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On 3-Hued Coloring of Graphs

Ting LIU, Lei SUN*
School of Mathematical Sciences, Shandong Normal University, Shangdong 250014*, P. R. China*

Abstract For integers $k > 0$, $r > 0$, a (k, r)-coloring of a graph G is a proper k-coloring of the vertices such that every vertex of degree d is adjacent to vertices with at least $\min\{d, r\}$ different colors. The r-hued chromatic number, denoted by $\chi_r(G)$, is the smallest integer k for which a graph G has a (k, r)-coloring. Define a graph G is r-normal, if $\chi_r(G) = \chi(G)$. In this paper, we present two sufficient conditions for a graph to be 3-normal, and the best upper bound of 3-hued chromatic number of a certain families of graphs.

Keywords r-hued chromatic number; 3-normal graph; triangle.

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1. Introduction

Graphs in this paper are simple and finite. For a graph G and $v \in V(G)$, $d_G(v)$ and $N_G(v)$ denote the degree of v in G and the set of vertices adjacent to v in G, respectively. $\delta(G)$ and $\Delta(G)$ denote the smallest degree and the largest degree in G, respectively. We say that a set of vertices are independent if there is no edge between these vertices. The independent number $\alpha(G)$ of a graph G is the size of a largest independent set of G.

For an integer $k > 0$, let $\overline{k} = \{1, 2, \ldots, k\}$. A proper k-coloring of a graph G is a map $c : V(G) \longrightarrow \overline{k}$ such that if $u, v \in V(G)$ are adjacent vertices in G, then $c(u) \neq c(v)$. The smallest k such that G has a proper k-coloring is the chromatic number of G, denoted by $\chi(G)$.

Let G be a graph, $k > 0$ be an integer, $\overline{k} = \{1, 2, \ldots, k\}$, and $c : V(G) \longrightarrow \overline{k}$ be a map. We denote by $c^{-1}(i)$ the vertex set which receives the color i. For $S \subseteq V(G)$, define $c(S) = \{c(u) | u \in S\}$. If for a vertex v with degree at least 2, $|c(N(v))| = 1$, then v is called a bad vertex, otherwise it is called a good vertex. We refer to [2] for undefined terminologies and notations.

Definition 1.1 ([8]) *For integers $k > 0$ and $r > 0$, a proper (k, r)-coloring of a graph G is a map $c : V(G) \longrightarrow \overline{k}$ satisfying both the following:*

(C1) $c(u) \neq c(v)$ for every edge $uv \in E(G)$; and

(C2) $|c(N_G(v))| \geq \min\{|N_G(v)|, r\}$ for any $v \in V(G)$.

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* Corresponding author
E-mail address: Liuting_2010@163.com (Ting LIU); Lsun89@163.com (Lei SUN)

For a fixed number $r > 0$, the r-hued chromatic number of G, denoted by $\chi_r(G)$, is the smallest k such that G has a (k, r)-coloring.

By the definition of $\chi_r(G)$, it follows immediately that $\chi(G) = \chi_1(G)$, and so r-hued coloring is a generalization of the classical graph coloring.

Recently, the dynamic coloring of graphs has been studied extensively by several authors, for instance see [1, 3–8].

Definition 1.2 ([5]) *A graph G is defined as normal if $\chi_2(G) = \chi(G)$. For $r \geq 3$, we can similarly define that a graph G is r-normal if $\chi_r(G) = \chi(G)$.*

2. Several sufficient conditions

In this section, we give several sufficient conditions of normal graph in 3-hued coloring.

Lemma 2.1 *For any $v \in V(G)$, if there exists an odd cycle in the subgraph of G induced by the neighbors of v, then G is 3-normal graph.*

Proof For any $v \in V(G)$, there is at least an odd cycle whose all vertices are joined to v. For every $k \geq 1$, $\chi(C_{2k+1}) = 3$, so every proper coloring of G is also a 3-hued coloring of G, then G is a 3-normal graph. □

Theorem 2.1 *For any $x, y \in V(G)$, and $xy \in E(G)$, if $d(x) + d(y) \geq n + 2$, and G does not contain an even cycle without a chord as an induced subgraph, then G is a 3-normal graph.*

Proof If $n \leq 3$, such graphs do not exist.

