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Application of Bipolar Fuzzy Sets in Semirings

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Abstract On the basis of concepts of bipolar fuzzy sets, we establish a new framework of bipolar fuzzy subsemirings (resp., ideals) which is a generalization of traditional fuzzy subsemirings (resp., ideals) in semirings. The concepts of bipolar fuzzy subsemirings (resp., ideals) are introduced and related properties are investigated by means of positive t-cut, negative s-cut and equivalence relation. Particularly, the notion of a normal bipolar fuzzy ideal is given, and some basic properties are studied in this paper.

Keywords bipolar fuzzy subsemiring (resp., ideal); semiring; equivalence relation; normal bipolar fuzzy ideal.

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1. Introduction

With the application of traditional fuzzy sets presented by Zadeh [11] in the fields of algebraic structures, the study of fuzzy algebras has achieved great success. Many wonderful and valuable results have been obtained by some mathematical researchers, such as Rosenfeld [9], Mordeson, Malik [8], Shum [1] and Zhan [12]. However, in traditional fuzzy sets, the membership degrees of elements are all restricted to the interval $[0, 1]$, which leads to a great difficulty in expressing the difference of the irrelevant elements from the contrary elements in fuzzy sets. In order to avoid this problem, Lee [6] introduced the concept of bipolar fuzzy sets which is an extension of the traditional fuzzy sets. Recently, based on the results of bipolar fuzzy sets, more and more researchers have devoted themselves to applying some results of bipolar fuzzy sets to algebraic structures [5, 7].

On the other hand, in the past several decades, studies on the subject of semirings introduced by Vandiver [10] have attracted researchers' widespread interest, and related results emerged in a large amount [2–4]. While, so far, to our knowledge, bipolar fuzzy sets have not been widely exploited in semirings. So, it is reasonable and necessary to consider a new framework of bipolar fuzzy subsemirings (resp., ideals). In this paper, we will give the concepts of bipolar fuzzy subsemirings (resp., ideals) and investigate related properties. The rest of this article is organized as follows. In Section 2, we introduce the basic notions which will be used in the paper. In Section 3, we present the concepts of bipolar fuzzy subsemirings (resp., ideals), and

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discuss related properties. In Section 4, the characterization of the maps from the set of bipolar fuzzy ideals to the set of ideals are investigated by means of equivalence relations. Finally, we characterize normal bipolar fuzzy ideals in Section 5.

2. Preliminaries

In this section, we review some concepts regarding semiring [8] and bipolar fuzzy set [4].

Suppose that $(S,+)$ and $(S,\cdot)$ are two semigroups, then the algebraic system $(S,+,\cdot)$ is called a semiring, in which the two algebraic structures are connected by the distributive laws: $a\cdot(b+c)=a\cdot b+a\cdot c$ and $(b+c)\cdot a=b\cdot a+c\cdot a$ for all $a,b,c\in S$. A non-empty subset A of a semiring S is called a subsemiring of S if A is closed with respect to the addition and multiplication in $(S,+,\cdot)$. A non-empty subset I of a semiring S is called a left (resp., right) ideal of S if not only I is closed with respect to the addition in $(S,+,\cdot)$ but also $SI\subseteq I$(resp., $IS\subseteq I$). Further, I is called an ideal of S if it is both a left and a right ideal of S.

Let S be a universe of discourse. Denotes $J_{[0,1]}=\{\mu^P|\mu^P: S\to[0,1]\}$, and $J_{[-1,0]}=\{\mu^N|\mu^N: S\to[-1,0]\}$. For every $\mu_A^P\in J_{[0,1]}$ and $\mu_A^N\in J_{[-1,0]}$, we call $A=\{(x,\mu_A^P(x),\mu_A^N(x))\mid x\in S\}$ a bipolar-valued fuzzy set in S, where $\mu_A^P(x)$ is called a positive membership degree which denotes the satisfaction degree of an element x to some specific property about the bipolar-valued fuzzy set A, and $\mu_A^N(x)$ is called a negative membership degree which denotes the satisfaction degree of x to some implicit counter-property about the bipolar-valued fuzzy set A. For the sake of simplicity, we shall use the symbol $A=(\mu_A^P,\mu_A^N)$ for the bipolar-valued fuzzy set $A=\{(x,\mu_A^P(x),\mu_A^N(x))\mid x\in S\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

3. Bipolar fuzzy subsemirings and bipolar fuzzy ideals

Throughout this paper, S and T are semirings unless otherwise specified.

