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On a Kind of Series Summation Utilizing C-Numbers

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Abstract This paper provides a pair of summation formulas for a kind of combinatorial series involving $\binom{ak+b}{m}$ as a factor of the summand. The construction of formulas is based on a certain series transformation formula [2, 7, 9] and by making use of the C-numbers [3]. Various consequences and examples including several remarkable classic identities are presented to illustrate some applications of the formulas obtained.

Keywords combinatorial series; summation formula; difference operator; generalized Stirling number; C-number.

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1. Introduction

We are concerned chiefly with a summation problem for a kind of combinatorial series of the form

$$S(f)=\sum_{k\geq 0}\binom{ak+b}{m}c_k t^k$$

where $a,b\in\mathbb{R}$ with $a\neq 0$, $m\in\mathbb{N}$, and

$$f(t)=\sum_{k\geq 0}c_k t^k=\sum_{k\geq 0}f^{(k)}(0)\frac{t^k}{k!}$$

generally stands for a formal power series while $f^{(k)}(0)$ denotes the formal derivatives of $f(t)$ at $t=0$. The term "combinatorial series" means that the number of terms in the series may be infinite or finite. Our consideration of $S(f)$ is motivated by the two facts, firstly, that various combinatorial sums or series with the factor $\binom{ak+b}{m}$ involved in the summands could not be generally evaluated to a closed form in terms of "elementary terms" as described in Comtet's book [1, p.216], and secondly that a discrete random variable X assuming values $k\in\mathbb{N}$ with probabilities p_k has the binomial moment and the mth moment of the following forms

$$\beta_m(X)=\sum_{k\geq 0}\binom{k}{m}p_k \quad\text{and}\quad \mu_m(X)=\sum_{k\geq 0}k^m p_k$$

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respectively, where $\binom{k}{m} = \binom{ak+b}{m}$ with $(a,b)=(1,0)$, and k^m is given by the limit

$$\lim_{a\to\infty} \frac{m!}{a^m}\binom{ak+b}{m} = k^m.$$

Since we are interested in the exact evaluation of $S(f)$, not using it as a formal series for manipulations, we will confine ourselves to the case where $f(t)$ is a real analytic function having a power series expansion within a convergence interval. We will show that the summation problem of $S(f)$ could be treated by using a series transformation formula (first given by He et al's [2]), and by making use of the C-numbers (also called non-central generalized factorial coefficients) fully developed by Charalambides-Koutras [3, 4]. The basic tools to be used will be given as preliminaries (§2). Shortly, we will also establish a summation formula for a type of series similar to $S(f)$, in which t^k is replaced by the falling factorial $(t)_k$, where $(t)_0 = 1$ and $(t)_k = t(t-1)\cdots(t-k+1)$ $(k \geq 1)$.

Our main result is a pair of theorems together with several corollaries and some summation formulas (§3). In the last section (§4) we will show by examples that some well–known formulas/identities due respectively to Euler, Stirling, Dobinski, Vandermonde, Carlitz-Li(Shanlan), and Knuth, et al. are all implied by the main theorems or the corresponding corollaries as presented in §3.

2. Preliminaries

It is known that the C-numbers $C(n,k;a,b)$ may be defined by the following relation (cf. Koutras [4] or Charalambides [5, p.317])

$$(a\,t+b)_n = \sum_{k=0}^{n} C(n,k;a,b)(t)_k, \quad n \in \mathbb{N} \tag{2.1}$$

with $C(0,0;a,b)=1$ and $C(n,k;a,b)=0$ for $k>n$. As for more interesting applications of $C(n,k;a,b)$ in combinatorics and statistics, we refer the reader to Charalambides and Koutras [3, 4, 6]. Using Newton's interpolation formula we may denote

$$C(n,k;a,b) = \frac{1}{k!}\Delta^k (at+b)_n\big|_{t=0} \tag{2.2}$$

where $\Delta^k\phi(0)$ denotes the kth difference of $\phi(t)$ at 0, and the difference operator Δ is defined by

$$\Delta\phi(t) = \phi(t+1) - \phi(t),\ \ \Delta^{k+1} = \Delta(\Delta^k), \quad k \geq 0$$

with $\Delta^0 = \mathbf{1}$ denoting the identity operator, viz., $\mathbf{1}\phi(t) = \phi(t)$. In view of (2.2), the C-number can be explicitly expressed by

$$C(n,k;a,b) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(aj+b)_n. \tag{2.3}$$

Moreover, the following recurrence relation is useful for computations:

$$C(n+1,k;a,b) = (ak+b-n)\ C(n,k;a,b) + a\ C(n,k-1;a,b) \tag{2.4}$$

where $n+1 \geq k \geq 1$ and $C(n,0;a,b) = (b)_n$.