Assume that $n \geq 4$. For any $x, y \in V(G)$, and $xy \in E(G)$, we have $d(x) + d(y) \geq n + 2$.

Suppose $d(x) = 2$ and $y \in N(x)$. We have $d(x) + d(y) \leq 2 + n - 1 = n + 1$, a contradiction. So G does not contain a vertex whose degree is 2.

For any $x \in V(G)$, we assume $d(x) \geq 3$. Let H be a subgraph of G induced by the neighbors of v. Next we shall show that there must exist an odd cycle in H. For any $y \in N(x)$, we have $d(x) + d(y) \geq n + 2$. So $d(y) \geq n + 2 - d(x)$. Then y is joined to at least two vertices in $N(x)$. That is $d_H(y) \geq 2$. So there must exist a cycle in H. Since G does not contain an even cycle without a chord, H does not contain an even cycle without a chord either. Therefore, H must contain an odd cycle. By Lemma 1, we have that G is a 3-normal graph. □

Theorem 2.2 *If $\alpha(G) = \alpha$, $\Delta(G) \leq \lceil \frac{n-3\alpha}{\alpha-1} \rceil - 1$, then G is a 3-normal graph.*

Proof Let c be a proper coloring of G such that $c(v) = j = \min\{i|$ there is no neighbors of v in $c^{-1}(i)\}$. If v is a bad vertex, then $2 \leq d(v) \leq \min\{\triangle(G), 2\alpha\}$, and $c(v) = 1$ or $c(v) = 2$, or $c(v) = 3$.

Case 1 Assume v_1 is a bad vertex and $c(v_1) = 1$.

By the construction of c and $2 \leq d(v_1) \leq \min\{\triangle(G), 2\alpha\}$, if v is not in $\{v_i|c(v_i) = 1\} \cup \{v_i|c(v_i) \in c(N(v_1))\}$, then there is at least one vertex in $V_1' = \{v_i|c(v_i) = 1\} \backslash \{v_1\}$ joined to v.

So there must exist one vertex $v_{j_1} \in V_1'$ such that $d(v_{j_1}) \geq \lceil \frac{n-3\alpha}{\alpha-1} \rceil$. It is a contradiction.

Case 2 Assume v_2 is a bad vertex and $c(v_2) = 2$.

By the construction of c and $2 \leq d(v_2) \leq \min\{\triangle(G), 2\alpha\}$, if v is not in $\{v_i|c(v_i) = 2\} \cup \{v_i|c(v_i) \in c(N(v_2))\}$, then there is at least one vertex in $V_2^{'} = \{v_i|c(v_i) = 2\} \backslash \{v_2\}$ joined to v. So there must exist one vertex $v_{j_2} \in V_2'$ such that $d(v_{j_2}) \geq \lceil \frac{n-3\alpha}{\alpha-1} \rceil + 1$. It is a contradiction.

Case 3 Assume v_3 is a bad vertex and $c(v_3) = 3$.

By the construction of c and $2 \leq d(v_3) \leq \min\{\triangle(G), 2\alpha\}$, if v is not in $\{v_i|c(v_i) = 3\} \cup \{v_i|c(v_i) \in c(N(v_3))\}$, then there is at least one vertex in $V_3' = \{v_i|c(v_i) = 3\} \backslash \{v_3\}$ joined to v. So there must exist one vertex $v_{j_3} \in V_3'$ such that $d(v_{j_3}) \geq \lceil \frac{n-3\alpha}{\alpha-1} \rceil + 2$. It is a contradiction.

So there is no bad vertex in G. Then G is a 3-normal graph. □

3. The best upper bound

In this section, we give the best upper bound of 3-hued chromatic number of a certain families of graphs.

