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The Filtration Dimensions of Generalized Power Series Algebras

Hailou YAO*, **Ying GUO**
College of Applied Sciences, Beijing University of Technology, Beijing 100124*, P. R. China*

Abstract In this paper, we study the properties of generalized power series modules and the filtration dimensions of generalized power series algebras. We obtain that $[[\triangle^{S,\leq}]]$-gfd$([[A^{S,\leq}]]) = \triangle$-gfd$(A)$ if A is an R-module where R is a perfect and coherent commutative algebra, and $(R,\leq)$ is standardly stratified.

Keywords standardly stratified algebra; generalized power series algebra; generalized power series module; filtration dimension.

MR(2010) Subject Classification 16G20; 16G60

1. Introduction

As a generalization of power series rings, Ribenboim introduced the notion of generalized power series rings in [1]. In fact, given a coefficient ring R and an ordered monoid $(S,\leq)$, Higman [2] carried out this construction firstly, and also, Neumann [3] investigated many special examples of these kinds of rings. Ribenboim [1] considered the ring consisting of functions from $(S,\leq)$ to R, having support which is artinian and narrow. This restriction enables one to perform the construction and gives rise to rings which are concrete enough to allow further study of their properties. Ribenboim in [4] gave the condition under which a generalized power series ring is noetherian, and provided some interesting examples of generalized power series rings. Varadarajan [5] made a further research on noetherian generalized power series rings in 2001, and investigated generalized power series modules [6] in the same year. Recently, Liu [7, 8] studied the properties of generalized power series rings and gave the necessary and sufficient conditions for a genralized power series ring to be reduced (2-primal, Dedekind finite, clean, uniquely clean, Baer) if and only if its coefficient ring is reduced (2-primal, Dedekind finite, clean, uniquely clean, Baer, respectively).

In this paper we study generalized power series algebras by another way– homological method which is different from the above. We make research on modules over generalized power series algebras and modules over their coefficient algebras through comparing their filtration dimensions. It has been several years to study algebras by way of filtration dimension. Here are

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* Corresponding author
E-mail address: yaohl@bjut.edu.cn (Hailou YAO); gykyxx@emails.bjut.edu.cn (Ying GUO)

some concise reviews about them. Scott [9] introduced the concept of quasi-hereditary algebras in studying semisimple complex Lie algebras and highest weight module category of algebraic groups in 1987. As a generalization of quasi-hereditary algebras, properly standardly stratified algebras and standardly stratified algebras were introduced by Cline, Parshall, Scott [10] and Dlab [11]. Many algebraists became more and more interested in this research field. For example, in order to calculate the global dimensions of GL_2- and GL_3-algebras, Parker [12] introduced the concept of ∇-(or $\triangle$-) good filtration dimension for a quasi-hereditary algebra and showed that the global dimension of a Schur algebra for GL_2 and GL_3 is twice the good filtration dimension in 2001. Zhu and Caenepeel [13] investigated these kinds of dimensions for standardly stratified algebras and properly stratified algebras and gave several characterizations of $\overline{\nabla}$-good filtration dimensions and $\overline{\triangle}$-good filtration dimensions in 2004. Recently, Wang and Zhu [14] studied these dimensions for standardly stratified algebras and Ringel's dual and obtained that the $\overline{\nabla}$-good filtration dimension of a standardly stratified algebra is equal to the $\overline{\triangle}$-good filtration dimension of its Ringel's dual. On these foundation of the investigations above we make research on the properties of generalized power series modules and the filtration dimensions of generalized power series algebras, and establish the relation between filtration dimensions of R-modules and the corresponding generalized power series $[[R^{S,\leq}]]$-modules.

2. Preliminaries

Let R be a commutative artinian ring, and A be a basic artinian algebra over R. Denote by A-mod the category of finitely generated A-modules, and by gf the composition of maps $f: M_1 \to M_2$ and $g: M_2 \to M_3$. The subcategories considered are full and closed under isomorphism.

Definition 2.1 *Given a class θ in A-mod, we denote by $\mathcal{F}(\theta)$ the full subcategory of all A-modules which have a θ-filtration, that is, a θ-filtration*

$$0 = M_t \subseteq M_{t-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = M,$$

such that each factor $M_{i-1}/M_i (1 \leq i \leq n)$ is isomorphic to an object in θ for $1 \leq i \leq t$. The modules in $\mathcal{F}(\theta)$ are said to be θ-good modules, and the category $\mathcal{F}(\theta)$ is said to be the θ-good module category.

Definition 2.2 *Let $(A, \leq)$ be the algebra together with a fixed ordering on a complete set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotents (given by the natural ordering of indices). For $1 \leq i \leq n$, let $P(i) = Ae_i$ be an indecomposable projective A-module, and $S(i)$ be the simple top of $P(i)$. The standard module $\triangle(i) = \triangle_A(i)$ is by definition the maximal factor module of $P(i)$ without composition factors $S(j)(j > i)$. $\overline{\triangle}(i) = \overline{\triangle}_A(i)$ is said to be a proper standard module, if it is the maximal factor module of $\triangle(i)$ such that the multiplicity condition $[\overline{\triangle}(i) : S(i)] = \dim_k \mathrm{Hom}_A(P(i), \overline{\triangle}(i)) = 1$ holds.*

Dually we have the notions of costandard modules $\nabla(i)$ and proper costandard modules $\overline{\nabla}(i)$.

Definition 2.3 *The pair $(A, \leq)$ is said to be standardly stratified if ${}_AA \in F(\triangle)$.*

The pair $(A, \leq)$ is said to be properly standardly stratified if ${}_AA \in F(\triangle)$ and ${}_AA \in F(\overline{\triangle})$.