Recalling the generalized factorial $(t|\alpha)_n$ defined by

$$(t|\alpha)_0 = 1,\ (t|\alpha)_n = t(t-\alpha)\cdots(t-n\alpha+\alpha),\quad n \geq 1$$

with $(t|1)_n = (t)_n$, we should mention that the so-called three-parameter generalized Stirling number $S(n,k;\alpha,\beta,\gamma)$ with $(\alpha,\beta,\gamma) \neq (0,0,0)$, put forward first by Hsu-Shiue in [7], is given by

$$(t+\gamma|\alpha)_n = \sum_{k=0}^{n} S(n,k;\alpha,\beta,\gamma)(t|\beta)_k. \tag{2.5}$$

A comparison of (2.1) with (2.5) yields, in the case $(\alpha,\beta,\gamma) = (1/a, 1, 1/b)$, that

$$C(n,k;a,b) = a^n S(n,k;1/a,1,b/a).$$

Consequently we have (loc. cit.)

$$\lim_{a\to\infty} a^{-n} C(n,k;a,b) = S(n,k;0,1,0) = \left\{ {n \atop k} \right\} \tag{2.6}$$

where $\left\{ {n \atop k} \right\}$ is Knuth's notation (cf. [8, p.259]) for the Stirling number of the second kind, viz.

$$\left\{ {n \atop k} \right\} := \frac{1}{k!}\left(\Delta^k t^n\right)\Big|_{t=0}.$$

Also, in the next section we will utilize a series transformation formula of the form

$$\sum_{k=0}^{\infty} \phi(k) f^{(k)}(0)\frac{t^k}{k!} = \sum_{k=0}^{\infty} \Delta^k \phi(0) f^{(k)}(t)\frac{t^k}{k!}. \tag{2.7}$$

Clearly (2.7) gives the formal Maclaurin series expansion for the case $\phi(t) \equiv 1$. Formula (2.7) was obtained and utilized formally in [2]. Convergence results with applications have been presented in our recent paper [9].

3. Theorems and corollaries

Let us state our main theorems as follows.

Theorem 3.1 *Let $f(t)$ be a real function having Maclaurin series expansion for $|t| < \rho$. Then the following summation formula*

$$\sum_{k=0}^{\infty} \binom{ak+b}{m} f^{(k)}(0)\frac{t^k}{k!} = \frac{1}{m!}\sum_{j=0}^{m} C(m,j;a,b) f^{(j)}(t)\ t^j \tag{3.1}$$

is valid for $|t| < \rho$, where $a \neq 0$ and $m \in \mathbb{N}$.

Theorem 3.2 *Let $g(t)$ be a real function defined on $\mathbb{R} = (-\infty, \infty)$. Then the following summation formula*

$$\sum_{k=0}^{\infty} \binom{ak+b}{m}\binom{x}{k} \Delta^k g(0) = \frac{1}{m!}\sum_{j=0}^{m} C(m,j;a,b)(x)_j \Delta^j g(x-j) \tag{3.2}$$

is valid under the condition $\overline{\lim}_k |\Delta^k g(0)|^{1/k} < 1$, *where* $a \neq 0$ *and* $x \in \mathbb{R}$.

Here it is obvious that $\binom{ak+b}{m}$ may be viewed as the weight coefficients, so that (3.1) and (3.2) just stand for the summation formulas for the weighted Maclaurin series expansion and the weighted Newton interpolation series, respectively, because the Maclaurin expansion and the Newton series just correspond to the special case $m = 0$.

