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Longest Cycles in 2-Connected Quasi-Claw-Free Graphs

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Abstract A graph G is called quasi-claw-free if it satisfies the property: $d(x,y)=2 \Rightarrow$ there exists a vertex $u \in N(x) \cap N(y)$ such that $N[u] \subseteq N[x] \cup N[y]$. In this paper, we show that every 2-connected quasi-claw-free graph of order n with $G \notin \mathcal{F}$ contains a cycle of length at least $\min\{3\delta+2, n\}$, where $\mathcal{F}$ is a family of graphs.

Keywords quasi-claw-free graph; claw-free graph.

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1. Introduction

Graphs considered in this paper are simple and finite. We use [2] for notation and terminology not defined here. We denote by $\delta(G)$ (or δ) the minimum degree of a graph G. For a subgraph H of a graph G and a subset S of $V(G)$, we denote by $G-H$ and $G[S]$ the induced subgraphs of G by $V(G)-V(H)$ and S, respectively. Let $N(u)$ denote the set of the neighbors of u and $N[u]=N(u)\cup\{u\}$. Let $N_H(S)=\bigcup_{x\in S} N_H(x)$ and $d_H(S)=|N_H(S)|$. For A and B in $V(G)$, let $E(A,B)=\{uv \in E(G): u \in A \text{ and } v \in B\}$. For a cycle C with a fixed orientation, and two vertices x and y on C, we define the segment $C[x,y]=xCy$ to be the set of vertices on C from x to y (including x and y) and $C^-[y,x]=yC^-x$ to be a traversal of $C[x,y]$ in the opposite sense according to the orientation of C. Let x^+ and x^- denote the successor and the predecessor of x according to the orientation of C, respectively, and x^{++} and x^{--} denote the successor and the predecessor of x^+ and x^-, respectively. We define $C(x,y)=C[x,y]-\{x,y\}$.

In this paper, $\mathcal{F}$ denotes the family of graphs as follows: if $G\in\mathcal{F}$, then G can be decomposed into three subgraphs G_1, G_2, and G_3 such that $V(G_i)\cap V(G_j)=\emptyset$ and $E(V(G_i),V(G_j))=\{u_iu_j, v_iv_j\}$, where $1\le i\ne j\le 3, u_i, v_i\in V(G_i), u_j, v_j \in V(G_j)$ and $u_i\ne v_i, u_j\ne v_j$.

A graph is called claw-free if it does not contain a copy of $K_{1,3}$ as an induced subgraph. Define $J(x,y)=\{u\in N(x)\cap N(y): N[u]\subseteq N[x]\cup N[y]\}$. A graph G is quasi-claw-free if it satisfies the property: $d(x,y)=2 \Rightarrow J(x,y)\ne\emptyset$. Clearly, a claw-free graph is quasi-claw-free, but not every quasi-claw-free graph is claw-free.

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Ainouche [1], Li [9, 10], Zhan [14] and Qu and Wang [13] gave some properties of Hamiltonicity and vertex pancyclicity of quasi-claw-free graphs, but there are few results on the circumference of quasi-claw-free graphs.

Theorem 1.1 ([9]) *Let G be a 2-connected quasi-claw-free graph of order n. If $\delta \geq n/4$, then G is hamiltonian or $G \in \mathcal{F}$.*

In this paper, we mainly consider the circumferences of 2-connected quasi-claw-free graphs not in $\mathcal{F}$.

Theorem 1.2 *Every 2-connected quasi-claw-free graph of order n with $G \notin \mathcal{F}$ contains a cycle of length at least* $\min\{3\delta+2, n\}$.

2. Some lemmas

By the proof of Lemma 2.1 in Zhan [14], we can get the following lemma.

Lemma 2.1 *Let G be a connected quasi-claw-free graph and C a longest cycle of G. Suppose that H is a component of $G-C$ and $c_i \in N_C(H)$, $1 \leq i \leq d_C(H)$. Then the following facts hold,*

(1) $J(y_i, c_i^+) = J(y_i, c_i^-) = \{c_i\}$ and $c_i^- c_i^+ \in E(G)$, where $y_i \in N_H(c_i)$.

(2) $N_H(c_i)$ is a complete subgraphs of G, and $c_i^- x, c_i^+ x \in E(G)$, $x \in N_C(c_i) - N_C(H)$.

For a pair of vertices x and y in a connected graph G, let $L_G(x,y)$ be the length of a longest (x,y)-path P in G. If G is connected with $|V(G)| \geq 2$, then we set $L(G) = \min\{L_G(x,y) : x, y \in V(G), x \neq y\}$.

Lemma 2.2 ([4]) *Let G be a 2-connected graph. Then there exist two distinct vertices $v_1, v_2 \in V(G)$ such that $L(G) \geq d(v_i)$, for $i \in \{1,2\}$.*

Lemma 2.3 ([11]) *Let G be a 2-connected graph on at least 2δ vertices. Then G has a cycle of length at least 2δ containing x and y for any two vertices x and y in G.*

Lemma 2.4 ([3]) *Let G be a 2-connected graph. If every longest path P in G has the property that the sum of the degrees of the two end-vertices of P is at least $|V(P)|+1$, then G is hamilton-connected.*

Given a subgraph H of a connected graph G such that $d(v) < |V(G)| - 1$ for some vertex v in $V(H)$, let $k_H = \min\{|N_{G-S}(S)| : \emptyset \subseteq S \subseteq V(H)$ and $N_{G-S}(S) \cup S \neq V(G)\}$. Clearly, $k_H \geq k_G$.

A pair of distinct vertices x, y in $N_{G-H}(H)$ is a useful pair if $|N_H(x,y)| \geq 2$. H is strongly linked in G if for each useful pair x and y, there exists a hamiltonian path $P = P[x', y']$ in H such that $x \in N(x')$ and $y \in N(y')$, otherwise H is weakly linked in G.

Lemma 2.5 ([5]) *Let G be a 2-connected graph, C a longest cycle of G and H a component of $G-C$. Suppose that H is not hamilton-connected and $k_H \geq 3$.*

(1) If H is 2-connected nonhamiltonian, then there exist nonadjacent vertices v and w in H such that $|V(C)| \geq 2d(v) + 2d(w)$;

(2) If H is hamiltonian and weakly linked in G, then there exist nonadjacent vertices v and w in H such that $|V(C)| \geq 2d(v) + 2d(w) + \min\{(1/2)|V(H)|, 6\}$.