Definition 2.4 *Let Λ be an associative algebra. A full subcategory $\mathcal{C}$ of Λ-mod is said to be contravariantly finite in Λ-mod if for any $X \in \Lambda$-mod, there is a module $F_X \in \mathcal{C}$ with a morphism $f : F_X \to X$ such that the restriction of $\mathrm{Hom}(-, f)$ to $\mathcal{C}$ is surjective. Such a morphism f is said to be a right $\mathcal{C}$- approximation of X. Dually, one can give the notion of a covariantly finite subcategory. A subcategory of Λ-mod is said to be functorially finite in Λ-mod if it is both contravariantly finite and covariantly finite.*

Now we will introduce some preliminaries about generalized power series algebras.

Throughout the next part of this section and the next section without special statement we always denote by R a commutative algebra over a field K with unit element 1, and by S an additive monoid, i.e., S is a commutative monoid and the operation is denoted by addition sign "+″. In this case the unit element in S is denoted by 0. We assume that S is endowed with a compatible strict order relation $\leq$ which is not necessarily a total order.

Definition 2.5 *Let $(S, \leq)$ be a set endowed with an order relation (which is not necessarily a total order). $(S, \leq)$ is said to be artinian if there does not exist an infinite strictly decreasing sequence of elements in S: $s_1 > s_2 > s_3 > \cdots$.*

Similarly, $(S, \leq)$ is said to be noetherian if there does not exist an infinite strictly increasing sequence of elements in S: $s_1 < s_2 < s_3 < \cdots$.

Finally, $(S, \leq)$ is said to be narrow if each subset of pairwise order incomparable elements of S is finite.

Definition 2.6 *Suppose that S is an additive monoid with zero element 0. Let $\leq$ be an order relation which is compatible: if $s \leq t$ and $u \in S$, then $s + u \leq t + u$. Then $(S, \leq)$ is called an ordered monoid. An ordered monoid $(S, \leq)$ is called a strictly ordered monoid (and $\leq$ is called a strict order) whenever it satisfies the condition: if $s < t$ and $u \in S$, then $s + u < t + u$.*

Definition 2.7 *Suppose that S is a strictly ordered monoid (written additively). Let R be a commutative algebra over a field K with unit element 1, and denote by R^S the set of all mappings $f : S \to R$. If $f \in R^S$, let the support of f be $\mathrm{supp}(f) = \{s \in S | f(s) \neq 0\}$. Define $[[R^{S,\leq}]] = \{f \in R^S | \mathrm{supp}(f)$ is artinian and narrow $\}$.*

For any f, g in $[[R^{S,\leq}]]$ and $s \in S$, the set $X_s(f, g) = \{(u, v) \in S \times S | u + v = s, f(u) \neq 0$ and $g(v) \neq 0\}$ turns out to be finite. With pointwise addition and multiplication (called convolution) defined by

$$fg(s) = \sum_{(u,v) \in X_s(f,g)} f(u)g(v),$$

$[[R^{S,\leq}]]$ turns out to be an associative commutative algebra over K with unit element e, where $e(0) = 1$ and for all $s \in S$ with $s \neq 0, e(s) = 0$. The map $r \longmapsto re$ naturally embeds R as a subalgebra of $[[R^{S,\leq}]]$.

For any R-module, we define $[[R^{S,\leq}]]$-module $[[M^{S,\leq}]] : [[M^{S,\leq}]] = \{\varphi : S \to M | \mathrm{supp}(\varphi)$

is artinian and narrow }. For any f in $[[R^{S,\leq}]]$, φ in $[[M^{S,\leq}]]$ and $s\in S$, the set $X_s(f,\varphi)=\{(u,v)\in S\times S|u+v=s, f(u)\neq 0 \text{ and } \varphi(v)\neq 0\}$ turns out to be finite. With pointwise addition and scalar product defined by

$$f\varphi(s)=\sum_{(u,v)\in X_s(f,\varphi)} f(u)\varphi(v)$$

$[[M^{S,\leq}]]$ turns out to be a unital $[[R^{S,\leq}]]$-module. For any element $m\in M$, let φ_m be a map from S to M with $\varphi_m(0)=m$ and $\varphi_m(s)=0$ for all the other $s\in S$. The map $m\longmapsto \varphi_m$ naturally embeds M as a submodule of $[[M^{S,\leq}]]$.

We call $[[R^{S,\leq}]]$ a generalized power series algebra, and $[[M^{S,\leq}]]$ a generalized power series module.

Example 2.1 If $S=\mathbb{N}$, the set of the natural numbers with the usual order, then $A=[[\mathbb{R}^{\mathbb{N},\leq}]]\cong\mathbb{R}[[x]]$, where $\mathbb{R}[[x]]$ is the ring of formal power series in one indeterminate and coefficients in $\mathbb{R}$ and $\mathbb{R}$ is the real field.

Example 2.2 Let $\mathbb{R}$ be a ring, and $\mathbb{N}$ be the set of natural numbers. Consider the multiplicative monoid $\mathbb{N}_{\geq 1}$, endowed with the usual order $\leq$. Then $A=[[\mathbb{R}^{\mathbb{N}_{\geq 1},\leq}]]$ is the ring of arithmetical function with values in $\mathbb{R}$, endowed with the Dirichlet convolution:

$$(f*g)(n)=\sum_{d|n} f(d)g(\frac{n}{d})$$

for each $n\geq 1$.

Remark *Varadarajan pointed out in [6] that Ribenboim's proofs in [4] are valid for the ring* $[[R^{s,\leq}]]$ *to be noetherian even when R is noncommutative in the case of sided noetherianness.*

3. Some lemmas on $[[R^{S,\leq}]]$-modules

We will give some lemmas in this section in order to prove the main results.