Definition 3.1 *An xy-path P is a graph such that: (1) if $v \in V(P)$ and $v \neq x, y$, then $d(v) = 2$; (2) $d(x), d(y) \geq 3$, denoted by P^*.*

Lemma 3.1 ([8]) *If G is a connected graph and $\delta(G) = 2$, then there is a path P^* whose length is at least 2, or G is a cycle.*

Lemma 3.2 *Let G be a connected r-regular graph. If every two adjacent vertices are in a triangle, then G is K_4.*

Proof $\forall v \in V(G)$. Let $N(v) = \{v_1, v_2, v_3\}$. Since $d(v) = d(v_1) = d(v_2) = d(v_3) = 3$, without loss of generality, we may assume $v_1v_2, v_2v_3 \in E(G)$. Suppose $N(v_1) = \{v, v_2, v_4\}$, then v_1, v_4 must be in a triangle. Therefore, v_4, v are adjacent or v_4, v_2 are adjacent. In this condition, $d(v) = 4$ or $d(v_2) = 4$. It is a contradiction. So $N(v_1) = \{v, v_2, v_3\}$, then G is K_4. □

Theorem 3.1 *Let G be a simple graph, $\Delta \leq 3$. If every two adjacent 3-vertices are in a triangle, then $\chi_3(G) \leq 6$.*

Proof Without loss of generality, we may assume that G is a connected graph. The proof is by induction on $n = |V(G)|$. We use $L(v)$ to denote the available color set for $v \in V(G)$.

When $|V(G)| \leq 6$, the result is easily verified. Suppose $|V(G)| \geq 7$.

Case 1 G has a cut vertex v.

Then there are i connected subgraphs $G_1, G_2, \ldots, G_i$ such that $\bigcap_{j=1}^{j=i} G_j = v$. By induction, Every G_j has a $(6,3)$-coloring $c_j : V(G_j) \longrightarrow \overline{6}$, $j = 1, 2, \ldots, i$. Without loss of generality, we may assume $c_1(v) = c_2(v) = \cdots = c_i(v)$. Because G_j is connected, by changing the colors, we can make the neighbors of v receive different colors. That is a 3-hued coloring of G, $c : V(G) \longrightarrow \overline{6}$, such that $c(v_m) = c_j(v_m)$, $\forall v_m \in G_j$, $j = 1, 2, \ldots, i$.

Case 2 G is 2-connected and $\delta = 2$.

Case 2.1 $G \cong C_n$.

When $n \equiv 0 \pmod 3$, $\chi_3(C_n) = \chi_2(C_n) = 3$; when $n = 5$, $\chi_3(C_n) = \chi_2(C_n) = 5$; for the other cases, $\chi_3(C_n) = \chi_2(C_n) = 4$.

Case 2.2 G has a path $P^* = v_1v_2 \cdots v_m$, for some $m \geq 4$.

Let $G' = G - \{v_2, \ldots, v_{m-1}\}$. By induction, G' has a (6,3)-coloring $c' : V(G') \longrightarrow \overline{6}$. Since G is 2-connected, we have $v_1 \neq v_m$, otherwise $v_1 = v_m$ is a cut vertex. Suppose $N(v_1) = \{v_2, a, b\}$, $N(v_m) = \{v_{m-1}, c, d\}$, we use

$i_2 \in \{1, 2, \ldots, 6\} \backslash \{c(a), c(b), c(v_1)\}$ to color v_2;

$i_3 \in \{1, 2, \ldots, 6\} \backslash \{c(v_1), c(v_2)\}$ to color v_3;

$\cdots$;

$i_j \in \{1, 2, \ldots, 6\} \backslash \{c(v_{j-2}), c(v_{j-1})\}$ to color v_j, $j = 1, \cdots m - 2$;

$\cdots$;

$i_{m-1} \in \{1, 2, \ldots, 6\} \backslash \{c(v_{m-3}), c(v_{m-2}), c(v_m), c(c), c(d)\}$ to color v_{m-1}.

Case 2.3 G has a path $P^* = v_1v_2 \cdots v_m$, for $m = 3$.

Note that in this case there could not exist an edge xy in G such that $d(x) = d(y) = 2$. Suppose $d(v) = 2$, $N(v) = \{x, y\}$, $d(x) = d(y) = 3$, $N(x) \backslash \{v\} = \{a, b\}$, $N(y) \backslash \{v\} = \{c, d\}$. Since G is simple, we have $x \neq y$.

Case 2.3.1 $xy \in E(G)$.

Let $G' = G - v$. By induction, G' has a $(6, 3)$-coloring $c' : V(G') \longrightarrow \overline{6}$, $c'(x) \neq c'(y)$. We use $i \in \{1, 2, \ldots, 6\} \backslash \{c(x), c(y), c(a), c(c)\}$ to color v.