Proofs of Theorems 3.1 and 3.2 Certainly, (2.7) is applicable to (3.1) by taking $\phi(t) = \binom{at+b}{m}$. To justify (3.1) as a formal identity, it suffices to verify the relation

$$\Delta^j \binom{at+b}{m}\bigg|_{t=0} = \frac{j!}{m!} C(m, j; a, b). \tag{3.3}$$

Actually this follows at once from the definition of C-numbers:

$$\text{LHS of (3.3)} = \frac{j!}{m!}\left[\frac{1}{j!}\Delta^j (at+b)_m\right]_{t=0} = \text{RHS of (3.3)}.$$

Furthermore, notice that

$$\lim_{k\to\infty} \left|\binom{ak+b}{m}^{1/k}\right| = 1.$$

Thus, according to Cauchy-Hadamard's formula for the radius of convergence, we see that the power series on the LHS of (3.1) has the same radius of convergence as that of the power series expansion of $f(t)$. This confirms the truth of Theorem 3.1.

To prove Theorem 3.2, let us first show that formula (3.2) could be deduced from Theorem 3.1 with two steps. First, by the choice $f(t) = (1+t)^x$ with a fixed real number $x \neq 0$, we can specialize formula (3.1) to the form

$$\sum_{k=0}^{\infty} \binom{ak+b}{m}\binom{x}{k} t^k = \frac{1}{m!}\sum_{j=0}^{m} C(m, j; a, b)(x)_j (1+t)^{x-j} t^j. \tag{3.4}$$

The second step is to construct an operator analogue of (3.4) by using the symbolic operator approach expounded in [2, 10]. To do this, we need only to make the substitution $t \mapsto \Delta$ on both sides of (3.4). Using the basic relation for the shift operator $E = \Delta + \mathbf{1}$, we see that $(1+t)^{x-j} t^j \mapsto \Delta^j (\mathbf{1}+\Delta)^{x-j} = \Delta^j E^{x-j}$. Consequently the desired result is given by

$$\sum_{k=0}^{\infty} \binom{ak+b}{m}\binom{x}{k} \Delta^k = \frac{1}{m!}\sum_{j=0}^{m} C(m, j; a, b)(x)_j E^{x-j} \Delta^j, \tag{3.5}$$

where all the operators involved are still assumed to be acting on functions of t. Thus, (3.2) is obtained by applying the operator series (3.5) to $g(t)$, and noting that $E^{x-j} g(0) = g(x-j)$. Finally, by Cauchy's root test we see that the convergence condition mentioned in Theorem 3.2 sufficiently ensures the absolute convergence of the series on the LHS of (3.2). This completes the proof of Theorem 3.2.

Theorems 3.1 and 3.2 have a number of particular consequences of some interest, which may be stated as corollaries.

Corollary 3.3 *Let* $f(t)$ *have a Maclaurin expansion for* $|t| < \rho$ *with* $\rho > 1$. *Then there holds* *the summation formula*

$$\sum_{k=0}^{\infty}\binom{ak+b}{m}\frac{f^{(k)}(0)}{k!}=\frac{1}{m!}\sum_{j=0}^{m}C(m,j;a,b)f^{(j)}(1). \tag{3.6}$$

Corollary 3.4 *With the same conditions as in Corollary 3.3, there holds the formula*

$$\sum_{k=0}^{\infty}f^{(k)}(0)\frac{k^m}{k!}=\sum_{j=0}^{m}\left\{ {m \atop j} \right\}f^{(j)}(1). \tag{3.7}$$

Corollary 3.5 (Grunert's formula) ([11]) *Let $f(t)$ have a Maclaurin expansion for $|t|<\rho$. Then the following formula is valid for $|t|<\rho$:*

$$\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^m f(t):=\sum_{k=0}^{\infty}k^m f^{(k)}(0)\frac{t^k}{k!}=\sum_{j=0}^{m}\left\{ {m \atop j} \right\}f^{(j)}(t)t^j. \tag{3.8}$$

Actually, both (3.7) and (3.8) are the limits with $a\to\infty$ of (3.6) and (3.1), respectively, after multiplying both sides of (3.6) and (3.1) by the number $m!a^{-m}$. All that is needed is the simple fact as shown in [7]. We remark that the operator $\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^m$ on the leftmost side of (3.8) plays an important role in Analysis and Combinatorics. As regards this aspect, we refer the reader to [12, §6.6], [13] and references therein for more details.

