|\cdot|$ and $\langle,\rangle$ denote the standard Euclidean norm and inner product. The Funk metric Θ on B^n is a special Randers metric expressed in the form

$$\Theta=\overline{\alpha}+\overline{\beta},$$

where

$$\overline{\alpha}=\frac{\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}{1-|x|^2},\quad \overline{\beta}=\frac{\langle x,y\rangle}{1-|x|^2}.$$

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Recently, Shen [1] and Yang [2] respectively studied the following projectively flat Finsler metrics:

$$F=\alpha+\varepsilon\beta+k\frac{\beta^2}{\alpha}. \tag{1.1}$$

In [3], Shen gave a necessary and sufficient condition for the following Finsler metric to be projectively flat:

$$F=\alpha+\varepsilon\beta+2k\frac{\beta^2}{\alpha}-\frac{k^2\beta^4}{3\alpha^3}. \tag{1.2}$$

The above discussion leads us to study the following function F on the tangent bundle TM of a manifold M,

$$F=\alpha(1+a_1s+a_2s^2+a_4s^4+\cdots+a_{2n}s^{2n}),\quad s=\frac{\beta}{\alpha}, \tag{1.3}$$

where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta=b_i(x)y^i$ is a 1-form on M and $a_1,a_2,a_4,\dots,a_{2n}$ are constants. By observation and calculations, we find a simple sufficient condition for F in (1.3) to be projectively flat, if $a_2,\dots,a_{2n}$ in (1.3) satisfy

$$a_{2k}=(-1)^{k-1}\lambda_{k-1}a_2^k,\lambda_1=\frac{n-1}{6n},\dots,\lambda_l=\frac{(2l-1)(1-\frac{l}{n})}{(l+1)(2l+1)}\lambda_{l-1}, \tag{1.4}$$

where $k=2,\dots,n,l=2,\dots,n-1$.

In this paper, we shall first prove the following:

Theorem 1.1 *Let $a_2\neq 0$, and F in (1.3) be a Finsler metric on a manifold M satisfying (1.4). F is projectively flat if and only if*

(i)

$$b_{i|j}=\tau[(a_2^{-1}+2b^2)a_{ij}-\frac{2n+1}{n}b_ib_j], \tag{1.5}$$

(ii) the spray coefficients G^i_α of α are in the form:

$$G^i_\alpha=\theta y^i-\tau\alpha^2b^i, \tag{1.6}$$

where $b:=\|\beta_x\|_\alpha$, $b_{i|j}$ denotes the covariant derivatives of β with respect to α, $\tau=\tau(x)$ is a scalar function and $\theta=\theta_i(x)y^i$ is a 1-form on M. In this case

$$G^i=(\theta+\tau\chi\alpha)y^i, \tag{1.7}$$

where

$$\chi:=\frac{\phi'(1-\frac{1}{n}a_2s^2)}{2a_2\phi}-s,\quad s=\frac{\alpha}{\beta}, \tag{1.8}$$

where $\phi=1+a_1s+a_2s^2+a_4s^4+\cdots+a_{2n}s^{2n}$.

About this class of projectively flat metric, we have

Corollary 1.2 *Suppose that $F=\alpha+a_1\beta+a_2\frac{\beta^2}{\alpha}+\cdots+a_{2n}\frac{\beta^{2n}}{\alpha^{2n-1}}$ with $a_2\neq 0$ is projectively flat with constant flag curvature $\mathbf{K}=\lambda=constant$, then $\lambda=0$.*

Corollary 1.3 *Let $F=\alpha+a_1\beta+a_2\frac{\beta^2}{\alpha}+\cdots+a_{2n}\frac{\beta^{2n}}{\alpha^{2n-1}}$, where $a_2\neq 0$. Suppose that F is a locally projectively flat metric with zero flag curvature. If $\tau=0$, then α is flat metric and β is* *parallel with respect to α. In this case, F is locally Minkowskian.*

Remark 1 When $n = 1, a_1 = \varepsilon, a_2 = k$, Theorem 1.1 is reduced to Theorem 3.1 in [1]. When $n = 2, a_1 = \varepsilon, a_2 = 2k, \tau = 2k\widetilde{\tau}, \chi = \frac{1}{k}\widetilde{\chi}$, Theorem 1.1 is reduced to Theorem 1.1 in [3], where $\widetilde{\tau}$ and $\widetilde{\chi}$ are equivalent to τ and χ in [3], respectively.