Case 2.3.2 x, y are not adjacent vertices and $\{a, b\} \cap \{c, d\} \neq \emptyset$.

Without loss of generality, we may assume $a \in N(x) \cap N(y) \backslash \{v\}$. Let $G' = G - v + xy$. By induction G' has a $(6, 3)$-coloring $c' : V(G') \longrightarrow \overline{6}$. We use $i \in \{1, 2, \ldots, 6\} \backslash \{c(x), c(y), c(a), c(b), c(d)\}$ to color v.

Case 2.3.3 x, y are not adjacent vertices and $\{a, b\} \cap \{c, d\} = \emptyset$. Without loss of generality, we may assume $d(a) \leq d(b)$.

Case 2.3.3.1 $d(a) = d(b) = 3$ and $N(a) \backslash \{b\} = N(b) \backslash \{a\} = \{x, e\}$.

Because G is connected, $d(e) = 3$. Let $G' = G \backslash \{x, v\}$, $|L(x)| = |\{1, 2, \ldots, 6\} \backslash \{c(a), c(b), c(e)\}| = 3$. By induction, G' has a $(6, 3)$-coloring $c' : V(G') \longrightarrow \overline{6}$. If $A = L(x) \cap \{c(c), c(d)\} \neq \emptyset$, we may assume that $c(c) = i \in A$. Let $c(x) = i$. And we use $j \in \{1, 2, \ldots, 6\} \backslash \{c(x), c(y), c(a), c(b), c(d)\}$ to color v. If $A = L(X) \cap \{c(c), c(d)\} = \emptyset$, we use $i \in L(x) \backslash \{c(y)\}$ to color x, $L(x) \backslash \{c(x), c(y)\}$ to color v.

Case 2.3.3.2 $d(a) = d(b) = 3$, $N(a) = \{x, b, e\}$, $N(b) = \{x, a, f\}$ and $e \neq f$.

We have $d(e) = d(f) = 2$. Suppose $N(e) = \{a, g\}$, $N(f) = \{b, h\}$. Let $G' = G - \{a, b, x, v\}$. By induction, G' has a $(6,3)$-coloring $c' : V(G') \longrightarrow \overline{6}$. Let $|L(a)| = |\{1, 2, \ldots, 6\}\backslash\{c(e), c(g)\}| \geq 4$, $|L(b)| = |\{1, 2, \ldots, 6\}\backslash\{c(f), c(h)\}| \geq 4$. Then $A = L(a) \cap \{c(y), c(c), c(d)\} \neq \emptyset$. Let $c(a) = i \in A$. We use $j \in L(b)\backslash\{c(a), c(e)\}$ to color b, $\{1, 2, \ldots, 6\}\backslash\{c(a), c(b), c(e), c(f), c(y)\}$ to color x, $\{1, 2, \ldots, 6\}\backslash\{c(x), c(y), c(b), c(c), c(d)\}$ to color v.

Case 2.3.3.3 $d(a) = 2$,$d(b) = 3$.

Suppose $N(b) = \{x, a, f\}$. Let $G' = G\backslash\{x, v\}$. By induction, G' has a $(6,3)$-coloring $c' : V(G') \longrightarrow \overline{6}$. Let $|L(x)| = |\{1, 2, \ldots, 6\}\backslash\{c(a), c(b), c(f)\}| = 3$. If $A = L(x) \cap \{c(c), c(d)\} \neq \emptyset$, we may assume $c(c) = i \in A$. Let $c(x) = i$. We may use $j \in \{1, 2, \ldots, 6\}\backslash\{c(x), c(y), c(a), c(b), c(d)\}$ to color v. If $A = L(x) \cap \{c(c), c(d)\} = \emptyset$, we may use $i \in L(x)\backslash\{c(y)\}$ to color x, $L(x)\backslash\{c(x), c(y)\}$ to color v.

Case 2.3.3.4 $d(a) = d(b) = 2$ and $d(c) = d(d) = 2$.

Note that in this case a, b are not adjacent, otherwise there is a contradiction with $m = 2$. If $N(a) = N(b) = \{x, z\}$,$N(z) = \{a, b, z'\}$, let $G' = G - \{a\} + xz$. By induction, G' has an $(6,3)$-coloring $c' : V(G') \longrightarrow \overline{6}$. We may use $\{1, 2, \ldots, 6\}\backslash\{c(z), c(z'), c(b), c(x), c(v)\}$ to color a. If $N(a) = \{e, x\}$, $N(b) = \{f, x\}$, and $e \neq f$, we can get $d(e) = d(f) = 3$.