Amongst other interesting cases of Theorem 3.2 are the following two corollaries.

Corollary 3.6 *For $m\in\mathbb{N}$, there holds*

$$\sum_{k=0}^{\infty}(-1)^k\binom{ak+b}{m}\Delta^k g(0)=\frac{1}{m!}\sum_{j=0}^{m}C(m,j;a,b)\Delta^j g(-1-j). \tag{3.9}$$

This last formula is obviously the special case $x=-1$ of (3.2).

Corollary 3.7 *For $m\in\mathbb{N}$, there holds*

$$\sum_{k=0}^{\infty}\binom{k+b}{m}\binom{x}{k}\Delta^k g(0)=\sum_{j=0}^{m}\binom{b}{m-j}\binom{x}{j}\Delta^j g(x-j). \tag{3.10}$$

Observing that for $a=1$, (3.3) gives

$$\frac{j!}{m!}C(m,j;1,b)=\Delta^j\binom{t+b}{m}\bigg|_{t=0}=\binom{b}{m-j},$$

and formula (3.10) is thus a direct consequence of (3.2) with $a=1$.

Note that $\Delta\binom{-t}{k}=-\binom{-t-1}{k-1}, k\geq 1$, so that

$$(t\Delta)\binom{-t}{k}=\frac{(-t)(-t-1)_{k-1}}{(k-1)!}=k\binom{-t}{k}$$

and it follows by induction that

$$(t\Delta)^m\binom{-t}{k}=k^m\binom{-t}{k},\quad m\in\mathbb{N}. \tag{3.11}$$

Thus as an analogue to the case of the operator $(td/dt)^n$, Theorem 3.2 also implies

Corollary 3.8 *For $m \in \mathbb{N}$, there holds*

$$\big(t\Delta\big)^m g(-t) := \sum_{k=0}^{\infty} k^m \Delta^k g(0) \binom{-t}{k} = \sum_{j=0}^{m} \left\{ {m \atop j} \right\} (-t)_j \, \Delta^j g(-t-j). \tag{3.12}$$

Recall that Melzak's formula [14] can be restated in a slight different way as

Lemma 3.9 *Let $g(t) = \sum_{i=0}^{m} a_i t^i$. Then, for arbitrary $n \geq m$, there holds*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{g(y-k)}{z+k} = \frac{g(y+z)}{z\binom{z+n}{n}}, \quad z \neq 0, -1, -2, \ldots, -n. \tag{3.13}$$

Clearly, the LHS of (3.13) may be written as a difference $(-1)^n \Delta^n (g(y-t)/(z+t))|_{t=0}$. Thus, combining Melzak's formula with Theorem 3.2, we come to a rather general summation formula immediately.

Corollary 3.10 *For $p \in \mathbb{N}$, let $g(t)$ be a polynomial of degree p and $h(t) = g(y-t)/(z+t)$. Then we have*

$$\sum_{k=0}^{p-1} \binom{ak+b}{m} \binom{x}{k} \Delta^k h(0) + \frac{g(y+z)}{z} \sum_{k=p}^{\infty} (-1)^k \binom{ak+b}{m} \frac{\binom{x}{k}}{\binom{z+k}{k}}$$
$$= \frac{1}{m!} \sum_{j=0}^{m} C(m,j;a,b)(x)_j \Delta^j h(x-j) \tag{3.14}$$

provided that $z \geq m+2$.

Proof It suffices to apply Lemma 3.9 to the partial series $\sum_{k=p}^{\infty} \binom{ak+b}{m} \binom{x}{k} \Delta^k h(0)$ on the LHS of (3.2) and note that the resulting series on the LHS of (3.14) converges absolutely for $z \geq m+2$, since

$$1 \Big/ \binom{z+k}{k} \leq 1 \Big/ \binom{m+2+k}{k} = 1 \Big/ \binom{m+2+k}{m+2} = \mathcal{O}(k^{-m-2}). \quad \square$$