Remark 2 When $n = 1, a_1 = \varepsilon, a_2 = k$, Corollaries 1.2 and 1.3 are reduced to Lemma 4.1 Proposition 4.2 in [1], respectively. When $n = 2, a_1 = \varepsilon, a_2 = 2k$, Corollaries 1.2 and 1.3 are reduced to Lemma 5.1 and Proposition 5.2 in [3], respectively.

2. (α, β)-metrics

The Finsler metric in (1.3) is a special (α, β)-metric. By definition, an (α, β)-metric is expressed in the following form,

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form, $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \leq b < b_0.$$

It is known that F is a Finsler metric if and only if $\|\beta_x\|_\alpha < b_0$ for any $x \in M$ (see [4]). Let G^i and G^i_α denote the spray coefficients of F and α, respectively, given by

$$G^i = \frac{g^{il}}{4}\{[F^2]_{x^k y^l} y^k - [F^2]_{x^k}\}, \quad G^i_\alpha = \frac{a^{il}}{4}\{[\alpha^2]_{x^k y^l} y^k - [\alpha^2]_{x^k}\},$$

where $(g^{ij}) := (\frac{1}{2}[F^2]_{y^i y^j})$ and $(a^{ij}) := (a_{ij})^{-1}$. We have the following

Lemma 2.1 *The geodesic coefficients G^i are related to G^i_α by*

$$G^i = G^i_\alpha + \alpha Q s^i_0 + J(-2Q\alpha s_0 + r_{00})\frac{y^i}{\alpha} + H(-2Q\alpha s_0 + r_{00})(b^i - s\frac{y^i}{\alpha}), \tag{2.1}$$

where

$$Q := \frac{\phi'}{\phi - s\phi'},$$
$$J := \frac{\phi'(\phi - s\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')},$$
$$H := \frac{\phi''}{2((\phi - s\phi') + (b^2 - s^2)\phi'')},$$

where $s := \frac{\beta}{\alpha}$ and $b := \|\beta_x\|_\alpha$. The formula (2.1) is given in [4].

It is well-known that a Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathrm{R}^n$ is projectively flat if and only if

$$F_{x^k y^l} y^k - F_{x^l} = 0. \tag{2.2}$$

This is due to Hamel [5]. From [1] and [5], we have

Lemma 2.2 *An (α, β)-metric $F = \alpha\phi(s)$, where $s = \frac{\beta}{\alpha}$, is projectively flat on an open subset*

$\mathcal{U} \subset \mathrm{R}^n$ *if and only if*

$$(a_{mi}\alpha^2 - y_m y_i)G^i_\alpha + \alpha^3 Q s_{i0} + H\alpha(-2\alpha Q s_0 + r_{00})(b_i\alpha - s y_i) = 0. \tag{2.3}$$

3. Polynomial (α, β)-metrics

If an (α, β)-metric is Finsler metric and has the following form:

$$F := \alpha(1 + a_1 s + a_2 s^2 + \cdots + a_n s^n) = \alpha(1 + \frac{a_1\beta}{\alpha} + \frac{a_2\beta^2}{\alpha^2} + \cdots + \frac{a_n\beta^n}{\alpha^n}), \tag{3.1}$$

we call it a polynomial (α, β)-metric. In this paper, we study the polynomial (α, β)-metric having the form (3.1).