If $N(a) \cap N(c) = \{e\}$.

Suppose $N(d) = \{y, g\}$. Let $G' = G\backslash\{x, v, y\}$. By induction, G' has a $(6,3)$-coloring $c' : V(G') \longrightarrow \overline{6}$. We may use $i \in \{1, 2, \ldots, 6\}\backslash\{c(a), c(b), c(e), c(f)\}$ to color x, $j \in \{1, 2, \ldots, 6\}\backslash\{c(c), c(d), c(e), c(g), c(x)\}$ to color y. And let $c(v) = c(e)$.

If $N(a) \cap N(c) = N(b) \cap N(d) = \emptyset$.

Suppose $N(a) = \{e, x\}, N(b) = \{f, x\}, N(c) = \{y, g\}, N(d) = \{y, h\}$. Let $G' = G\backslash\{a, b, x, v, y, c, d\}$. By induction, G' has a $(6,3)$-coloring $c' : V(G') \longrightarrow \overline{6}$. Suppose $|L(a)| = |\{1, 2, \ldots, 6\}\backslash(\{c(e)\} \cup c(N_G(e)))| = 3$, $|L(b)| = |\{1, 2, \ldots, 6\}\backslash(\{c(f)\} \cup c(N_G(f)))| = 3$, $|L(c)| = |\{1, 2, \ldots, 6\}\backslash(\{c(g)\} \cup c(N_G(g)))| = 3$, $|L(d)| = |\{1, 2, \ldots, 6\}\backslash(\{c(h)\} \cup c(N_G(h)))| = 3$, if $L(a) \cap L(c) = \emptyset$ or $L(b) \cap L(d) = \emptyset$. Without loss of generality, we may assume $A = L(a) \cap L(c) \neq \emptyset$. Let $c(a) = c(c) = i_1 \in A$. We may use $i_2 \in \{1, 2, \ldots, 6\}\backslash\{c(a), c(b), c(e), c(f)\}$ to color x, $i_3 \in \{1, 2, \ldots, 6\}\backslash\{c(c), c(d), c(g), c(h)\}$ to color y, $i_4 \in \{1,\ 2, \ldots, 6\}\backslash\{c(x), c(y), c(a), c(b), c(d)\}$ to color v. If $L(a) \cap L(c) = L(b) \cap L(d) = \emptyset$, we may assume $L(a) = L(b) = \{1, 2, 3\}$, $L(c) = L(d) = \{4, 5, 6\}$. Let $c(a) = 1$, $c(y) = c(b) \in \{1, 2, 3\}\backslash\{c(g), c(h)\}$, $c(x) = \{1, 2, 3\}\backslash\{c(a), c(b)\}$. We may use $i \in \{1, 2, \ldots, 6\}\backslash\{c(x), c(y), c(a), c(c), c(d)\}$ to color v.

Case 3 G is 2-connected and $\delta = 3$.

Since $\Delta \leq 3$, we have $\Delta = \delta = 3$. Since every two adjacent vertices are in a triangle, by Lemma 3.3, we can conclude $G = K_4$. The result is right.

The upper bound in the Theorem 3.4 is best possible. There exists a graph G with $\chi_3(G) = 6$. Suppose $V(G) = \{v_1, v_2, \ldots, v_6\}$, $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, v_6v_1, v_6v_2, v_6v_3\}$. Since there are six vertices in $V(G)$, we have $\chi_3(G) \leq 6$. Next, we will show that $\chi_3(G) \geq 6$. Suppose $\chi_3(G) = 5, c : V(G) \longrightarrow \overline{5}$ is a $(5,3)$-coloring. Without loss of generality, we may assume that

$c(v_1)=1, c(v_2)=2, c(v_5)=3, c(v_6)=4$. Then $c(v_3)=5$, otherwise, there will be a bad vertex. On this condition, no matter which color we choose from $\{1,2,\ldots,5\}$ to v_4, there is a bad vertex. Then $\chi_3(G)\geq 6$. So $\chi_3(G)=6$. $\square$

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