Remark 3.11 As is suggested by Newton's backward interpolation formula, there is a summation formula parallel to (3.2) of Theorem 3.2, namely

$$\sum_{k=0}^{\infty} \binom{ak+b}{m} \binom{x+k-1}{k} \Delta^k g(-k) = \frac{1}{m!} \sum_{j=0}^{m} C(m,j;a,b)(x+j-1)_j \Delta^j g(x) \tag{3.15}$$

the last formula being valid under the condition $\overline{\lim}_k |\Delta^k g(-k)|^{1/k} < 1$. The proof follows exactly the similar lines as that of proving (3.2). More precisely, the first step is to get a formal series in powers of t by setting $f(t) = (1-t)^{-x}$ in Theorem 3.1; and the second step is to replace t by the operator ΔE^{-1}, where ΔE^{-1} commutes with E^α and $(\Delta E^{-1})t = 1$, so that ΔE^{-1} is also a "delta operator". Consequently formula (3.15) is obtainable by applying the resulted operator series to get $g(t)$ and using the operator relations

$$(\Delta E^{-1})^k g(0) = \Delta^k g(-k), \;\; (1-\Delta E^{-1})^{-x-j} (\Delta E^{-1})^j g(0) = E^x \Delta^j g(0) = \Delta^j g(x).$$

As a consequence of (3.15), also an analogue to (3.12) in Corollary 3.8, we have

$$\sum_{k=0}^{\infty} k^m \Delta^k g(-k)\binom{t+k-1}{k} = \sum_{j=0}^{m} \left\{ {m \atop j} \right\}(t+j-1)_j \, \Delta^j g(t). \tag{3.16}$$

By use of (3.11) we easily see that (3.16) implies Jordan's formula (cf. Charalambides [5, Example 8.5]) via (3.15) with $x \to t$ and $m = 0$:

$$\big(t\Delta\big)^m g(t) = \sum_{j=0}^{m} \left\{ {m \atop j} \right\}(t+j-1)_j \, \Delta^j g(t). \tag{3.17}$$

4. Selected examples

In what follows we will present several examples illustrating some applications of the basic source formula (3.1) and its consequences (corollaries, etc.) established in the previous section.

Example 4.1 In (3.6) taking $f(t) = \frac{1-t^{n+1}}{1-t} = 1 + t + t^2 + \cdots + t^n$, $|t| < 1$, we have

$$f^{(k)}(0) = k!\,(k \le n), \;\; f^{(j)}(1) = j!\binom{n+1}{j+1}.$$

Thus (3.6) leads to

$$\sum_{k=0}^{n} \binom{ak+b}{m} = \frac{1}{m!}\sum_{j=0}^{m} C(m,j;a,b) j!\binom{n+1}{j+1}. \tag{4.1}$$

In particular, from (4.1) and (2.6) or directly from (2.7), we easily deduce that

$$\sum_{k=0}^{n} k^m = \sum_{j=0}^{m} j!\left\{ {m \atop j} \right\}\binom{n+1}{j+1}. \tag{4.2}$$

This is the familiar formula of Stirling for the higher-order arithmetic progression.

Example 4.2 In (3.6), take $f(t) = e^t$. Then

$$f^{(k)}(0) = 1 \;\text{ and }\; f^{(j)}(1) = e,$$

which reduce (3.6) and (3.7) to the following two identities respectively

$$\frac{m!}{e}\sum_{k=0}^{\infty} \frac{1}{k!}\binom{ak+b}{m} = \sum_{j=0}^{m} C(m,j;a,b); \tag{4.3}$$

$$\frac{1}{e}\sum_{k=0}^{\infty} \frac{k^m}{k!} = \sum_{j=0}^{m} \left\{ {m \atop j} \right\} = \omega(m), \tag{4.4}$$

where the last equality is known as Dobinski's identity for the Bell numbers $\omega(m)$ (cf. [8, p.373, Exer.15]). Analogously, take $f(t) = e^{-t}$ in (3.6). Then

$$f^{(k)}(0) = (-1)^k \;\text{ and }\; f^{(j)}(1) = (-1)^j/e,$$

reducing (3.7) to

$$e\sum_{k=0}^{\infty}(-1)^k\frac{k^m}{k!} = \sum_{j=0}^{m}(-1)^j\left\{ {m \atop j} \right\} = D_m. \tag{4.5}$$

The finite sum D_m on the right is known as the Dobinski number.