By Lemma 2.1 and the proof of Li [7], we can obtain Lemmas 2.6 and 2.7.

Lemma 2.6 *Let C be a longest cycle in an m-connected ($m \geq 2$) quasi-claw-free graph G, and H be a component of $G - C$ such that $|V(H)| \geq 3$. If H is hamilton-connected, then there exists some vertex v in H such that $|V(C)| \geq s(d(v) - s + 4) + (k_H - s)(|V(H)| - s + 3) \geq s(d(v) - s + 4) + (m - s)(|V(H)| - s + 3)$, where $0 \leq s \leq |V(H)| + 3$.*

Lemma 2.7 *Let C be a longest cycle of a 2-connected quasi-claw-free graph G and H be a component of $G - C$.*

(1) If H is strongly linked in G but not hamilton-connected, then there exist non-adjacent vertices v and w in H such that $|V(C)| \geq 2(d(v) + d(w)) - 2$ and $|V(G)| \geq 3(d(v) + d(w)) - 6$;

(2) If H is not 2-connected, then there exist nonadjacent vertices v and w such that $|V(C)| \geq 2(d(v) + d(w)) + 4$.

3. Proof of Theorem 1.2

Proof of Theorem 1.2 Let G be a 2-connected non-hamiltonian quasi-claw-free graph of order n satisfying the conditions of Theorem 1.3 and C a longest cycle of G with a chosen orientation. Assume that H is a component of $G - C$ and $N_C(H) = \{c_1, c_2, \ldots, c_m\}$, where c_i is labeled in the order of the direction of C and i modulo m. Then by Lemma 2.1(1), $c_i^- c_i^+ \in E(G)$ ($i \in \{1, \ldots, m\}$). Suppose that Theorem 1.3 is not true. Then $|V(C)| \leq 3\delta + 1$. Let $X = \{i : |N_H(c_i, c_{i+1})| \geq 2\}$ and $S_i = C(c_i^+, c_{i+1}^-)$. If $|V(H)|$=1, then by the maximality of C and $c_i^- c_i^+ \in E(G)$ ($i \in \{1, \ldots, m\}$), $|V(C)| \geq 4\delta$, a contradiction. Since G is 2-connected and $|V(H)| \geq 2$, $|X| \geq 2$. By Lemma 2.7(2), any component of $G - C$ is 2-connected, otherwise $|V(C)| \geq 4\delta + 4$, a contradiction. Without loss of generality, assume $M = \{v_i c_i \in E(G) : v_i \in V(H), c_i \in N_C(H)\}$ is a maximum matching in $E(V(H), V(C))$. Then $|M| \geq 2$. Since H is 2-connected and G is a simple graph, there exist at least two internally-disjoint (v_1, v_2)-paths in H and $|V(H)| \geq 3$. Obviously, there is a (v_1, v_2)-path of order at least 3 in H and then by the maximality of C, $|V(C)| \geq 12$. It follows that $3\delta + 1 \geq |V(C)| \geq 12$, which implies $\delta \geq 4$. Next we consider three cases to complete the proof of Theorem 1.2.

Case 1 H is not hamiltonian.

By Lemma 2.5(1), $k_H = 2$. Again by the proof of Lemma 2.5(1) (in [5], Corollary 6.2), $|X| = |M| = 2$. Since $M = \{v_1 c_1, v_2 c_2\}$, $N_C(H) = \{v_1, v_2\}$. Let P be a longest (v_1, v_2)-path in H. We can get the following claims.

Proposition 1.1 If $d_C(H) = 2$, then $N_C(c_1) - \{c_2, c_1^-, c_1^+\} = N_C(c_2) - \{c_1, c_2^-, c_2^+\} = \emptyset$.

Proof Suppose $d_C(H) = 2$. Then for any vertex $x \in V(H)$, $d_H(x) \geq \delta - 2$. By Lemma 2.2, $|V(P)| \geq \delta - 1$. Without loss of generality, assume $y \in N(c_1) \cap S_1$. Suppose $N(y^-) \cap V(H') \neq \emptyset$, where H' is a component of $G - C - H$. By Lemma 2.1(2), $yc_1^-, yc_1^+ \in E(G)$. We obtain

$|C(y,c_2^-)|\geq|V(P)|$, otherwise there exists a longer cycle $C'=c_1P[v_1,v_2]c_2c_2^-C[c_2^+,c_1^-]C^-[y,c_1]$ than C, a contradiction. If $y^-c_2^+\in E(G)$, then there is a longer cycle $C'=c_1P[v_1,v_2]C^-[c_2,y]$ $C[c_1^+,y^-]C[c_2^+,c_1]$ than C, a contradiction. Thus $y^-c_2^+\notin E(G)$, similarly $y^-c_2^-\notin E(G)$. If $y^-c_2\in E(G)$, then by Lemma 2.1(2), $y^-c_2^+,y^-c_2^-\in E(G)$, a contradiction. Let y_1 and y_2 be the neighbors of y^- closest to c_2^- and c_2^+ on S_1 and S_2, respectively. Then $a=|C(y_1,c_2^-)|\geq|V(P)|$, otherwise there is a longer cycle $C'=c_1P[v_1,v_2]c_2c_2^-C[c_2^+,c_1^-]C[y,y_1]$ $C^-[y^-,c_1]$ than C, a contradiction. Similarly, $b=|C(c_2^+,y_2)|\geq|V(P)|$. Obviously, $d_C(H')\geq 2$ and H' is 2-connected, then there exist three vertices u_1,y',u_2 such that $y'\in V(C)$, $u_1\in N_{H'}(y^-),u_2\in N_{H'}(y')$. Let P' be the longest (u_1,u_2)-path in H' with an orientation from u_1 to u_2. By Lemma 2.1(2), $G[N_{H'}(y^-)]$ and $G[N_{H'}(y')]$ are two complete graphs and $N_{G-C-H'}(y^-)=N_{G-C-H'}(y')=\emptyset$. Thus $|V(P')|\geq d_{H'}(y^-)$ and $y'\notin N_C(H)$. By Lemma 2.1(1), $y^{--}y,y'^-y'^+\in E(G)$. If $y'=c_1^-$, then we can obtain a longer cycle $C'=y'P'^-[u_2,u_1]C^-[y^-,c_1]C[y,y']$ than C, a contradiction. Similarly, $y'\notin\{c_1^+,\ c_2^-,\ c_2^+\}$. Now we consider the location of y'.