First, we hope β is closed when F in (3.1) is projectively flat. From (2.3), F must be in the form (1.3), that is,

$$a_{2k-1} = 0, \quad k = 2, \ldots, n.$$

Secondly, from [1] and [3], we assume

$$a_{2k} = (-1)^{k-1}\lambda_{k-1} a_2^k, \quad k = 2, \ldots, n.$$

Then, by direct calculations, we have

$$H = \frac{U}{V},$$

where

$$U = a_2 - 6\lambda_1 a_2^2 s^2 + \cdots + (-1)^{n-1} n(2n-1)\lambda_{n-1} a_2^n s^{2n-2}, \tag{3.2}$$

$$\begin{aligned} V =& (1 + 2a_2 b^2) - (12\lambda_1 a_2^2 b^2 + 3a_2)s^2 + \cdots + [(-1)^{n-1} 2n(2n-1)\lambda_{n-1} a_2^n b^2 - \\ & (-1)^{n-2}(2n-1)(2n-3)\lambda_{n-2} a_2^{n-1}] s^{2n-2} - (-1)^{n-1}(2n+1)(2n-1)\lambda_{n-1} a_2^n s^{2n}. \end{aligned} \tag{3.3}$$

Since β is closed, equation (2.3) is reduced to the following

$$(a_{mi}\alpha^2 - y_m y_i)G^i_\alpha + H\alpha(b_i\alpha - s y_i) r_{00} = 0. \tag{3.4}$$

We can get r_{00} from (3.4), but to have a simple form, from [1] and [3], we should make

$$H = \frac{a_2(1 - 6\lambda_1 a_2 s^2 + \cdots + (-1)^{n-1} n(2n-1)\lambda_{n-1} a_2^{n-1} s^{2n-2})}{(1 + 2a_2 b^2 - \frac{2n+1}{n} a_2 s^2)(1 - 6\lambda_1 a_2 s^2 + \cdots + (-1)^{n-1} n(2n-1)\lambda_{n-1} a_2^{n-1} s^{2n-2})}. \tag{3.5}$$

From

$$\begin{aligned} &(1 + 2a_2 b^2 - \frac{2n+1}{n} a_2 s^2)(1 - 6\lambda_1 a_2 s^2 + \cdots + (-1)^{n-1} n(2n-1)\lambda_{n-1} a_2^{n-1} s^{2n-2}) \\ &= (1 + 2a_2 b^2) - (12\lambda_1 a_2^2 b^2 + 3a_2)s^2 + \cdots + [(-1)^{n-1} 2n(2n-1)\lambda_{n-1} a_2^n b^2 - (-1)^{n-2} \\ &\quad (2n-1)(2n-3)\lambda_{n-2} a_2^{n-1}] s^{2n-2} - (-1)^{n-1}(2n+1)(2n-1)\lambda_{n-1} a_2^n s^{2n}, \end{aligned}$$

we get

$$\lambda_1 = \frac{n-1}{6n}, \ldots, \lambda_l = \frac{(2l-1)(1 - \frac{l}{n})}{(l+1)(2l+1)}\lambda_{l-1}, \quad l = 2, \ldots, n-1. \tag{3.6}$$

So we study the Finsler metric satisfying condition (1.4) with the form in (1.3).

4. A class of projectively flat finsler metrics

By Lemma 2.1 and (1.4), the spray coefficients G^i of F are given by (2.1) with

$$Q := \frac{a_1\alpha^{2n} + 2a_2\alpha^{2n-1}\beta + \cdots + 2na_{2n}\alpha\beta^{2n-1}}{\alpha^{2n} - a_2\alpha^{2n-2}\beta^2 - \cdots - (2n-1)a_{2n}\beta^{2n}}, \tag{4.1}$$

$$J := \frac{\phi'(1 - \frac{1}{n}a_2 s^2)}{2\phi\left(1 + 2a_2 b^2 - \frac{2n+1}{n}a_2 s^2\right)}, \tag{4.2}$$

$$H := \frac{a_2\alpha^2}{(1 + 2a_2b^2)\alpha^2 - \frac{2n+1}{n}a_2\beta^2}, \tag{4.3}$$

Equation (2.3) is reduced to the following equation:

$$\begin{aligned}0 =&(\alpha^{2n} - \cdots - (2n-1)a_{2n}\beta^{2n})\big((1+2a_2b^2)\alpha^2 - \frac{2n+1}{n}a_2\beta^2\big)(a_{mi}\alpha^2 - y_m y_i)G^i_\alpha + \\ &\big((1+2a_2b^2)\alpha^2 - \frac{2n+1}{n}a_2\beta^2\big)\alpha^3(a_1\alpha^{2n} + \cdots + 2na_{2n}\alpha\beta^{2n-1})s_{i0} - \\ &2a_2\alpha^3(a_1\alpha^{2n} + 2a_2\alpha^{2n-1}\beta + \cdots + 2na_{2n}\alpha\beta^{2n-1})s_0(b_i\alpha^2 - \beta y_i) + \\ &(\alpha^{2n} - a_2\alpha^{2n-2}\beta^2 - \cdots - (2n-1)a_{2n}\beta^{2n})a_2\alpha^2 r_{00}(b_i\alpha^2 - \beta y_i).\end{aligned} \tag{4.4}$$