Example 4.3 In (3.1) taking $f(t) = \frac{1}{1-t}$, $|t| < 1$, we have

$$f^{(k)}(0) = k!, \quad f^{(j)}(t) = j!(1-t)^{-1-j}.$$

Thus (3.1) alone and (3.1) together with (2.6) yield respectively the following two formulas

$$m! \sum_{k=0}^{\infty} \binom{ak+b}{m} t^k = \sum_{j=0}^{m} C(m, j; a, b) \frac{j! t^j}{(1-t)^{j+1}} \tag{4.6}$$

and

$$\sum_{k=0}^{\infty} k^m t^k = \sum_{j=0}^{m} \left\{ {m \atop j} \right\} \frac{j! t^j}{(1-t)^{j+1}}, \tag{4.7}$$

where (4.7) is the well-known formula of Euler for the arithmetic-geometric series. More interestingly, a replacement of t by $\frac{t}{1+t}$ in (4.7) gives an equivalent form

$$\sum_{k=0}^{\infty} k^m \frac{t^k}{(1+t)^{k+1}} = \sum_{j=0}^{m} \left\{ {m \atop j} \right\} j! t^j, \quad t > -1/2. \tag{4.8}$$

Example 4.4 Once more, set $f(t) = (1+t)^n$ in (3.1). Then we find

$$f^{(k)}(0) = k! \binom{n}{k}, \quad f^{(j)}(t) = j! \binom{n}{j} (1+t)^{n-j}.$$

A substitution of these results into (3.1) yields at once

$$m! \sum_{k=0}^{n} \binom{ak+b}{m} \binom{n}{k} t^k = (1+t)^n \sum_{j=0}^{m} C(m, j; a, b) \binom{n}{j} \frac{j! t^j}{(1+t)^j}, \quad t \neq -1. \tag{4.9}$$

This identity is easily recognized to be a natural extension of formula (1.126) of Gould's formulary [15].

Example 4.5 Taking $f(t) = (t)_n = t(t-1)(t-2)\cdots(t-n+1)$, we have

$$\frac{f^{(k)}(0)}{k!} = (-1)^{n-k} \left[{n \atop k} \right],$$

where $\left[{n \atop k} \right]$ is known as the Stirling number of the first kind (in Knuth's notation [8, p.259]). It is easily found that

$$f^{(j)}(1) = j! \left((-1)^{n-j-1} \left[{n-1 \atop j} \right] + (-1)^{n-j} \left[{n-1 \atop j-1} \right] \right).$$

Thus formula (3.6) is specialized to be

$$\sum_{k=0}^{n} (-1)^k \left[{n \atop k} \right] \binom{ak+b}{m} = \frac{1}{m!} \sum_{j=0}^{m} (-1)^j C(m, j; a, b) j! \left(\left[{n-1 \atop j-1} \right] - \left[{n-1 \atop j} \right] \right). \tag{4.10}$$

Example 4.6 Let α and $\beta \in \mathbb{R}$ with $\alpha\beta \neq 0$. Then using formula (3.2) with $x \mapsto \alpha$ and taking

$g(t) = \binom{\beta+t}{n}$, we may get

$$\sum_{k=0}^{n}\binom{ak+b}{m}\binom{\alpha}{k}\binom{\beta}{n-k} = \frac{1}{m!}\sum_{j=0}^{m} C(m,j;a,b)j!\binom{\alpha}{j}\binom{\alpha+\beta-j}{n-j}. \qquad (4.11)$$

Upon multiplying both sides of (4.11) by $m!a^{-m}$ and then letting $a \to \infty$, we have

$$\sum_{k=0}^{n} k^m\binom{\alpha}{k}\binom{\beta}{n-k} = \sum_{j=0}^{m} j!\left\{ {m \atop j} \right\}\binom{\alpha}{j}\binom{\alpha+\beta-j}{n-j}. \qquad (4.12)$$