(a) Suppose $y'\in C(y_1,c_2^-)$. Then we obtain $d=|C(y'^+,c_2^-)|\geq|V(P)|+|V(P')|$, otherwise there is a longer cycle $C'=c_1P[v_1,v_2]c_2c_2^-C[c_2^+,c_1^-]C[y,y'^-]y'^+y'P'^-[u_2,u_1]C^-[y^-,c_1]$ than C. Recall that $b=|C(c_2^+,y_2)|\geq|V(P)|$, $|V(P')|\geq d_{H'}(y^-)$, $|V(P)|\geq\delta-1$ and $y^-c_2,y^-c_2^+,y^-c_2^-\notin E(G)$. It follows that $|V(C)|\geq|C[y_2,y_1]|+b+d+|\{c_2,c_2^-,c_2^+,\ y^-\}|\geq 2|V(P)|+d_{H'}(y^-)+d_C(y^-)+4\geq 3\delta+2$, a contradiction. Thus $y'\notin C(y_1,c_2^-)$, a contradiction. Similarly, we have (b) as follows.

(b) $y'\notin C(c_2^+,y_2)$.

(c) Suppose $y'\in C[y,y_1]$. Then without loss of generality, assume $N_C(H')\cap C(y,y')=\emptyset$. By the maximality of C, obviously $y'^-,\ y'^+\notin N(y^-)$. Let y_3 be the neighbor of y^- closest to y' on $C(y,y')$. Then $N_C(y^-)$ is contained in $A=C[y_2,y_3]\cup C[y',y_1]$. We obtain $e=|C(y_3,y'^-)|\geq|V(P')|$, otherwise there is a longer cycle $C'=y^-P'[u_1,u_2]y'y'^-C[y'^+,y^{--}]C[y,y_3]y^-$ than C, a contradiction. Recall that $a=|C(y_1,c_2^-)|\geq|V(P)|$, $b=|C(c_2^+,y_2)|\geq|V(P)|$, $|V(P)|\geq\delta-1$ and $y^-c_2,y^-c_2^+,y^-c_2^-\notin E(G)$. Thus $|V(C)|\geq e+|A|+a+b+|\{y^-,c_2^-,c_2,c_2^+\}|\geq d_{H'}(y^-)+d_C(y^-)+2|V(P)|+4\geq 3\delta+2$, a contradiction. Thus $y'\notin C[y,y_1]$. Similarly, we can obtain (d).

(d) $y'\notin C[y_2,y^-]$.

From (a)-(d), we obtain $N(y^-)\cap V(H')=\emptyset$. It follows that $N(y^-)\cap V(G-C)=\emptyset$. If $c_2y^-\in E(G)$, then there is a longer cycle $C'=c_1P[v_1,v_2]c_2C^-[y^-,c_1^+]$ $C^-[c_1^-,c_2^+]C^-[c_2^-,y]c_1$ than C, a contradiction. Similarly, $c_2^-,c_2^+\notin N(y^-)$. Thus $N[y^-]$ is contained in $C[y_2,y_1]$. Recall that $a=|C(y_1,c_2^-)|\geq|V(P)|$, $b=|C(c_2^+,y_2)|\geq|V(P)|$ and $|V(P)|\geq\delta-1$. It follows that $|V(C)|\geq|C[y_2,y_1]|+a+b+|\{y^-,c_2^-,c_2^+,c_2\}|\geq d(y^-)+2|V(P)|+4\geq 3\delta+2$, a contradiction. Thus Proposition 1.1 is true. □

Suppose that W_1 is a complete graph, $W=W_1\cup\{z_1,z_2\}$ and z_i is adjacent to all the vertices of W_1 $(i\in\{1,2\})$. Let $z_1[W_1]z_2$ denote a hamiltonian (z_1,z_2)-path of W.

Proposition 1.2 If H is not hamiltonian, then $d_C(H)\geq 3$.

Proof Suppose $d_C(H)=2$. Then $d_H(x)\geq\delta-2$ for any vertex $x\in V(H)$. By Lemma 2.4, $|V(H)|\geq 2(\delta-2)$. By Lemma 2.3, there is a cycle in H of length at least $2(\delta-2)$ containing v_1 and v_2. Without loss of generality, assume that C' is a longest cycle in H containing v_1

and v_2 with a chosen orientation. Then $|V(C')| \geq 2(\delta - 2)$. Let $C' = w_1 w_2 \ldots w_t w_1$ and w_i ($i \in \{1, 2, \ldots, t\}$) be labeled in the order of the direction of C'. Then $t \geq 2\delta - 4$. Moreover let $w_1 = v_1, w_i = v_2$ ($2 \leq i \leq t$). If $w_t c_2 \in E(G)$, then $|V(P)| \geq |C[w_1, w_t]| \geq 2\delta - 4$. It follows that $|V(C)| \geq 2|V(P)| + 6 \geq 4\delta - 2$, a contradiction. Thus $w_t c_2 \notin E(G)$. Similarly, $w_2 c_2, w_{i-1} c_1, w_{i+1} c_1 \notin E(G)$. If $i \leq \delta/2$, then $t - i \geq (3\delta/2) - 4$, $|V(P)| \geq |C[w_i, w_1]| \geq (3\delta/2) - 2$ and $|V(C)| \geq 2|V(P)| + 6 \geq 3\delta + 2$, a contradiction. Thus $i \geq (\delta/2) + 1$.

Without loss of generality, assume $w_j \notin N(c_2)$ for $2 \leq j \leq i - 1$ and $c_1 c_2 \in E(G)$. Then by Proposition 1.1, $d_H(c_2) \geq \delta - 3$. Obviously, $c_1^- c_2 \notin E(G)$, i.e., $d(c_1^-, c_2) = 2$. Suppose $w \in J(c_1^-, c_2) \cap V(G - C)$. By Lemma 2.1(2), $w \in V(H)$ and then $c_1^- w \in E(G)$, i.e., $c_1^- \in N_C(H)$, a contradiction. Thus $J(c_1^-, c_2) \subseteq V(C)$ and by Proposition 1.1, $J(c_1^-, c_2) = \{c_1\}$. For any vertex $x_1 \in N_H(c_1)$, obviously, $x_1 c_1^- \notin E(G)$ and then $x_1 c_2 \in E(G)$. Similarly, $x_2 \in N_H(c_1)$ for any vertex $x_2 \in N_H(c_2)$. Thus $N_H(c_1) = N_H(c_2)$. Recall that $w_1 \in N_H(c_1)$, $w_i \in N_H(c_2)$. Let $T = N_H(c_1)$. By Lemma 2.1(2), T is a complete graph and then $w_1 w_i \in E(G)$.