Lemma 3.1 *If R is a perfect and coherent commutative ring with unit element 1, then* $[[R^{S,\leq}]]$ *is flat as a right R-module.*

Proof As a right R-module, $[[R^{S,\leq}]]\cong\prod_{s\in S} R$ is projective, and so it is flat. □

Lemma 3.2 *Let R be a commutative algebra with unit element 1,* $(S,\leq)$ *be a strictly ordered monoid,* $[[R^{S,\leq}]]$ *be a generalized power series algebra, and* $[[M^{S,\leq}]]$ *be an* $[[R^{S,\leq}]]$*-module. Then we have the following*

(a) *If M is an injective* $[[R^{S,\leq}]]$*-module, then M is an injective R-module;*

(b) *If M is a flat R-module, then* $[[R^{S,\leq}]]\otimes_R M$ *is a flat* $[[R^{S,\leq}]]$*-module;*

(c) *If M is a flat* $[[R^{S,\leq}]]$*-module, then M is a flat R-module;*

(d) *If M is an injective R-module, then* $\mathrm{Hom}_R([[R^{S,\leq}]],M)$ *is an injective* $[[R^{S,\leq}]]$*-module.*

Proof (a) If M is an injective $[[R^{S,\leq}]]$-module, then M is an injective $\prod R$-module since

$[[R^{S,\leq}]] \cong \prod_{s\in S}({}_R R)$. For any left ideal I in R we have that $\prod_{s\in S} I$ is a left ideal in $\prod_{s\in S} R$, and so every homomorphism $g' \in \mathrm{Hom}(\prod_{s\in S} I, M)$ can be extended to a homomorphism $f' \in \mathrm{Hom}(\prod_{s\in S} R, M)$ such that

$$f'(a_1, a_2, \dots, a_s, \dots) = g'(a_1, a_2, \dots, a_s, \dots)$$

in case $(a_1, a_2, \dots, a_s, \dots) \in \prod_{s\in S} I$ according to Baer's criterion. Thus, for any left ideal I in R, every homomorphism $g \in \mathrm{Hom}(I, M)$ can be extended to a homomorphism $f \in \mathrm{Hom}(R, M)$ such that $g(a) = f(a)$ for $a \in I$. Hence, M is an injective R-module.

(b) If M is a flat R-module, then $[[R^{S,\leq}]] \otimes_R M$ is a flat $[[R^{S,\leq}]]$-module according to Theorem 4 on page 147 in [15].

(c) Obviously, if M is an $[[R^{S,\leq}]]$-module, then M is an R-module. If M is a flat $[[R^{S,\leq}]]$-module, then $\mathrm{Hom}_Z(M, Q/Z)$ is an injective $[[R^{S,\leq}]]$-module by Theorem 31 on page 147 in [16], where Z is the ring of integers, and Q is a commutative additive group consisting of all rational numbers, and so Q is a $Z-$module. From (a) in this lemma we know that $\mathrm{Hom}_Z(M, Q/Z)$ is an injective R-module. Thus, M is a flat R-module according to Theorem 31 on page 147 in [16].

(d) Since R is a commutative ring, the generalized power series ring $[[R^{S,\leq}]]$ can be regarded as an algebra over R. Therefore, we know that $\mathrm{Hom}_R([[R^{S,\leq}]], M)$ is an injective $[[R^{S,\leq}]]$-module from Lemma 3 on page 79 in [16]. □

Lemma 3.3 *Let R be a commutative algebra with unit element 1, $(S, \leq)$ be a strictly ordered monoid. If A is an R-module, and B is a submodule of A, then* $\frac{[[A^{S,\leq}]]}{[[B^{S,\leq}]]} \cong [[(\frac{A}{B})^{S,\leq}]]$.

Proof We define a homomorphism φ from $\frac{[[A^{S,\leq}]]}{[[B^{S,\leq}]]}$ to $[[(\frac{A}{B})^{S,\leq}]]$ such that $\varphi(\overline{f}) = \widetilde{f}$, where $\overline{f} = f + [[B^{S,\leq}]]$, $f \in [[A^{S,\leq}]]$ and $\widetilde{f}$ is given by $\widetilde{f(s)} = f(s) + B$, a mapping from S to $\frac{A}{B}$.

Firstly, we prove that φ is well-defined. If $\overline{f} = \overline{g}$, then $f + [[B^{S,\leq}]] = g + [[B^{S,\leq}]]$. Thus, for any $s \in S$, we have that $f(s) - g(s) \in B$, and so, $f(s) + B = g(s) + B$. Hence, $\widetilde{f} = \widetilde{g}$. It is easy to check that φ is a homomorphism.

Secondly, we prove that φ is an isomorphism. If $\widetilde{f} = 0$, then $f(s) + B = 0$ for any $s \in S$. Thus, $f(s) \in B$, i.e., $f \in [[B^{S,\leq}]]$, that is, $\overline{f} = 0$. So, $\mathrm{Ker}\,\varphi = 0$, i.e. φ is injective. Now we prove that φ is surjective. For any $\widetilde{f} \in [[(\frac{A}{B})^{S,\leq}]]$, we have that $\overline{f} \in \frac{[[A^{S,\leq}]]}{[[B^{S,\leq}]]}$ corresponds to $\widetilde{f}$ under φ. Thus, φ is surjective. Therefore, φ is an isomorphism. This completes the proof. □

Lemma 3.4 *Let $(R, \leq)$ be a standardly stratified commutative algebra, Define*

(i) $[[\triangle(i)^{S,\leq}]] = [[R^{S,\leq}]] \otimes_R \triangle(i)$; *(ii)* $[[\overline{\triangle}(i)^{S,\leq}]] = [[R^{S,\leq}]] \otimes_R \overline{\triangle}(i)$;

(iii) $[[\nabla(i)^{S,\leq}]] = [[R^{S,\leq}]] \otimes_R \nabla(i)$; *(iv)* $[[\overline{\nabla}(i)^{S,\leq}]] = [[R^{S,\leq}]] \otimes_R \overline{\nabla}(i)$.