Evidently, both (4.11) and (4.12) with $m = 0$ are consistent with the classical Vandermonde convolution identity. Furthermore, the special case $\beta = -1$ of (4.12) gives

$$\sum_{k=0}^{n}(-1)^{n-k}k^m\binom{\alpha}{k} = \sum_{j=0}^{m} j!\left\{ {m \atop j} \right\}\binom{\alpha}{j}\binom{\alpha-1-j}{n-j} \qquad (4.13)$$

while $\alpha = \beta = n$ reveals formula (3.77) of Gould's formulary [15]:

$$\sum_{k=0}^{n} k^m\binom{n}{k}^2 = \sum_{j=0}^{m} j!\left\{ {m \atop j} \right\}\binom{n}{j}\binom{2n-j}{n}. \qquad (4.14)$$

Example 4.7 Making the substitutions $x \mapsto n$ and $b \mapsto m$, we derive from (3.10) the equality

$$\sum_{k=0}^{\infty}\binom{m+k}{m}\binom{n}{k}\Delta^k g(0) = \sum_{j=0}^{m}\binom{m}{j}\binom{n}{j}\Delta^j g(n-j). \qquad (4.15)$$

It is worth mentioning that the last equality implies the following Carlitz's generalization of the age-old identity of Li Shanlan (as the special case $m = n$ of (4.16) below) (cf. Egorychev [16, p.170, Eq.(5.71)] or [1, p.173, Exer.38])

$$\sum_{j=0}^{\min\{m,n\}}\binom{m}{j}\binom{n}{j}\binom{p+m+n-j}{p-j} = \binom{p+m}{m}\binom{p+n}{n}, \quad p \in \mathbb{N}. \qquad (4.16)$$

Indeed, by taking $g(t) = \binom{p+m+t}{p}$, it is easy to compute

$$\Delta^k g(0) = \binom{p+m}{p-k} \quad \text{and}$$
$$\Delta^j g(n-j) = \binom{p+m+t}{p-j}\bigg|_{t=n-j} = \binom{p+m+n-j}{p-j}.$$

Once equipped with these results, the LHS of (4.16) is easily checked to be consistent with the RHS of (4.15) while the corresponding LHS further evaluates to

$$\text{LHS of (4.15)} = \sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{m}\binom{p+m}{p-k} = \sum_{k=0}^{n}\binom{m+k}{m}\binom{p+m}{m+k}\binom{n}{k}$$
$$= \binom{p+m}{m}\sum_{k=0}^{n}\binom{p}{k}\binom{n}{k} = \binom{p+m}{m}\binom{p+n}{n}.$$

Example 4.8 It may be shown that an identity due to Knuth (cf. Gould [15, (3.155)]) of the form (with $0 < j < n$)

$$\sum_{k=j}^{n}\binom{s+k}{k}\binom{k}{j}=\binom{s+j}{j}\binom{n+s+1}{n-j} \tag{4.17}$$

is also a particular consequence of (3.2). Indeed, it can be embedded in (3.10) with $b=0, m\mapsto j, x\mapsto -s-1$, and $g(t)=\binom{n-t}{n}$. In these cases, we easily find that $\binom{x}{k}=\binom{-s-1}{k}=(-1)^k\binom{s+k}{k}$ and

$$\Delta^k g(0)=(-1)^k\binom{n-t-k}{n-k}\bigg|_{t=0}=(-1)^k\binom{n-k}{n-k}=(-1)^k,\quad 0\le k\le n.$$

Accordingly, the LHS of (3.10) is evaluated to be the LHS of (4.17). The RHS of (3.10) (with $m\mapsto j, j\mapsto k$) simultaneously gives rise to

$$\sum_{k=0}^{j}\binom{0}{j-k}\binom{s+k}{k}\binom{n-k-t}{n-k}\bigg|_{t=-s-1-k}=\binom{s+j}{j}\binom{n+s+1}{n-j}.$$