Suppose $T - V(C') \neq \emptyset$. Since T is a complete graph, w_1, $w_i \in N_{C'}(T - C')$. By the maximality of C' and the proof of Lemma 2.1 in [4], we obtain that Lemma 2.1 also holds for C'. Thus $w_t w_2$, $w_{i-1} w_{i+1} \in E(G)$. Let $T' = T - \{w_1, w_i\}$. Obviously, T' is also a complete graph. Recall that $w_{i+1} c_1 \notin E(G)$, i.e., $w_{i+1} \notin V(T)$. Let $P' = w_1 w_t C[w_2, w_{i-1}] w_{i+1} w_i [T' \setminus \{w\}] w$, where $w \in V(T')$. Obviously, $w c_2 \in E(G)$. Recall that $i \geq (\delta/2) + 1$. Then $|V(P')| \geq i + |\{w_t, w_{i+1}\}| + d_H(c_2) - 2 \geq (3\delta/2) - 2$. Replacing the path P by P', we obtain $|V(C)| \geq 2|V(P')| + 6 \geq 3\delta + 2$, a contradiction.

Suppose $T \subseteq V(C')$. Obviously, $w_1 c_2 \in E(G)$. Recall that $w_2 c_2 \notin E(G)$. Then $d(w_2, c_2) = 2$. Moreover, recall that $w_j \notin N(c_2)$ for $2 \leq j \leq i - 1$ and $w_t c_2 \notin E(G)$. Then $z \in C'[w_i, w_t) \cup \{w_1\}$ for any vertex $z \in J(w_2, c_2)$. Without loss of generality, assume $w_j \in J(w_2, c_2)$ ($i + 1 \leq j \leq t - 1$). Then $w_{j+1} c_2 \in E(G)$ or $w_{j+1} w_2 \in E(G)$. Similarly, $w_{j-1} c_2 \in E(G)$ or $w_{j-1} w_2 \in E(G)$. If $w_{j+1} w_2 \in E(G)$, then there is a (w_1, w_j)-path $P' = C'^-[w_1, w_{j+1}] C[w_2, w_j]$ of order at least $2\delta - 4$. Replacing P by P', we obtain $|V(C)| \geq 2|V(P')| + 6 \geq 4\delta - 2$, a contradiction. Thus $w_{j+1} w_2 \notin E(G)$ and then $w_{j+1} c_2 \in E(G)$. Similarly, $w_{j-1} c_2 \notin E(G)$ and $w_{j-1} w_2 \in E(G)$. By Lemma 2.1(2), $w_{j+1} w_i \in E(G)$. We obtain $w_{j-1} \neq w_{i+1}$, otherwise there is a (w_1, w_j)-path $P' = C'^-[w_1, w_{j+1}] C'^-[w_i, w_2] w_{j-1} w_j$ of order at least $2\delta - 4$, and we can get a contradiction as above. Let $T_1 = T - C'[w_i, w_j]$. Then we can get a path $P' = C'[w_1, w_{j-1}] w_j [T_1 \setminus \{w\}] w$, where $w \in T_1$. Obviously, $A = \{w_{i+1}, w_{j-1}\}$ is contained in $V(P')$ and $A \cap T = \emptyset$. Thus $|V(P')| \geq i + d_H(c_2) - |\{w_1, w_i\}| + |A| \geq (3\delta/2) - 2$, and then replacing P by P', we obtain $|V(C)| \geq 2|V(P')| + 6 \geq 3\delta + 2$, a contradiction. It follows that Propositon 1.2 is true. □

Proposition 1.3 If H is not hamiltonian, then $N_H(c_1) - \{v_1, v_2\} = N_H(c_2) - \{v_1, v_2\} = \emptyset$.

Proof Suppose $N_H(c_1) - \{v_1, v_2\} \neq \emptyset$. Recall that $|M| = 2$ and $d_C(H) \geq 3$. Then $N_H(x) = \{v_2\}$ for any vertex $x \in N_C(H) - \{c_1\}$ and $d_H(z) \geq \delta - 1$ for any vertex $z \in V(H) - \{v_2\}$. Thus $d_C(c_2) \geq \delta - 1$ and by Lemma 2.2, $|V(P)| \geq \delta$. Let y_1 and y_2 be the neighbors of c_2 closest to c_1^- and c_1^+ on $C(c_2^+, c_1^-)$ and $C(c_1^+, c_2^-)$, respectively. Then we can obtain $a = |C(y_1, c_1^-)| \geq |V(P)|$ and $b = |C(c_1^+, y_2)| \geq |V(P)|$. Obviously, $c_1^- c_2, c_1^+ c_2 \notin E(G)$. It follows that $|V(C)| \geq |C[y_2, y_1]| + a + b \geq d_C(c_2) + |\{c_1^-, c_1^+, c_2\}| + 2|V(P)| = 3\delta + 2$, a contradiction.

Thus $N_H(c_1) - \{v_1, v_2\} = \emptyset$. By symmetry, $N_H(c_2) - \{v_1, v_2\} = \emptyset$. □

Since $|M| = 2$ and by Proposition 1.3, we can obtain the following result.

Proposition 1.4 For any vertex $w \in V(H) - \{v_1, v_2\}$, $N_C(w) = \emptyset$.

Proposition 1.5 If H is not hamiltonian, then $N_H(v_1) \cap N_H(v_2) \neq \emptyset$.

Proof Suppose $N_H(v_1) \cap N_H(v_2) = \emptyset$. By Proposition 3, $d(c_1, x) = 2$ for any vertex $x \in N_H(v_1) - \{v_2\}$. By Propositions 1.3 and 1.4, $J(x, c_1) \subseteq \{v_1, v_2\}$, If $v_2 \in J(x, c_1)$, then $x \in N_H(v_1) \cap N_H(v_2)$, a contradiction. Thus $J(x, c_1) = \{v_1\}$ and by Proposition 1.3, for any two distinct vertices $x_1, x_2 \in N_H(v_1) - \{v_2\}$, $x_1x_2 \in E(G)$. Thus $N_H(v_1) - \{v_2\}$ is a complete graph. Similarly, $N_H(v_2) - \{v_1\}$ is a complete graph. It follows that $|V(P)| \geq d_H(v_1) + d_H(v_2)$. Since $|X| = 2$, $|V(C)| \geq 3d_C(v_1) + 3d_C(v_2) - 3 + 2|V(P)| > 2(d_C(v_1) + d_H(v_1)) + 2(d_C(v_2) + d_H(v_2)) - 1 \geq 4\delta - 1$, a contradiction. □

We call $\mathcal{T}$ a string of a graph G if for two distant subgraphs W_i, W_j of $\mathcal{T}$, there is a sequence $i, i+1, \ldots, j-1, j$ such that $E(W_i, W_{i+1}) \neq \emptyset$, $E(W_{i+1}, W_{i+2}) \neq \emptyset, \ldots, E(W_{j-1}, W_j) \neq \emptyset$.