The coefficients of α must be zero (note: α^{even} is a polynomial in y^i). We obtain

$$a_1\alpha^{2n+3}\big((1+2a_2b^2)\alpha^2 - \frac{2n+1}{n}a_2\beta^2\big)s_{i0} = 2a_1a_2\alpha^{2n+3}s_0(b_i\alpha^2 - \beta y_i). \tag{4.5}$$

Suppose that $a_1 \neq 0$. Then

$$\big((1+2a_2b^2)\alpha^2 - \frac{2n+1}{n}a_2\beta^2\big)s_{i0} = 2a_2 s_0(b_i\alpha^2 - \beta y_i). \tag{4.6}$$

Contracting (4.6) with b^i yields

$$(\alpha^2 - \frac{1}{n}a_2\beta^2)s_0 = 0. \tag{4.7}$$

From (4.1) and (4.2), we have

$$\alpha^2 - \frac{1}{n}a_2\beta^2 \neq 0.$$

Thus $s_0 = 0$. Then it follows from (4.6) that

$$s_{i0} = 0. \tag{4.8}$$

Thus β is closed.

Suppose that $a_1 = 0$. Then (4.4) is reduced to the following

$$\begin{aligned}&\big\{\big((1+2a_2b^2)\alpha^2 - \frac{2n+1}{n}a_2\beta^2\big)(a_{mi}\alpha^2 - y_m y_i)G^i_\alpha + a_2\alpha^2 r_{00}(b_i\alpha^2 - \beta y_i)\big\} \\ &\quad(\alpha^{2n} - a_2\alpha^{2n-2}\beta^2 - \cdots - (2n-1)a_{2n}\beta^{2n}) \\ &= -\big\{s_{i0}\big((1+2a_2b^2)\alpha^2 - \frac{2n+1}{n}a_2\beta^2\big) - 2a_2s_0(b_i\alpha^2 - \beta y_i)\big\} \\ &\quad\alpha^3(2a_2\alpha^{2n-1}\beta + \cdots + 2na_{2n}\alpha\beta^{2n-1}).\end{aligned} \tag{4.9}$$

Note that $\alpha^3(2a_2\alpha^{2n-1}\beta + \cdots + 2na_{2n}\alpha\beta^{2n-1})$ is not divisible by $\alpha^{2n} - a_2\alpha^{2n-2}\beta^2 - \cdots - (2n-1)a_{2n}\beta^{2n}$. Thus $s_{i0}((1+2a_2b^2)\alpha^2 - \frac{2n+1}{n}a_2\beta^2) - 2a_2s_0(b_i\alpha^2 - \beta y_i)$ is divisible by $\alpha^{2n} - a_2\alpha^{2n-2}\beta^2 -$

$\cdots - (2n-1)a_{2n}\beta^{2n}$. But this is impossible unless

$$s_{i0}\big((1+2a_2b^2)\alpha^2 - \frac{2n+1}{n}a_2\beta^2\big) = 2a_2s_0(b_i\alpha^2 - \beta y_i). \tag{4.10}$$

By the discussion under the supposition $a_1 \neq 0$, we have β is closed, too.