Then we have the following:

(a) If $M \in \mathcal{F}_R(\triangle)$, *then* $[[M^{S,\leq}]] \in \mathcal{F}_{[[R^{S,\leq}]]}([[\triangle^{S,\leq}]])$;

(b) If $M \in \mathcal{F}_R(\overline{\triangle})$, *then* $[[M^{S,\leq}]] \in \mathcal{F}_{[[R^{S,\leq}]]}([[\overline{\triangle}^{S,\leq}]])$;

(c) If $M \in \mathcal{F}_R(\nabla)$, *then* $[[M^{S,\leq}]] \in \mathcal{F}_{[[R^{S,\leq}]]}([[\nabla^{S,\leq}]])$;

(d) If $M \in \mathcal{F}_R(\overline{\nabla})$, *then* $[[M^{S,\leq}]] \in \mathcal{F}_{[[R^{S,\leq}]]}([[\overline{\nabla}^{S,\leq}]])$.

Proof (a) If $M \in \mathcal{F}_R(\triangle)$, then there exists a filtration chain $0 = M_n \subseteq M_{n-1} \subseteq \cdots \subseteq M_1 \subseteq$

$M_0 = M$ such that $\frac{M_i}{M_{i+1}} \cong \triangle(j)$ holds for some $j = 1, 2, \ldots, n$ $(i = 1, 2, \ldots, n)$. Thus , we have the following chain

$$0 = [[R^{S,\leq}]] \otimes_R M_n \subseteq [[R^{S,\leq}]] \otimes_R M_{n-1} \subseteq \cdots [[R^{S,\leq}]] \otimes_R M_0 = [[R^{S,\leq}]] \otimes_R M$$

i.e., the chain $0 = [[M_n^{S,\leq}]] \subseteq [[M_{n-1}^{S,\leq}]] \subseteq \cdots \subseteq [[M_1^{S,\leq}]] \subseteq [[M_0^{S,\leq}]] = [[M^{S,\leq}]]$ is desired. In fact, thanks to Lemma 3.3 we have that

$$\frac{[[M_i^{S,\leq}]]}{[[M_{i+1}^{S,\leq}]]} \cong [[(\frac{M_i}{M_{i+1}})^{S,\leq}]] \cong [[\triangle(j)^{S,\leq}]]$$

holds for some $j = 1, 2, \ldots, n$ $(i = 1, 2, \ldots, n)$. Thus, $[[M^{S,\leq}]] \in \mathcal{F}_{[[R^{S,\leq}]]}([[\triangle^{S,\leq}]])$.

Similarly to the proof of (a), we can prove (b), (c) and (d). □

The following lemma can be proved easily.

Lemma 3.5 *Let $(R, \leq)$ be a standardly stratified commutative algebra. If $[[B^{S,\leq}]] \in \mathcal{F}_{[[R^{S,\leq}]]}([[\triangle^{S,\leq}]])$, then $B \in \mathcal{F}_R(\triangle)$.*

Lemma 3.6 *Assume that R is an associative ring with unit element 1, and A is a left R-module, then we have the following*

$$(\prod_{\lambda\in\Lambda} R) \otimes_R A \cong \prod_{\lambda\in\Lambda} (R \otimes_R A) \cong \prod_{\lambda\in\Lambda} A \tag{1}$$

where Λ is a finite or infinite set.

Proof In case Λ is finite, it is obvious that the lemma holds. So, we consider the case of infiniteness. From the definition of products and tensor products, it is easy to obtain the following homomorphism,

$$g : (\prod_{\lambda\in\Lambda} R) \otimes_R A \longrightarrow \prod_{\lambda\in\Lambda} (R \otimes_R A) \tag{2}$$

$$(\cdots r_\lambda \cdots) \otimes a \longmapsto (\cdots r_\lambda \otimes a \cdots)$$

It is easy to see that g is an isomorphism when A is the regular module R.

One can prove that g is also an isomorphism when A is a free left R-module, similarly.

Now, for any left R-module A, there exist free left R-modules F_1 and F_2 such that

$$F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} A \longrightarrow 0 \tag{3}$$

is an exact sequence. Thus, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
(\prod_{\lambda\in\Lambda} R)\otimes_R F_2 & \xrightarrow{I\otimes f_2} & (\prod_{\lambda\in\Lambda} R)\otimes_R F_1 & \xrightarrow{I\otimes f_1} & (\prod_{\lambda\in\Lambda} R)\otimes_R A & \longrightarrow & 0 & \longrightarrow & 0 \\
{\scriptstyle g_2}\downarrow & & {\scriptstyle g_1}\downarrow & & {\scriptstyle g}\downarrow & & \| & & \| \\
\prod_{\lambda\in\Lambda}(R\otimes_R F_2) & \xrightarrow{\prod_{\lambda\in\Lambda}(I\otimes f_2)} & \prod_{\lambda\in\Lambda}(R\otimes_R F_1) & \xrightarrow{\prod_{\lambda\in\Lambda}(I\otimes f_1)} & \prod_{\lambda\in\Lambda}(R\otimes_R A) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Diagram 3.1 Five Lemma Diagram

where g_2, g_1 and g are as ones in (2), and moreover g_1 and g_2 are isomorphisms. From Five Lemma we learn that g is an isomorphism. This completes the proof. □

Remark (i) Dually, in case M is a right R-module, we also have

$$M \otimes_R (\prod_{\lambda\in\Lambda} R) \cong \prod_{\lambda\in\Lambda}(M \otimes_R R) \cong \prod_{\lambda\in\Lambda} M.$$

(ii) For the general case, the formula does not hold (one can refer to page 152, in [13]);

(iii) If M_λ is a free right R-module($\lambda \in \Lambda$), and A is a left R-module, then one can similarly prove

$$(\prod_{\lambda\in\Lambda} M_\lambda) \otimes_R A \cong \prod_{\lambda\in\Lambda}(M_\lambda \otimes_R A).$$

Lemma 3.7 $\mathcal{F}_{[[R^{S,\leq}]}([[\triangle^{S,\leq}]])$ *is contravariantly finite in the subcategory consisting of all modules in the form of* $[[A^{S,\leq}]$.