Proposition 1.6 There exists some vertex $x \in N_H(v_1) \cup N_H(v_2)$ and a (v_1, v_2)-path P' in H such that $N[x] \subseteq V(P')$.

Proof Since $|X| = 2$ and $d_C(H) \geq 3$, $v_1c_2 \in E(G)$ and $v_2c_1 \in E(G)$ cannot hold at the same time. Without loss of generality, suppose $v_1c_2 \in E(G)$. Then $c_1v_2 \notin E(G)$ and for any vertex $z \in N_H(v_1) - \{v_2\}$, by Propositions 1.3 and 1.4, $J(c_1, z) = \{v_1\}$. For any two vertices $z_1, z_2 \in N_H(v_1) - \{v_2\}$, since $J(c_1, z_i) = \{v_1\}$ and $z_ic_1 \notin E(G)$ $(i \in \{1, 2\})$, $z_1z_2 \in E(G)$. Thus $H_1 = G[N_H(v_1) - \{v_2\}]$ is a complete graph. By Proposition 1.3, for any vertex $z \in N_H(v_2) - \{v_1\}$, $d(z, c_2) = 2$, and moreover by Proposition 1.4, $J(c_2, z) \in \{v_1, v_2\}$. If $v_1 \in J(c_2, z)$, then $c_1c_2 \in E(G)$ or $c_1z \in E(G)$. By Propositon 1.4, $c_1z \notin E(G)$ and then $c_1c_2 \in E(G)$. By Lemma 2.1(1), $J(c_2^+, v_2) = \{c_2\}$, then $c_2^+c_1 \in E(G)$ or $v_2c_1 \in E(G)$. Recall that $v_2c_1 \notin E(G)$ and by the maximality of C, $c_2^+c_1 \notin E(G)$. It follows that $v_1 \notin J(c_2, z)$ and then $J(c_2, z) = \{v_2\}$ for any vertex $z \in N_H(v_2) - \{v_1\}$. Thus similarly to v_1, $G[N_H(v_2) - \{v_1\}]$ is a complete graph. Let $H_2 = G[N_H(v_2) - \{v_1\}] - H_1$. Then H_2 is a complete graph. By Proposition 1.5, if there exists some vertex $x \in N_H(v_1) \cap N_H(v_2)$ such that $N(x) \subseteq (H_1 \cup H_2)$, then we can easily obtain a (v_1, v_2)-path P' containing all the vertices of $N[x]$.

Suppose $x \in N_H(v_1) \cap N_H(v_2)$ and $N(x) - (H_1 \cup H_2) \neq \emptyset$. Let $W = N(x) - (H_1 \cup H_2)$. Then for any vertex $y \in W$, $d(y, v_2) = 2$. By Propositon 1.4, $J(v_2, y) \in N_H(v_2)$. Suppose $z \in W$, $x_1 \in J(v_2, z)$ and $x_1 \in H_2$. Then $W' = \{z : z \in W, x_1 \in J(z, v_2)\}$ is a complete graph. Without loss of generality, assume $|J(v_2, y)| = 1$ for any vertex $y \in W$. Let $T = \{a : a \in H_1 \cup H_2, \{a\} = J(y', v_2)$ for some vertex $y' \in W\}$. Assume $T = \{a_1, a_2, \ldots, a_m\}$, $W_1 = \{y : y \in W, \{a_1\} = J(y, v_2)\}$, $W_2 = \{y : y \in W, \{a_2\} = J(y, v_2)\}, \ldots, W_m = \{y : y \in W, \{a_m\} = J(y, v_2)\}$. Obviously, $W = W_1 \cup W_2 \cup \cdots \cup W_m$. We divide $W_1, W_2, \ldots, W_m$ into k maximal strings $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$. Without loss of generality, assume $\mathcal{A}_1 = \{W_1, W_2, \ldots, W_p\}$. Now we use induction to prove if p is odd, then for any vertex $w_p \in W_p$, there is a path (w_1, w_p)-path P_1 containing all the vertices of $W_1 \cup \cdots \cup W_p$ such that $w_1 \in W_1$, and $V(P_1) \cap V(W_i) \neq \emptyset$, $i \in \{2, 3, \ldots, m\}$, and if p is even, then there is a (w_1, w_p)-path P_1 containing all the vertices of $W_1 \cup \cdots \cup W_p$ such that $w_1 \in W_1$, $w_p = a_p$, and $V(P_1) \cap V(W_i) \neq \emptyset$, $i \in \{2, 3, \ldots, m\}$. Assume that the induction is true for any integer $l \leq p$, and $w_{p-1}w_p \in E(W_{p-1}, W_p)$. If p is odd, then by induction there is a (w_1, w_{p-1})-path P_1' containing all the vertices of $W_1 \cup W_2 \cdots \cup W_{p-1}$. Then we can get a (w_1, a_p)-path $w_1 P_1' w_{p-1} w_p [W_p \setminus \{w_p, w\}] w a_p$ containing all the vertices of $W_1 \cup W_2 \cdots \cup W_p$. If p is even, then there is a (w_1, a_{p-1})-path P_1' containing all the vertices of $W_1 \cup W_2 \cup \cdots W_{p-1}$. Then for any vertex $w \in W_p$, there is a (w_1, w)-path $w_1 P_1' a_{p-1} a_p [W_p \setminus \{w\}] w$ containing all the vertices of $W_1 \cup W_2 \cdots \cup W_p$. Let (w_q, w_t) be a path P_2 containing all the vertices of $\mathcal{A}_2$. Obviously, $d(w_p, w_q) = 2$. Assume $a \in J(w_p, w_q)$. Then by the maximal division of strings, $a \notin W_{p+1} \cup \cdots \cup W_m$. If $a \in W_i$ $(i \in \{1, 2, \ldots, p\}$, then $a_i w_p \in E(G)$ or $a_i w_q \in E(G)$. If $a_i w_p \in E(G)$, then $w_p \in W_i$ by the definition of W_i, a contradiction. Similarly, $a_i w_q \notin E(G)$. It follows that $a \notin W$ and then there is a path $w_1 P_1 w_p a w_q P_2 w_t$ containing all the vertices of $\mathcal{A}_1 \cup \mathcal{A}_2$. Then we can easily get a path P'' containing all the vertices of W. It follows that we can get a (v_1, v_2)-path P' containing all the vertices of $N[x]$.