Definition 3.1 *A bipolar fuzzy set $A=(\mu_A^P,\mu_A^N)$ of S is called a bipolar fuzzy subsemiring of S if for all $x,y\in S$:*

(1a) $\mu_A^P(x+y)\geq\min\{\mu_A^P(x),\mu_A^P(y)\}$ *and* $\mu_A^N(x+y)\leq\max\{\mu_A^N(x),\mu_A^N(y)\}$,

(2a) $\mu_A^P(xy)\geq\min\{\mu_A^P(x),\mu_A^P(y)\}$ *and* $\mu_A^N(xy)\leq\max\{\mu_A^N(x),\mu_A^N(y)\}$.

Example 3.2 Consider a semiring $S=\{0,1,2,3\}$ with the following tables:

$+$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	2

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	1
3	0	1	1	1

Define a bipolar fuzzy set $A=(\mu_A^P,\mu_A^N)$ as follows:

	0	1	2	3
μ_A^P	0.5	0.3	0.3	0.3
μ_A^N	-0.7	-0.2	-0.2	-0.2

Then by routine calculations, we know that $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy subsemiring of S.

Definition 3.3 *A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ of S is called a bipolar fuzzy left (resp., right) ideal of S if for all $x, y \in S$ there hold (1a) and*

(2b) $\mu_A^P(xy) \geq \mu_A^P(y)$ *(resp.,* $\mu_A^P(xy) \geq \mu_A^P(x)$*) and* $\mu_A^N(xy) \leq \mu_A^N(y)$ (resp., $\mu_A^N(xy) \leq \mu_A^N(x)$).

If a bipolar fuzzy set is not only a bipolar fuzzy left ideal but also a bipolar fuzzy right ideal of S, then we call it a bipolar fuzzy ideal of S. In this paper, the collection of all bipolar fuzzy ideals of S is denoted by $\mathrm{BFI}(S)$. We note that a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ of S is a bipolar fuzzy ideal of S if and only if it satisfies $(1a)$ and

(3c) $\mu_A^P(xy) \geq \max\{\mu_A^P(x), \mu_A^P(y)\}$ and $\mu_A^N(xy) \leq \min\{\mu_A^N(x), \mu_A^N(y)\}$ for all $x, y \in S$.

Example 3.4 The set N consisting of all non-zero positive integers is a semiring with respect to usual addition and multiplication. Define a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ of N as:

$$\mu_A^P(x) = \begin{cases} 0, & \text{if } 0 < x < 3, \\ 0.5, & \text{if } 3 \leq x < 6, \\ 0.8, & \text{if } x \geq 6 \end{cases}$$

and

$$\mu_A^N(x) = \begin{cases} -0.1, & \text{if } 0 < x < 3, \\ -0.5, & \text{if } 3 \leq x < 6, \\ -1, & \text{if } x \geq 6. \end{cases}$$

Then it is easy to show that A is a bipolar fuzzy ideal of N.

Definition 3.5 *Let $A = (\mu_A^P, \mu_A^N) \in \mathrm{BFI}(S)$, $B = (\mu_B^P, \mu_B^N) \in \mathrm{BFI}(S)$. Then we call $A \subseteq B$ if $\mu_A^P(x) \leq \mu_B^P(x)$ and $\mu_A^N(x) \geq \mu_B^N(x)$ for all $x \in A$.*

Proposition 3.6 *Let A be a non-empty subset of S. Then a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ defined by*

$$\mu_A^P(x) = \begin{cases} m_1, & \text{if } x \in A, \\ m_2, & \text{otherwise} \end{cases} \quad \text{and} \quad \mu_A^N(x) = \begin{cases} n_1, & \text{if } x \in A, \\ n_2, & \text{otherwise,} \end{cases}$$

where $0 \leq m_2 \leq m_1 \leq 1$, $-1 \leq n_1 \leq n_2 \leq 0$, is a bipolar fuzzy left (resp., right) ideal of S if and only if A is a left (resp., right) ideal of S.

From the definition, we can easily derive the following proposition.

Proposition 3.7 *In a semiring S, every bipolar fuzzy ideal of S is a bipolar fuzzy subsemiring.*

However, the converse of Proposition 3.7 is not true in general, which can be shown in the following example.

Example 3.8 Let $S = \{0, a, b\}$ be a set with an addition operation $(+)$ and a multiplication $(\cdot)$ defined as follows:

$+$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

$\cdot$	0	a	b
0	0	b	b
a	b	a	b
b	b	b	b

Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set of S defined as

	0	a	b
μ_A^P	0.3	0.5	0.7
μ_A^N	-0.3	-0.7	-0.5

By routine calculations, we know that $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy subsemiring, but not a bipolar fuzzy ideal.