Proof Since $\mathcal{F}_R(\triangle)$ is contravariantly finite, there exists a right $\mathcal{F}_R(\triangle)$- approximation $f : C \longrightarrow A$ of A for each R-module A. Thus, for any $B \in \mathcal{F}_R(\triangle)$ we have the following exact sequence

$$\mathrm{Hom}_R(B, C) \xrightarrow{\mathrm{Hom}(B,f)} \mathrm{Hom}_R(B, A) \longrightarrow 0.$$

Now we prove that $1_{[[R^{S,\leq}]]} \otimes f : [[R^{S,\leq}]] \otimes_R C \simeq [[C^{S,\leq}]] \longrightarrow [[R^{S,\leq}]] \otimes_R A \simeq [[A^{S,\leq}]]$ is a right $\mathcal{F}_{[[R^{S,\leq}]]}([[\triangle^{S,\leq}]])$-approximation of $[[A^{S,\leq}]]$. i.e., we prove that for any $[[B^{S,\leq}]] \in \mathcal{F}_{[[R^{S,\leq}]]}([[\triangle^{S,\leq}]])$ there exists the following exact sequence

$$\mathrm{Hom}_{[[R^{S,\leq}]]}([[B^{S,\leq}]],[[C^{S,\leq}]]) \xrightarrow{\mathrm{Hom}_{[[R^{S,\leq}]]}([[B^{S,\leq}]],1_{[[R^{S,\leq}]]}\otimes f)} \mathrm{Hom}_{[[R^{S,\leq}]]}([[B^{S,\leq}]],[[A^{S,\leq}]]) \longrightarrow 0.$$

As we have that

$$\begin{aligned}\mathrm{Hom}_{[[R^{S,\leq}]]}([[B^{S,\leq}]], [[A^{S,\leq}]]) &\simeq \mathrm{Hom}_{[[R^{S,\leq}]]}([[R^{S,\leq}]] \otimes_R B, [[A^{S,\leq}]]) \\ &\simeq \mathrm{Hom}_R(B, \mathrm{Hom}_{[[R^{S,\leq}]]}([[R^{S,\leq}]], [[A^{S,\leq}]])) \\ &\simeq \mathrm{Hom}_R(B, [[A^{S,\leq}]]),\end{aligned}$$

and that $\mathrm{Hom}_{[[R^{S,\leq}]]}([[B^{S,\leq}]], [[C^{S,\leq}]]) \simeq \mathrm{Hom}_R(B, [[C^{S,\leq}]])$, it suffices to prove that there exists the following exact sequence

$$\mathrm{Hom}_R(B, [[C^{S,\leq}]]) \xrightarrow{\mathrm{Hom}_R(B,1_{[[R^{S,\leq}]]}\otimes f)} \mathrm{Hom}_R(B, [[A^{S,\leq}]]) \longrightarrow 0.$$

That is to prove that for any given R-homomorphism $\xi : B \longrightarrow [[A^{S,\leq}]]$ there exists an R-homomorphism $\eta : B \longrightarrow [[C^{S,\leq}]]$ such that the following commutative diagram holds.

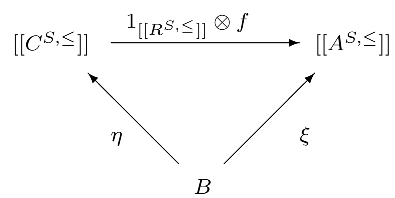

Diagram 3.2 Universal Property Diagram

As

$$[[C^{S,\leq}]] \simeq [[R^{S,\leq}]] \otimes_R C \simeq \prod_{i\in S} R \otimes_R C \simeq \prod_{i\in S} C$$

and

$$[[A^{S,\leq}]] \simeq [[R^{S,\leq}]] \otimes_R A \simeq \prod_{i\in S} R \otimes_R A \simeq \prod_{i\in S} A,$$

we can regard the R-homomorphism $1_{[[R^{S,\leq}]]} \otimes f : [[R^{S,\leq}]] \otimes_R C \longrightarrow [[R^{S,\leq}]] \otimes_R A$ as the R-homomorphism $\prod_{i\in S} f : \prod_{i\in S} C \longrightarrow \prod_{i\in S} A$. Now, let R-homomorphism $g : C \longrightarrow \prod_{i\in S} C$ be the embedding map and R-homomorphism $h : \prod_{i\in S} A \longrightarrow A$ be the projective map. Then $h(\prod_{i\in S} f)g = f$. Since $f : C \longrightarrow A$ is a right $\mathcal{F}_R(\triangle)$-approximation of A, and $B \in \mathcal{F}_R(\triangle)$, there exists a τ in $\mathrm{Hom}_R(B, C)$ such that the following commutative diagram holds:

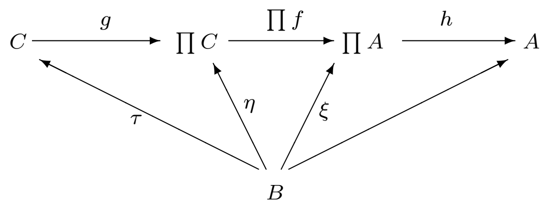

Diagram 3.3 Approximation Property Diagram

Now, it is sufficient to take $\eta = g\tau$. Thus, $1_{[[R^{S,\leq}]]} \otimes f : [[R^{S,\leq}]] \otimes_R C \longrightarrow [[R^{S,\leq}]] \otimes_R A$ is a right $\mathcal{F}_{[[R^{S,\leq}]]}([[\triangle^{S,\leq}]])$-approximation of $[[A^{S,\leq}]]$. This completes the proof. □

Similarly, we have:

Lemma 3.8 $\mathcal{F}_{[[R^{S,\leq}]]}([[\overline{\nabla}^{S,\leq}]])$ *is covariantly finite in the subcategory consisting of modules in the form of* $[[A^{S,\leq}]]$.