Similarly, if $v_1 c_2, v_2 c_1 \notin E(G)$, then Proposition 1.6 also holds. □

Now we complete the proof of Case 1. Since $|X| = 2$ and $d_C(H) \geq 3$, $v_1 c_2 \in E(G)$ and $v_2 c_1 \in E(G)$ cannot hold at the same time. Without loss of generality, assume $v_1 c_2 \notin E(G)$. Then by Proposition 1.3, $d_C(c_2) \geq \delta - 1$. Let y_1 and y_2 be the neighbors of c_2 closest to c_1^+ and c_1^- on S_1 and $C(c_2^+, c_1^-)$, respectively. Then by Propositon 1.6, there is some vertex $x \in N_H(v_1) \cup N_H(v_2)$ and a $(v_1,\, v_2)$-path P' such that $N[x] \subseteq V(P')$. Then $|C(c_1^+, y_1)| \geq d(x) + 1$ and $|C(y_2, c_1^-)| \geq d(x) + 1$. It follows that $|V(C)| \geq |C(c_1^+, y_1)| + |C(y_2, c_1^-)| + |C[y_1, y_2]| + |\{c_1^-, c_1^+, c_2\}| \geq d_C(c_2) + 2d(x) + 5 \geq 3\delta + 4$, a contradiction. It follows that =Case 1 of Theorem 1.2 cannot hold. □

Case 2 H is hamiltonian but not hamilton-connected.

Recall that M is a maximum matching in $E(V(H), V(C))$ and $X = \{i : |N_H(c_i, c_{i+1})| \geq 2\}$. If $|X| = 2$, then obviously, $|M| = 2$, and using a similar proof to Case 1, we obtain a contradiction. Thus $|X| \geq 3$. It follows that $d_C(H) \geq 3$. By Lemmas 2.5(2) and 2.7(1), $k_H = 2$. By the definition of k_H, there is a vertex set S in $V(H)$ such that $|N_{G-S}(S)| = 2$. Thus $|S| \geq \delta - 1$. If $S = V(H)$, then by $d_C(H) \geq 3$, we obtain $k_H \geq 3$, a contradiction. Let $N_{G-S}(S) = \{u_1, u_2\}$. Suppose $\{u_1, u_2\} \subseteq V(C)$. Then $N_{H-S}(C) \neq \emptyset$ since $d_C(H) \geq 3$. Obviously, $E(H - S, S) \neq \emptyset$ and then $k_H \geq 3$, a contradiction. Suppose $u_1 \in V(C), u_2 \notin V(C)$. Then $u_2 \in V(H - S)$. It follows that u_2 is a cut vertex of H, a contradiction with the connectedness of H. From above, we obtain $\{u_1, u_2\} \cap V(C) = \emptyset$. Let C' be a hamiltonian cycle of H. Without loss of generality, assume that $i \in X$, $v_i \in N_H(c_i), v_j \in N_H(c_{i+1}) (i \neq j)$, and P_{ij} is a longest (v_i, v_j)-path in H. Obviously, $v_i, v_j \in V(H - S)$. Since H is hamiltonian, S and $H - S$ are two segments of a hamiltonian cycle C' of H. Since $|N_{G-S}(S)| = 2$ and $N_{G-S}(S) \subseteq H - S$, $S \subseteq C'(v_i, v_j)$ or $S \subseteq C'(v_j, v_i)$. Without loss of generality, assume $S \subseteq C'(v_i, v_j)$. Then $|V(P_{ij})| \geq |C'(v_i, v_j)| \geq |S| \geq \delta - 1$. Since $|X| \geq 3$, suppose $i_1, i_2 \in X - \{i\}$ $(i_1 \neq i_2), v_{i_1} \in N_H(c_{i_1}), v_{i_2} \in N_H(c_{i_2})$. Then similarly to the above i and j, there is a (v_{i_1}, v_{j_1})-path $P_{i_1 j_1}$ and a (v_{i_2}, v_{j_2})-path $P_{i_2 j_2}$ in H such that $v_{j_1} \in N_H(c_{i_1+1})$, $v_{j_2} \in N_H(c_{i_2+1})$, $|V(P_{i_1 j_1})| \geq \delta - 1$, $|V(P_{i_2 j_2})| \geq \delta - 1$. Since $|X| \geq 3$, we obtain $|V(C)| \geq |V(P_{ij})| + |V(P_{i_1 j_1})| + |V(P_{i_2 j_2})| + |\{c_i, c_i^-, c_i^+, c_{i_1}, c_{i_1}^-, c_{i_1}^+, c_{i_2}, c_{i_2}^-, c_{i_2}^+\}| \geq 3\delta + 6,$

a contradiction. It follows that Case 2 of Theorem 1.2 cannot hold. □

Case 3 H is hamilton-connected.

Obviously, H is 3-connected. Suppose $d_C(H) \geq 3$. If $k_H \geq 3$, then taking $s = 3$ in Lemma 2.6, we obtain $|V(C)| \geq 3\delta+3$, a contradiction. If $k_H = 2$, then there is a vertex set S in H such that $|N_{G-S}(S)| = 2$ and by the proof of Case 2, $N_{G-S}(S) \subseteq H-S$. It follows that $|N_{H-S}(S)| = 2$, a contradiction with the connectedness of H. Thus $d_C(H) = 2$, i.e., $N_C(H) = \{c_1, c_2\}$, and then for any vertex $x \in V(H)$, $d(x) \geq \delta-2$. It follows that $|V(H)| \geq \delta-1$. Let $|V(H)| = h$. Recall that $v_1 \in N_H(c_1), v_2 \in N_H(c_2)$, $S_1 = C(c_1^+, c_2^-)$, $S_2 = C(c_2^+, c_1^-)$, and P is a hamiltonian (v_1, v_2)-path of H. Obviously, $|V(P)| = h$. By a similar argument to the proof of Proposition 1.1 of Case 1, we obtain the following result.

Proposition 3.1 The following properties hold,

(a) $N_C(c_1) - \{c_1^-, c_1^+, c_2\} = N_C(c_2) - \{c_2^-, c_2^+, c_1\} = \emptyset$.