Since β is closed, equation (3.4) is reduced to the following

$$\big((1+2a_2b^2)\alpha^2 - \frac{2n+1}{n}a_2\beta^2\big)(a_{mi}\alpha^2 - y_my_i)G^i_\alpha + a_2\alpha^2 r_{00}(b_i\alpha^2 - \beta y_i) = 0. \tag{4.11}$$

Contracting (4.11) with b^i, we get

$$\big((1+2a_2b^2)\alpha^2 - \frac{2n+1}{n}a_2\beta^2\big)(b_m\alpha^2 - y_m\beta)G^i_\alpha = -a_2\alpha^2(b^2\alpha^2 - \beta^2)r_{00}. \tag{4.12}$$

Note that $(1+2a_2b^2)\alpha^2 - \frac{2n+1}{n}a_2\beta^2$ is divisible by $b^2\alpha^2 - \beta^2$ if and only if

$$b^2 = \frac{n}{a_2}, \quad a_2 > 0. \tag{4.13}$$

In this case, from the definition of (α,β)-metric and equation (3.5), we know F is not a Finsler metric. Thus $(1+2a_2b^2)\alpha^2 - \frac{2n+1}{n}a_2\beta^2$ is not divisible by α^2 and $b^2\alpha^2-\beta^2$. Then $(b_m\alpha^2 - y_m\beta)G^i_\alpha$ is divisible by $\alpha^2(b^2\alpha^2-\beta^2)$. Therefore, there is a scalar function $\tau = \tau(x)$ such that

$$r_{00} = \frac{\tau}{a_2}[(1+2a_2b^2)\alpha^2 - \frac{2n+1}{n}a_2\beta^2]. \tag{4.14}$$

Then

$$b_{i|j} = \frac{\tau}{a_2}[(1+2a_2b^2)a_{ij} - \frac{2n+1}{n}a_2b_ib_j]. \tag{4.15}$$

By $s_{i0} = 0$ and (4.14), the formula (2.1) for G^i can be simplified to

$$G^i = G^i_\alpha + \tau\chi\alpha y^i + \tau\alpha^2 b^i, \tag{4.16}$$

where χ is given in (1.8). We know that F is projectively flat if and only if

$$G^i = Py^i.$$

By (4.16), this is equivalent to the following

$$G^i_\alpha = \theta y^i - \tau\alpha^2 b^i, \tag{4.17}$$

where $\theta = \theta_i(x)y^i$ is a 1-form. In this case, G^i is given by (1.7).

Below is a special example satisfying (1.5) and (1.6).

Example 4.1 Let $F = \alpha\phi(s)$ be an (α,β)-metric on an open subset $\mathcal{U} \in \mathrm{R}^n$. Define

$$\phi(s) = 1 + 2^n s + 2n\sum_{k=0}^{n-1}\frac{(-1)^k C_k^{n-1}s^{2k+2}}{(2k+1)(2k+2)}, \tag{4.18}$$

where

$$C_k^m := \frac{m(m-1)\cdots(m-k+1)}{k!},$$

and

$$s = \frac{\beta}{\alpha}, \quad \alpha := \frac{\zeta^n}{\omega^n}\tilde{\alpha}, \quad \beta := \frac{\zeta^n}{\omega^n}\tilde{\beta}, \quad \tilde{\alpha} = \frac{\eta}{\omega^2}, \quad \tilde{\beta} = \frac{\langle x,y\rangle}{\omega^2} + \frac{\langle a,y\rangle}{\zeta}, \tag{4.19}$$

where

$$\omega := \sqrt{1 - |x|^2}, \tag{4.20}$$

$$\zeta := 1 + \langle a, x \rangle, \tag{4.21}$$

$$\eta := \sqrt{(1 - |x|^2)|y|^2 + \langle x, y \rangle^2}, \tag{4.22}$$

where $a \in \mathrm{R}^n$ is a constant vector with $|a| < 1$. Thus F satisfies (1.4), and it is projectively flat. By direct calculations from (4.19)–(4.22), α and β satisfy (1.5) and (1.6).

5. Flag curvature

In this section, we shall study the following metric with constant flag curvature $\mathbf{K} = \lambda$,

$$F = \alpha + a_1\beta + a_2\frac{\beta^2}{\alpha} + \cdots + a_{2n}\frac{\beta^{2n}}{\alpha^{2n-1}},$$

where $a_1, \ldots, a_{2n}$ are constants with $a_2 \neq 0$. We assume that F is locally projectively flat so that in a local coordinate system the spray coefficients of F are in the form (1.7). It is known that if the spray coefficients of F are in the form $G^i = Py^i$, then F is of scalar curvature with flag curvature

$$\mathbf{K} = \frac{P^2 - P_{x^k}y^k}{F^2}.$$

Then

$$\mathbf{K} = \frac{(\theta + \tau\chi\alpha)^2 - \theta_{x^k}y^k - \tau_{x^k}y^k\chi\alpha - \tau\chi'(s)s_{x^k}y^k\alpha - \tau\chi\alpha_{x^k}y^k}{F^2}. \tag{5.1}$$