Definition 3.9 *Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set of S and $(s, t) \in [-1, 0] \times [0, 1]$. We define*

$$A_t^P = \{x \in S \mid \mu_A^P(x) \geq t\} \text{ and } A_s^N = \{x \in S \mid \mu_A^N(x) \leq s\},$$

and call them positive $t-cut$ of A and negative $s-cut$ of A, respectively. Further, for every $k \in [0, 1]$, the set $A_t^P \cap A_{-t}^N$ is called the $k-cut$ of A.

Theorem 3.10 *Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set of S. Then $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy subsemiring of S if and only if*

(i) for all $t \in [0, 1]$, $A_t^P \neq \emptyset \Rightarrow A_t^P$ is a subsemiring of S;

(ii) for all $s \in [-1, 0]$, $A_s^N \neq \emptyset \Rightarrow A_s^N$ is a subsemiring of S.

Proof Suppose that $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy subsemiring of S, and $t \in [0, 1]$ satisfying $A_t^P \neq \emptyset$. If $x, y \in A_t^P$, then we have $\mu_A^P(x) \geq t$ and $\mu_A^P(y) \geq t$ which imply that $\mu_A^P(x + y) \geq \min\{\mu_A^P(x), \mu_A^P(y)\} \geq t$ and $\mu_A^P(xy) \geq \min\{\mu_A^P(x), \mu_A^P(y)\} \geq t$. So, $x + y$ and $xy \in \mu_A^P$. Then μ_A^P is a subsemiring of S. Similarly, we can prove (ii).

Conversely, assume that (i) and (ii) are all valid. For any $x \in S$, letting $\mu_A^P(x) = t$, and $\mu_A^N(x) = s$, we can obtain that $x \in A_t^P \cap A_s^N$, then A_t^P and A_s^N are all non-empty. If $A = (\mu_A^P, \mu_A^N)$ is not a bipolar fuzzy subsemiring of S, then there exist $x_1, x_2 \in A_t^P$, and $t \in [0, 1]$ satisfying

$$\mu_A^P(x_1 + x_2) < t < \min\{\mu_A^P(x_1), \mu_A^P(x_2)\} \text{ and } \mu_A^P(x_1 x_2) < t < \min\{\mu_A^P(x_1), \mu_A^P(x_2)\}.$$

Therefore, x_1 and $x_2 \in A_t^P$, but $x_1 + x_2$ and $x_1 x_2 \notin A_t^P$. Similarly, we can get that x_1 and $x_2 \in A_s^N$, but $x_1 + x_2$ and $x_1 x_2 \notin A_s^N$. Those are all contradictions obviously. Consequently, $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy subsemiring of S.

Corollary 3.11 *If $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy subsemiring of S, then the k-cut of $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy subsemiring of S for all $k \in [0, 1]$.*

Theorem 3.12 *Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set of S. Then $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy ideal of S if and only if* *(i) for all $t \in [0,1]$ if $A_t^P \neq \emptyset \Rightarrow A_t^P$ is an ideal of S;*

(ii) for all $s \in [-1,0]$ if $A_s^N \neq \emptyset \Rightarrow A_s^N$ is an ideal of S.

Proof The proof is similar to that of Theorem 3.10.

For the sake of simplicity, for a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ we use the notation $S^{(t,s)} = \{x \in S \mid \mu_A^P(x) \geq t\} \cap \{x \in S \mid \mu_A^N(x) \leq s\}$.

Corollary 3.13 *If $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy ideal of S, then $S^{(t,s)}$ is an ideal of S for all $(t,s) \in [0,1] \times [-1,0]$. In particular, the non-empty k-cut of $A = (\mu_A^P, \mu_A^N)$ is an ideal of S for all $k \in [0,1]$.*

In general, $\{x \in S \mid \mu_A^P(x) \geq t\} \cup \{x \in S \mid \mu_A^N(x) \leq s\}$ is often not an ideal of S, which can be seen in the following example.