(b) $N(c_1^+) - \{c_1^-, c_1\} \subseteq S_1$.

(c) $N(c_2^-) - \{c_2^+, c_2\} \subseteq S_1$.

(d) $N(c_1^-) - \{c_1^+, c_1\} \subseteq S_2$.

(e) $N(c_2^+) - \{c_2^-, c_2\} \subseteq S_2$.

Proposition 3.2 If H is hamilton-connected, then $G[N(c_i^+) - \{c_i^-, c_i\}]$ and $G[N(c_i^-) - \{c_i^+, c_i\}]$ are complete graphs, $i \in \{1, 2\}$.

Proof By Proposition 3.1(a) and (b), $d(c_i, x) = 2$ and $J(c_i, x) = c_i^+$ for any vertex $x \in N(c_i^+) - \{c_i, c_i^-\}$. By Proposition 3.1(a), $y_1, y_2 \notin N(c_i)$, and then $y_1y_2 \in E(G)$ since $J(c_i, x) = c_i^+$ for any two distinct vertices $y_1, y_2 \in N(c_i^+) - \{c_i^-, c_i\}$. Thus $G[N(c_i^+) - \{c_i^-, c_i\}]$ is a complete graph. Similarly, $G[N(c_i^-) - \{c_i^+, c_i\}]$ is a complete graph. □

Proposition 3.3 If H is hamilton-connected, then $|S_i| \leq 2\delta - 4$, $i \in \{1, 2\}$.

Proof Otherwise, without loss of generality, suppose $|S_1| \geq 2\delta - 3$. Obviously, $|S_2| \geq |V(P)|$. Thus $|V(C)| \geq 2\delta - 3 + \delta - 1 + 6 \geq 3\delta + 2$, a contradiction. □

Let u and v be the neighbors of c_1^+ and c_2^- closest to c_2^- and c_1^+ on S_1, respectively. Moreover let w and f be the neighbors of c_2^+ and c_1^- closest to c_1^- and c_2^+ on S_2, respectively.

Proposition 3.4 For any vertex $x \in C(c_1^+, u)$ and any vertex $y \in S_2$, $xy \notin E(G)$.

Proof Otherwise, suppose $xy \in E(G)$ and $y \in C(f, c_1^-)$. Let x_1 and x_2 be the neighbors of c_1^+ closest to x on $C(c_1^+, x)$ and $C(x, u)$, respectively. Moreover, let y_1 and y_2 be the neighbors of c_1^- closest to y on $C(f, y)$ and $C(y, c_1^-)$, respectively. By Proposition 3.2, $x_1u \in E(G)$. Then $a = |C(y_1, y)| + |C(x_1, x)| + |C(u, c_2^-)| \geq h$, otherwise there is a longer cycle $C' = c_1P[v_1, v_2]c_2c_2^-C[c_2^+, y_1]\ C^-[c_1^-, y]C[x, u]C^-[x_1, c_1]$ than C, a contradiction. Similarly, $b = |C(y, y_2)| + |C(c_2^+, f)| + |C(x, x_2)| \geq h$. Thus $|V(C)| \geq d(c_1^-) + d(c_1^+) - |\{c_1\}| + a + b + |\{c_2\}| \geq 4\delta - 2$, a contradiction. It follows that $y \notin C(f, c_1^-)$. By Proposition 3.1(e), $y \neq c_2^+$. Suppose $y \in C(c_2^+, f)$. Then $f \neq c_2^+$ and by the maximality of C, $d = |C(c_2^+, y)| + |C(x, x_2)| \geq h$.

Suppose $xc_1^+ \in E(G)$. Then by Proposition 3.1(b), $d(c_1^+, y) = 2$ and $J(c_1^+, y) \subseteq C(c_1^+, u)$. Let $x' \in J(c_1^+, y)$. Then $x'^+c_1^+ \in E(G)$ or $x'^+y \in E(G)$. If $x'^+c_1^+ \in E(G)$, then by the maximality of C, $|C(c_2^+, y)| \geq h$. It follows that $|S_2| \geq |C(c_2^+, y)| + |C[f, c_1^-)| \geq 2\delta - 3$, a contradiction with Proposition 3.3. Thus $x'^+y \in E(G)$. Then let x'_2 be the neighbor closet to $C(x', u)$. By the maximality of C, $d' = |C(c_2^+, y)| + |C(x', x'_2)| \geq h + 1$. Thus $|V(C)| \geq d(c_1^-) + d(c_1^+) - |\{c_1\}| + d' + |\{c_2^-, c_2, y\}| \geq 3\delta + 2$, a contradiction. It follows that $x \notin N(c_1^+)$. Thus $|V(C)| \geq d(c_1^-) + d(c_1^+) - |\{c_1\}| + d + |\{c_2^-, c_2, x, y\}| \geq 3\delta + 2$, a contradiction. Thus $y \notin C(c_2^+, f)$. Suppose $y = f$. Then by Proposition 3.1(d), $d(c_1^-, x) = 2$ and $J(c_1^-, x) \subseteq S_2$. Let $z \in J(c_1^-, x)$. From above, $z \notin C(f, c_1^-) \cup C(c_2^+, f)$ since $zx \in E(G)$. Thus $z = f$ and then $f^+x \in E(G)$ or $f^+c^- \in E(G)$. From above, $f^+x \notin E(G)$, and by the choice of f, $f^+c_1^- \notin E(G)$. Thus Proposition 3.4 is true. □

Using a similar proof to Proposition 3.4, we can get the following result.

Proposition 3.5 If H is hamilton-connected, then $E(A, S_2) = E(B, S_1) = \emptyset$, where $A = C(v, c_2^-), B = C(f, c_1^-) \cup C(c_2^+, w)$.

Proposition 3.6 If H is hamilton-connected, then $wu, wv, uf, vf \notin E(G)$.