Observe that

$$\begin{aligned} s_{x^k}y^k &= \frac{r_{00}}{\alpha} + \frac{2}{\alpha^2}(b_m\alpha - sy_m)G^i_\alpha \\ &= \tau\left(\frac{1}{a_2} - \frac{1}{n}s^2\right)\alpha, \\ \alpha_{x^k}y^k &= \frac{2}{\alpha}G^m_\alpha y_m = 2(\theta - \tau\beta)\alpha. \end{aligned}$$

We obtain

$$\mathbf{K} = \frac{\theta^2 - \theta_{x^k}y^k + \tau^2\chi^2\alpha^2 - \tau_{x^k}y^k\chi\alpha - \tau^2(\frac{1}{a_2} - \frac{1}{n}s^2)\chi'(s)\alpha^2 + 2s\tau^2\chi\alpha^2}{F^2}. \tag{5.2}$$

Proposition 5.1 *Suppose that $F = \alpha + a_1\beta + a_2\frac{\beta^2}{\alpha} + \cdots + a_{2n}\frac{\beta^{2n}}{\alpha^{2n-1}}$ with $a_2 \neq 0$ is projectively flat with constant flag curvature $\mathbf{K} = \lambda = constant$, then $\lambda = 0$.*

Proof First by (5.2), the equation $\mathbf{K} = \lambda$ multiplied by $\alpha^{8n-4}F^4$ yields:

$$A\alpha^{2n+1} + B\alpha^{2n} - 4a_1a_{2n}^3\beta^{6n+1}\alpha^{2n-1}\lambda - \cdots - 4a_{2n-2}a_{2n}^3\beta^{8n-2}\lambda\alpha^2 - \lambda a_{2n}^4\beta^{8n} = 0, \tag{5.3}$$

where A and B are homogeneous polynomials in y of degree $6n-1$ and $6n$, respectively. Rewriting the above equation as

$$(A\alpha^2 - 4a_1a_{2n}^3\beta^{6n+1}\lambda)\alpha^{2n-1} + (B\alpha_{2n} - \cdots - 4a_{2n-2}a_{2n}^3\beta^{8n-2}\lambda\alpha^2 - a_{2n}^4\beta^{8n}\lambda) = 0, \tag{5.4}$$

we must have

$$A\alpha^2 - 4a_1a_{2n}^3\beta^{6n+1}\lambda = 0, \quad (B\alpha^{2n-2} - \cdots - 4a_{2n-2}a_{2n}^3\beta^{8n-2}\lambda)\alpha^2 = a_{2n}^4\beta^{8n}\lambda. \tag{5.5}$$

From (3.6), we know

$$a_4 = -\frac{n-1}{6n}a_2^2, \ a_{2l} = C_l(n)a_2^l, \quad 2 \le l \le n-1,$$

where $C_l(n)$ is a function associated with n and l. Since β^2 is not divisible by α and $a_2 \neq 0$, we conclude from the second identity in (5.5) that $\lambda = 0$. $\Box$

Now, we consider the case when $\tau = 0$. In this case, from (1.5) and (1.6), we have

$$b_{i|j} = 0, \quad G^i = G^i_\alpha = \theta y^i.$$

By Lemma 5.1, F has zero flag curvature. Thus α is locally isometric to the Euclidean metric. We have proved the following

Proposition 5.2 *Let* $F = \alpha + a_1\beta + a_2\frac{\beta^2}{\alpha} + \cdots + a_{2n}\frac{\beta^{2n}}{\alpha^{2n-1}}$, *where* $a_2 \neq 0$. *Suppose that* F *is a locally projectively flat metric with zero flag curvature. If* $\tau = 0$, *then* α *is flat metric and* β *is parallel with respect to* α. *In this case,* F *is locally Minkowskian.*

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