Lemma 3.9 *If R is a commutative ring with unit element 1 and A is an R-module, then we have* $[[A^{S,\leq}]] \cong [[R^{S,\leq}]] \otimes_R A$.

Proof From Lemma 3.6 we have

$$[[A^{S,\leq}]] \cong \prod_{s\in S} A \cong \prod_{s\in S} (R \otimes_R A) \cong (\prod_{s\in S} R) \otimes_R A \cong [[R^{S,\leq}]] \otimes_R A. \quad \square$$

4. On $[[\triangle^{S,\leq}]]$-gfd of $[[R^{S,\leq}]]$

In this section, let R be a perfect and coherent commutative algebra with unit element 1 (in fact, in this case R is artinian [17]), and $(R, \leq)$ be a standardly stratified algebra.

For any $[[R^{S,\leq}]]$-module $[[A^{S,\leq}]]$ there exists a finite $\mathcal{F}_{[[R^{S,\leq}]]}([[\triangle^{S,\leq}]])$-resolution,

$$\cdots \longrightarrow [[M_d^{S,\leq}]] \longrightarrow \cdots \longrightarrow [[M_0^{S,\leq}]] \longrightarrow [[A^{S,\leq}]] \longrightarrow 0,$$

where $[[M_i^{S,\leq}]] \in \mathcal{F}_{[[R^{S,\leq}]]}([[\triangle^{S,\leq}]])$, $i = 1, 2, \ldots, d$.

Definition 4.1 *Let R be a standardly stratified algebra. We define* $[[\triangle^{S,\leq}]]$-gfd$(A^{S,\leq}]])$ *as the minimal index d in all* $\mathcal{F}_{[[R^{S,\leq}]]}([[\triangle^{S,\leq}]])$*-resolutions. If such a minimal index d does not exist,we define* $[[\triangle^{S,\leq}]]$-gfd$(A^{S,\leq}]] = \infty$. *Furthermore, we define,*

$$[[\triangle^{S,\leq}]]\text{-gfd}([[R^{S,\leq}]]) = \sup\{[[\triangle^{S,\leq}]]\text{-gfd}([[A^{S,\leq}]]) | A \in \mathrm{mod}\, R\}.$$

Similarly, one can give the definitions of $[[\overline{\nabla}^{S,\leq}]]$-gfd$(A^{S,\leq}]])$ and $[[\overline{\nabla}^{S,\leq}]]$-gfd$(R^{S,\leq}]])$.

Theorem 4.1 *Let A be an R-module. Then $\triangle$-gfd$(A) = [[\triangle^{S,\leq}]]$-gfd$([[A^{S,\leq}]])$.*

Proof Considering an $\mathcal{F}_R(\triangle)$-resolution of A

$$\cdots \longrightarrow M_n \longrightarrow M_{n-1} \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow A \longrightarrow 0,$$

we obtain an $\mathcal{F}_{[[R^{S,\leq}]]}([[\triangle^{S,\leq}]])$-resolution of $[[A^{S,\leq}]]$ as follows

$$\begin{aligned}\cdots \longrightarrow [[R^{S,\leq}]] \otimes_R M_n \longrightarrow [[R^{S,\leq}]] \otimes_R M_{n-1} \cdots \longrightarrow [[R^{S,\leq}]] \otimes_R M_1 \\ \longrightarrow [[R^{S,\leq}]] \otimes_R M_0 \longrightarrow [[R^{S,\leq}]] \otimes_R A \longrightarrow 0.\end{aligned}$$

Thus, $[[\triangle^{S,\leq}]]$-gfd$([[A^{S,\leq}]]) \leq \triangle$-gfd$(A)$.

We now prove $\triangle$-gfd$(A) \leq [[\triangle^{S,\leq}]]$-gfd$([[A^{S,\leq}]])$.

Suppose $[[\triangle^{S,\leq}]]$-gfd$([[A^{S,\leq}]]) = n$. Then there exists an $\mathcal{F}_{[[R^{S,\leq}]]}([[\triangle^{S,\leq}]])$-resolution of $[[A^{S,\leq}]]$ as follows

$$0 \longrightarrow Q_n \longrightarrow Q_{n-1} \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow [[A^{S,\leq}]] \longrightarrow 0,$$

where $Q_i \in \mathcal{F}_{[[R^{S,\leq}]]}([[\triangle^{S,\leq}]])$ $(i = 1, 2, \ldots, n)$. As an R-module, we have that $Q_i \in \mathcal{F}_R(\triangle)$. Since $[[A^{S,\leq}]]$ is the direct product of $|S|$ copies of A, we know that $\triangle$-gfd$(\prod A) = n$, and

$$\triangle\text{-gfd}(A) \leq [[\triangle^{S,\leq}]]\text{-gfd}([[A^{S,\leq}]]).$$

Hence, $\triangle$-gfd$(A) = [[\triangle^{S,\leq}]]$-gfd$([[A^{S,\leq}]])$. $\square$

Similarly, the following holds.