Proof Suppose $wu \in E(G)$. Then by Proposition 3.1(b) and the choice of u, $d(c_1^+, w) = 2$ and $J(c_1^+, w) \subseteq C(c_1^+, u]$. Let $x \in J(c_1^+, w)$. If $x \in C(c_1^+, u)$, then we obtain a contradiction with Proposition 3.4. Thus $J(c_1^+, w) = \{u\}$ and then $u^+c_1^+ \in E(G)$ or $u^+w \in E(G)$. By the choice of u, $u^+c_1^+ \notin E(G)$ and then $u^+w \in E(G)$, $d(c_2^+, u^+) = 2$. Similarly, $J(c_2^+, u^+) = \{w\}$ and $w^+u^+ \in E(G)$. We obtain $a = |C(w^+, c_1^-)| + |C(u^+, c_2^-)| \geq h$, otherwise there is a longer cycle $C' = c_1P[v_1, v_2]c_2c_2^-C[c_2^+, w^+]C[u^+, c_1]$ than C, a contradiction. Thus $|V(C)| \geq a + |C[c_2^-, w]| + |C[c_1^-, u]| + |\{w^+, u^+\}| \geq a + d(c_2^+) + d(c_1^+) + |\{c_2^-, c_1^+, u^+, w^+\}| \geq 3\delta + 3$, a contradiction. Thus $wu \notin E(G)$. Similarly, $wv, uf, vf \notin E(G)$. □

Proposition 3.7 If H is hamilton-connected, then $E(S_1, S_2) = \emptyset$.

Proof By Proposition 3.3, $v = u^+$, $u = v$ or $u \in C(v, c_2^-)$. Similarly, $f = w^+$, $f = w$ or $w \in C(f, c_1^-)$. Suppose $xy \in E(G), x \in S_1, y \in S_2$. Then $x \in C(f, c_1^-) \cup C(c_2^+, w) \cup \{w, f\}$, and by Propositions 3.4-3.6, we obtain a contradiction. □

Proposition 3.8 For any vertex $x \in S_1$ and $y \in S_2$, there is no (x, y)-path in $G[H' \cup \{x, y\}]$, where H' is a component of $G - C$.

Proof Suppose that there is an (x, y)-path P with internal vertices in H'. Since $N_C(H) = \{c_1, c_2\}$, $H \cap H' = \emptyset$. Similar to H, H' is hamilton-connected and $h' = |V(H')| \geq \delta - 1$. Let P' be a hamiltonian (v'_1, v'_2)-path in H' such that $v'_1 \in N_{H'}(x), v'_2 \in N_{H'}(y)$. By Lemma 2.1(1), $x^-x^+, y^-y^+ \in E(G)$. Let $S'_1 = C(x^+, y^-)$ and $S'_2 = C(y^+, x^-)$. By the previous proof of Case 3, we obtain that Propositions 3.1–3.7 also hold for S'_1, S'_2, x and y. Obviously, $c_2 \in S'_1$, $c_1 \in S'_2$. Suppose $u = x^+$. Then by Proposition 3.1(b), $c_1^+ = x^-$. We obtain $a = |C(c_2^+, y^-)| \geq h + h'$, otherwise there exists a cycle $C' = xP'[v'_1, v'_2]yy^-C[y^+, c_1^-]c_1^+c_1P[v_1, v_2]c_2c_2^+C^-[c_2^-, x]$ longer than C, a contradiction. Then $|S'_1| > a \geq 2\delta - 2$, a contradiction with Proposition 3.3. Thus

$u \neq x^+$. Suppose $u = x^-$. Then by Proposition 3.3, $v \in C(c_1^+, u)$, $v = u = x^-$ or $v = u^+ = x$. If $v \in C(c_1^+, u)$, then $v \in S_1$, $c_2^+ \in S_1'$, $E(S_1', S_2') \neq \emptyset$, a contradiction with Proposition 3.7. Similarly, by Proposition 3.1(c) and Proposition 3.1(a), respectively, we obtain a contradiction if $v = u = x^-$ or $v = u^+ = x$. Thus $u \neq x^-$. Similarly, if $u \in C(c_1^+, x^-) \cup C(x^+, c_2^-) \cup \{x\}$, we obtain a contradiction. Thus Proposition 3.8 holds. □

Proposition 3.9 If H is hamilton-connected, then $N(x) \cap V(G-C) = N(y) \cap V(G-C) = \emptyset$ for any vertex $x \in S_1$ and any vertex $y \in S_2$.

Proof Suppose $z_1 \in N(x) \cap V(G-C)$. Then obviously, $z_1 \notin V(H)$. Let H' be a component of $G-C-H$ containing z_1. Similar to H, H' is hamilton-connected, $d_C(H') = 2$ and $|V(H')| = h' \geq \delta - 1$. Let $N_C(H') = \{c_1', c_2'\}$, $z_2 \in N_H(c_2')$, $x = c_1'$ and P' be a hamiltonian (z_1, z_2)-path in H'. By Lemma 2.1(1), $c_1'^- c_1'^+, c_2'^- c_2'^+ \in E(G)$. By Proposition 3.8, $c_2' \in S_1$. Without loss of generality, assume $c_1' \in C(c_1^+, c_2'^-)$. By the maximality of C, $|C(c_1'^+, c_2'^-)| \geq h'$. By Proposition 3.3, $c_1' \in C(c_1^+, u)$ and $c_2' \in C(v, c_2^-)$. Without loss of generality, assume $c_2' \in C(c_1', u]$. Let y_1 and y_2 be the neighbors of c_1^+ closest to c_1' and c_2' on $C(c_1^+, c_1')$ and $C(c_1', c_2')$, respectively. Then by Proposition 3.2, $y_1 y_2 \in E(G)$. Let $A = C(y_1, c_1'^-) \cup C(y_2, c_2'^-)$. Then $|A| \geq h'$, otherwise there is a longer cycle $C' = c_1' P'[z_1, z_2] c_2' c_2'^- C[c_2'^+, y_1] C^-[y_2, c_1'^+] c_1'^- c_1'$ than C, a contradiction. Obviously, $N(c_1^+) - \{c_1, c_1^-\}$ is contained in $S_1 - A$. Thus $|S_1| \geq |A| + d(c_1^+) - |\{c_1, c_1^-\}| \geq 2\delta - 3$, a contradiction with Proposition 3.3. It follows that $N(x) \cap V(G-C) = \emptyset$ for any vertex $x \in S_1$. Similarly, $N(y) \cap V(G-C) = \emptyset$ for any vertex $y \in S_2$. □

By Propositions 3.8 and 3.9, we obtain that $G - C$ has only one component H which is hamilton-connected. Let $G_1 = G[H \cup \{c_1, c_2\}]$, $G_2 = G[C[c_1^+, c_2^-]]$ and $G_3 = G[C[c_2^+, c_1^-]]$. Then by Propositions 3.7–3.9, we obtain $G_1 \cup G_2 \cup G_3 \in \mathcal{F}$. It follows that Case 3 of Theorem 1.2 cannot hold. By Cases 1–3, Theorem 1.2 is true. □

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