Example 4.9 Taking $g(t)=1/\binom{y+t}{z}, y>z\in\mathbb{N}$, and then by Frish's identity of [15, (4.2)], we have

$$\Delta^k g(t)=\frac{(-1)^k z}{(k+z)\binom{k+y+t}{k+z}}.$$

Consequently formula (3.2) yields

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{k+z}\binom{ak+b}{m}\frac{\binom{x}{k}}{\binom{k+y}{k+z}}=\frac{1}{m!}\sum_{j=0}^{m}(-1)^j C(m,j;a,b)\frac{j!}{j+z}\frac{\binom{x}{j}}{\binom{x+y}{z+j}}, \tag{4.18}$$

where the condition $y-z\ge m+1$ guarantees the absolute convergence of the LHS of (4.18). Letting $a\to\infty$ with $x=n$ and $z=1$, we recover immediately formula (4.4) of [15]

$$\frac{1}{y}\sum_{k=0}^{n}(-1)^k k^m\frac{\binom{n}{k}}{\binom{y+k}{k}}=\sum_{j=0}^{m}(-1)^j\left\{\begin{matrix}m\\ j\end{matrix}\right\}\frac{j!\binom{n}{j}}{(y+n-j)\binom{y+n}{j}}. \tag{4.19}$$

Example 4.10 Observe that for $z:\mathrm{Re}(z+1-k)>0, (z)_k=\Gamma(z+1)/\Gamma(z+1-k)$, $\Gamma(z)$ is the classical Gamma function, $\mathrm{Re}(z)$ denotes the real part of z. Assume x in (3.2) to be nonnegative integer so we may now choose $g(t)=\Gamma(y+1-t)/\Gamma(z+1-t)$ for $t:\mathrm{Re}(t)<\min\{x,\mathrm{Re}(y),\mathrm{Re}(z)\}$. Making use of formula (7.1) in [15], namely

$$\Delta^k g(t)=(-1)^k\frac{\Gamma(y-z+1)\Gamma(y+1-t-k)}{\Gamma(z+1-t)\Gamma(y-z+1-k)}$$

for $k\le\min\{\mathrm{Re}(y-z)+1,x\}$, we thereby deduce from formula (3.2) that for $x\le\mathrm{Re}(y-z)+1$

$$\frac{1}{\Gamma(z+1)}\sum_{k=0}^{x}(-1)^k\binom{ak+b}{m}\binom{x}{k}\frac{\Gamma(y-k+1)}{\Gamma(y-z+1-k)}$$
$$=\frac{\Gamma(y-x+1)}{m!}\sum_{j=0}^{m}C(m,j;a,b)\frac{(-1)^j j!\binom{x}{j}}{\Gamma(z-x+1+j)\Gamma(y-z+1-j)}. \tag{4.20}$$

From there, assuming $x=n\in\mathbb{N}$, we may reformulate this last formula in terms of binomial coefficients to get

$$\binom{y}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{ak+b}{m}\frac{\binom{y-z}{k}}{\binom{y}{k}}=\frac{1}{m!}\binom{z}{n}\sum_{j=0}^{m}C(m,j;a,b)\frac{(-1)^j j!\binom{n}{j}\binom{y-z}{j}}{\binom{z-n+j}{j}}. \tag{4.21}$$

By the same argument we may derive from (4.20)

$$\frac{\binom{y}{y-z}}{\binom{y-x}{y-z}}\sum_{k=0}^{\infty}(-1)^k\binom{ak+b}{m}\binom{x}{k}\frac{\binom{y-z}{k}}{\binom{y}{k}}=\frac{1}{m!}\sum_{j=0}^{m}C(m,j;a,b)\frac{(-1)^j j!\binom{x}{j}\binom{y-z}{j}}{\binom{z-x+j}{j}} \tag{4.22}$$

provided that $y-z$ is a nonnegative integer. If both y and z are also nonnegative integers, then (4.22) can further be simplified to

$$\sum_{k=0}^{y-z}(-1)^k\binom{ak+b}{m}\binom{x}{k}\binom{y-k}{z}=\frac{1}{m!}\sum_{j=0}^{m}C(m,j;a,b)(-1)^j(x)_j\binom{y-x}{y-z-j} \tag{4.23}$$

which turns out to be an equivalent form of formula (4.11).

We end our discussions by pointing out that in the present paper, as mentioned earlier, we have mostly been concerned with applications of Theorems 3.1 and 3.2 within the range of analytic functions. Once we consider the LHS of (3.1) as a formal power series, the RHS of (3.1) is none other than the exponential generating function for the number sequence $\{\binom{ak+b}{m}f^{(k)}(0)\}_{k\geq 0}$. As may be expected, the basic source formula (3.1) and its discrete companion (3.2) should be possibly used to find more other formulas/identities than those just displayed in the above-mentioned examples.

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