Theorem 4.2 *Let A be an R-module. Then $[[\overline{\nabla}^{S,\leq}]]$-gfd$([[A^{S,\leq}]]) = \overline{\nabla}$-gfd$(A)$.*

Theorem 4.3 *Let R be a perfect and coherent commutative algebra, and $(R, \leq)$ be a standardly stratified algebra. If A is an R-module, then $[[\triangle^{S,\leq}]]$-gfd$([[A^{S,\leq}]]) = d$ if and only if $\mathrm{Ext}^i_{[[R^{S,\leq}]]}([[A^{S,\leq}]], [[\overline{\nabla}(\lambda)^{S,\leq}]]) = 0$ for all $i > d$ and $\lambda \in \Lambda$, but there exists a $\lambda \in \Lambda$ such that*

$$\mathrm{Ext}^d_{[[R^{S,\leq}]]}([[A^{S,\leq}]], [[\overline{\nabla}(\lambda)^{S,\leq}]]) \neq 0.$$

Proof As an R-module we have that $[[\overline{\nabla}(\lambda)^{S,\leq}]] \cong \prod \overline{\nabla}(\lambda)$. Thus, we have that

$$[[\triangle^{S,\leq}]]\text{-gfd}([[A^{S,\leq}]]) = d \Longleftrightarrow \triangle\text{-gfd}(A) = d \Longleftrightarrow \mathrm{Ext}^i_R(A, \overline{\nabla}(\lambda)) = 0$$

for all $i > d$ and $\lambda \in \Lambda$, but there exists a $\lambda \in \Lambda$ such that $\mathrm{Ext}^d_R(A, \overline{\nabla}(\lambda)) \neq 0 \Longleftrightarrow$ $\mathrm{Ext}^i_R(A, \prod \overline{\nabla}(\lambda)) = 0$ for all $i > d$ and $\lambda \in \Lambda$, but there exists a $\lambda \in \Lambda$ such that $\mathrm{Ext}^d_R(A, \prod \overline{\nabla}(\lambda)) \neq 0$. Since

$$\begin{aligned}\mathrm{Ext}^i_{[[R^{S,\leq}]]}([[R^{S,\leq}]] \otimes_R A, [[\overline{\nabla}(\lambda)^{S,\leq}]]) &\cong \mathrm{Ext}^i_R(A, \mathrm{Hom}_{[[R^{S,\leq}]]}([[R^{S,\leq}]], [[\overline{\nabla}(\lambda)^{S,\leq}]])) \\ &\cong \mathrm{Ext}^i_R(A, [[\overline{\nabla}(\lambda)^{S,\leq}]]) \cong \mathrm{Ext}^i_R(A, \prod_{s\in S} \overline{\nabla}(\lambda)) \cong \prod_{s\in S} \mathrm{Ext}^i_R(A, \overline{\nabla}(\lambda)),\end{aligned}$$

we know that $[[\triangle^{S,\leq}]]$-gfd$([[A^{S,\leq}]]) = d$ if and only if

$$\mathrm{Ext}^i_{[[R^{S,\leq}]]}([[A^{S,\leq}]], [[\overline{\nabla}(\lambda)^{S,\leq}]]) = 0$$

for all $i > d$ and $\lambda \in \Lambda$, but there exists a $\lambda \in \Lambda$ such that

$$\mathrm{Ext}^d_{[[R^{S,\leq}]]}([[A^{S,\leq}]], [[\overline{\nabla}(\lambda)^{S,\leq}]]) \neq 0. \quad \square$$

Similarly, one can obtain the following

Theorem 4.4 *Let R be a perfect and coherent commutative algebra, and $(R, \leq)$ be a standardly stratified algebra. If A is an R-module, then $[[\overline{\nabla}^{S,\leq}]]$-gfd$([[A^{S,\leq}]]) = d$ if and only if $\mathrm{Ext}^i_{[[R^{S,\leq}]]}([[\triangle(\lambda)^{S,\leq}]], [[A^{S,\leq}]]) = 0$ for all $i > d$ and $\lambda \in \Lambda$, but there exists a $\lambda \in \Lambda$ such that*

$$\mathrm{Ext}^d_{[[R^{S,\leq}]]}([[\triangle(\lambda)^{S,\leq}]], [[A^{S,\leq}]]) \neq 0.$$

Theorem 4.5 *Let A, B and C be R-modules. If*

$$0 \to [[A^{S,\leq}]] \to [[B^{S,\leq}]] \to [[C^{S,\leq}]] \to 0$$

is an exact sequence, we have the following

(i) *If $[[\triangle^{S,\leq}]]$-gfd$([[B^{S,\leq}]]) > [[\triangle^{S,\leq}]]$-gfd$([[A^{S,\leq}]])$, then*

$$[[\triangle^{S,\leq}]]\text{-gfd}([[C^{S,\leq}]]) = [[\triangle^{S,\leq}]]\text{-gfd}([[B^{S,\leq}]]);$$

(ii) *If $[[\triangle^{S,\leq}]]$-gfd$([[B^{S,\leq}]]) < [[\triangle^{S,\leq}]]$-gfd$([[A^{S,\leq}]])$, then*

$$[[\triangle^{S,\leq}]]\text{-gfd}([[C^{S,\leq}]]) = [[\triangle^{S,\leq}]]\text{-gfd}([[A^{S,\leq}]]) + 1;$$

(iii) *If $[[\triangle^{S,\leq}]]$-gfd$([[B^{S,\leq}]]) = [[\triangle^{S,\leq}]]$-gfd$([[A^{S,\leq}]])$, then*

$$[[\triangle^{S,\leq}]]\text{-gfd}([[C^{S,\leq}]]) \leq [[\triangle^{S,\leq}]]\text{-gfd}([[A^{S,\leq}]]) + 1.$$

Proof For any $[[\overline{\nabla}(\lambda)^{S,\leq}]]$ and n, there exists a long exact sequence

$$\begin{aligned}\cdots &\to \mathrm{Ext}^n_{[[R^{S,\leq}]]}([[C^{S,\leq}]], [[\overline{\nabla}(\lambda)^{S,\leq}]]) \to \mathrm{Ext}^n_{[[R^{S,\leq}]]}([[B^{S,\leq}]], [[\overline{\nabla}(\lambda)^{S,\leq}]])\\ &\to \mathrm{Ext}^n_{[[R^{S,\leq}]]}([[A^{S,\leq}]], [[\overline{\nabla}(\lambda)^{S,\leq}]]) \to \mathrm{Ext}^{n+1}_{[[R^{S,\leq}]]}([[C^{S,\leq}]], [[\overline{\nabla}(\lambda)^{S,\leq}]])\\ &\to \mathrm{Ext}^{n+1}_{[[R^{S,\leq}]]}([[B^{S,\leq}]], [[\overline{\nabla}(\lambda)^{S,\leq}]]) \to \mathrm{Ext}^{n+1}_{[[R^{S,\leq}]]}([[A^{S,\leq}]], [[\overline{\nabla}(\lambda)^{S,\leq}]]) \to \cdots.\end{aligned}$$

Assume $[[\triangle^{S,\leq}]]$-gfd$([[B^{S,\leq}]]) = m$, and $[[\triangle^{S,\leq}]]$-gfd$([[A^{S,\leq}]]) = n$, we will discuss the following three cases.

Case 1 If $m > n$, then $\mathrm{Ext}^m_{[[R^{S,\leq}]]}([[A^{S,\leq}]], [[\overline{\nabla}(\lambda)^{S,\leq}]]) = 0$, but there exists a λ such that

$$\mathrm{Ext}^m_{[[R^{S,\leq}]]}([[B^{S,\leq}]], [[\overline{\nabla}(\lambda)^{S,\leq}]]) \neq 0.$$

According to the above long exact sequence it holds that

$$\mathrm{Ext}^m_{[[R^{S,\leq}]]}([[C^{S,\leq}]], [[\overline{\nabla}(\lambda)^{S,\leq}]]) \neq 0.$$

Thus, we have that

$$\mathrm{Ext}^{m+j}_{[[R^{S,\leq}]]}([[B^{S,\leq}]], [[\overline{\nabla}(\lambda)^{S,\leq}]]) \simeq \mathrm{Ext}^{m+j}_{[[R^{S,\leq}]]}([[C^{S,\leq}]], [[\overline{\nabla}(\lambda)^{S,\leq}]])$$

for any $j > 0$ and $\lambda \in \Lambda$. From Theorem 4.3 we learn that $[[\triangle^{S,\leq}]]$-gfd$([[C^{S,\leq}]]) = [[\triangle^{S,\leq}]]$-gfd$([[B^{S,\leq}]])$.

Case 2 If $m < n$, then $\mathrm{Ext}^n_{[[R^{S,\le}]]}([[B^{S,\le}]], [[\overline{\nabla}(\lambda)^{S,\le}]]) = 0$, but there exists a λ such that

$$\mathrm{Ext}^n_{[[R^{S,\le}]]}([[A^{S,\le}]], [[\overline{\nabla}(\lambda)^{S,\le}]]) \neq 0.$$

According to the above long exact sequence it holds that

$$\mathrm{Ext}^{n+1}_{[[R^{S,\le}]]}([[C^{S,\le}]], [[\overline{\nabla}(\lambda)^{S,\le}]]) \neq 0.$$

Thus, we have that

$$\mathrm{Ext}^{n+j}_{[[R^{S,\le}]]}([[C^{S,\le}]], [[\overline{\nabla}(\lambda)^{S,\le}]]) \simeq \mathrm{Ext}^{n+j-1}_{[[R^{S,\le}]]}([[A^{S,\le}]], [[\overline{\nabla}(\lambda)^{S,\le}]])$$

for any $j > 0$ and $\lambda \in \Lambda$. From Theorem 4.3 we learn that $[[\triangle^{S,\le}]]$-gfd$([[C^{S,\le}]]) = [[\triangle^{S,\le}]]$-gfd$([[A^{S,\le}]]) + 1$.

Case 3 If $m = n$, then $\mathrm{Ext}^{n+1}_{[[R^{S,\le}]]}([[B^{S\le}]], [[\overline{\nabla}(\lambda)^{S,\le}]]) \simeq \mathrm{Ext}^{n+1}_{[[R^{S,\le}]]}([[A^{S,\le}]], [[\overline{\nabla}(\lambda)^{S,\le}]]) = 0$. According to the above long exact sequence it holds that $\mathrm{Ext}^{n+2}_{[[R^{S,\le}]]}([[C^{S,\le}]],[[\overline{\nabla}(\lambda)^{S,\le}]]) = 0$. From Theorem 4.3 we learn that $[[\triangle^{S,\le}]]$-gfd$([[C^{S,\le}]]) \le [[\triangle^{S,\le}]]$-gfd$([[A^{S,\le}]]) + 1$. □

Similarly, one can obtain

Theorem 4.6 *Let A, B and C be R-modules. If*

$$0 \to [[A^{S,\le}]] \to [[B^{S,\le}]] \to [[C^{S,\le}]] \to 0$$

is an exact sequence, then we have the following

(i) *If* $[[\overline{\nabla}^{S,\le}]]$-gfd$([[B^{S,\le}]]) > [[\overline{\nabla}^{S,\le}]]$-gfd$([[C^{S,\le}]])$, *then*

$$[[\overline{\nabla}^{S,\le}]]\text{-gfd}([[A^{S,\le}]]) = [[\overline{\nabla}^{S,\le}]]\text{-gfd}([[C^{S,\le}]]);$$

(ii) *If* $[[\overline{\nabla}^{S,\le}]]$-gfd$([[B^{S,\le}]]) < [[\overline{\nabla}^{S,\le}]]$-gfd$([[C^{S,\le}]])$, *then*

$$[[\overline{\nabla}^{S,\le}]]\text{-gfd}([[A^{S,\le}]]) = [[\overline{\nabla}^{S,\le}]]\text{-gfd}([[C^{S,\le}]]) + 1;$$

(iii) *If* $[[\overline{\nabla}^{S,\le}]]$-gfd$([[B^{S,\le}]]) = [[\overline{\nabla}^{S,\le}]]$-gfd$([[C^{S,\le}]])$, *then*

$$[[\overline{\nabla}^{S,\le}]]\text{-gfd}([[A^{S,\le}]]) \le [[\overline{\nabla}^{S,\le}]]\text{-gfd}([[C^{S,\le}]]) + 1.$$

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