Example 3.14 For the semiring S defined in Example 3.2, let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set of S defined by

	0	1	2	3
μ_A^P	0.1	0.7	0.5	0.3
μ_A^N	-0.4	-0.9	-0.6	-0.3

We have

$$A_t^P = \begin{cases} \emptyset, & \text{if } 0.7 < t \leq 1, \\ \{1\}, & \text{if } 0.5 < t \leq 0.7, \\ \{1,2\}, & \text{if } 0.3 < t \leq 0.5, \\ \{1,2,3\}, & \text{if } 0.1 < t \leq 0.3, \\ S, & \text{if } 0 \leq t \leq 0.1 \end{cases} \quad \text{and} \quad A_s^N = \begin{cases} \emptyset, & \text{if } -1 \leq s < -0.9, \\ \{1\}, & \text{if } -0.9 \leq s < -0.6, \\ \{1,2\}, & \text{if } -0.6 \leq s < -0.4, \\ \{0,1,2\}, & \text{if } -0.4 \leq s < -0.3, \\ S, & \text{if } -0.3 \leq t \leq 0. \end{cases}$$

A routine calculation shows that $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy ideal of S, but $A_{0.6}^P \cup A_{-0.5}^N = \{1\} \cup \{1,2\} = \{1,2\}$ is not an ideal of S. Likewise, $A_{0.5}^P \cup A_{-0.5}^N = \{1,2\}$ is not an ideal of S.

Theorem 3.15 *If $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy ideal of S and $\mu_A^P(x) + \mu_A^N(x) \geq 0$, for all $x \in S$, then $A_k^P \cup A_{-k}^N$ is an ideal of S for all $k \in [0,1]$.*

Proof Let $k \in [0,1]$. Then we have $A_k^P \neq \emptyset$ and $A_{-k}^N \neq \emptyset$ which are all ideals of S by Theorem 3.12. For all $x_1, x_2 \in A_k^P \cup A_{-k}^N$ and $x \in S$, to complete the proof, we just need to consider the following four cases:

(i) $x_1 \in A_k^P, x_2 \in A_k^P$, (ii) $x_1 \in A_k^P, x_2 \in A_{-k}^N$, (iii) $x_1 \in A_{-k}^N, x_2 \in A_k^P$, (iv) $x_1 \in A_{-k}^N, x_2 \in A_{-k}^N$.

Case (i) implies that $\mu_A^P(x_1) \geq k, \mu_A^P(x_2) \geq k$. In fact, $A = (\mu_A^P, \mu_A^N) \in \mathrm{BFI}(S)$, so we have $\mu_A^P(x_1 + x_2) \geq \min\{\mu_A^P(x_1),\ \mu_A^P(x_2)\} \geq k$, $\mu_A^P(xx_1) \geq \mu_A^P(x_1) \geq k$ and $\mu_A^P(x_1x) \geq \mu_A^P(x_1) \geq k$. Therefore, $x_1 + x_2, xx_1$ and $x_1x \in A_k^P \subseteq A_k^P \cup A_{-k}^N$. The proof of case (iv) is similar to that of case (i). For case (ii), we have $\mu_A^P(x_1) \geq k$, $\mu_A^N(x_2) \leq -k$. Since $\mu_A^P(x) + \mu_A^N(x) \geq 0$, $\mu_A^P(x_2) \geq -\mu_A^N(x_2) \geq k$, $\mu_A^P(x_1 + x_2) \geq \min\{\mu_A^P(x_1),\ \mu_A^P(x_2)\} \geq \min\{\mu_A^P(x_1), -\mu_A^N(x_2)\} \geq k$, $\mu_A^P(xx_1) \geq \mu_A^P(x_1) \geq k$ and $\mu_A^P(x_1x) \geq \mu_A^P(x_1) \geq k$. Then $x_1 + x_2, xx_1, x_1x \in A_k^P \subseteq A_k^P \cup A_{-k}^N$. The proof of case (iii) is similar to that of case (ii). Hence $A_k^P \cup A_{-k}^N$ is an ideal of S. □

4. Equivalence relations on bipolar fuzzy ideals

For any $(t,s)\in[0,1]\times[-1,0]$, define two binary relations P^t and N^s on BFI(S) as follows: $(A,B)\in P^t \Leftrightarrow A_t^P = B_t^P$ and $(A,B)\in N^s \Leftrightarrow A_s^N = B_s^N$, for all $A=(\mu_A^P,\mu_A^N)$ and $B=(\mu_B^P,\mu_B^N)$. It is easy to know P^t and N^s are equivalence relations on BFI(S).

For any $A=(\mu_A^P,\mu_A^N)\in$ BFI(S), we use $[A]_{P^t}$ (resp., $[A]_{N^s}$) to denote the equivalence class of $A=(\mu_A^P,\mu_A^N)$ modular P^t(resp., N^s). For all $A=(\mu_A^P,\mu_A^N)\in$ BFI(S), the family of $[A]_{P^t}$ (resp., $[A]_{N^s}$) is denoted by BFI(S)$/P^t$ (resp., BFI(S)$/N^s$). So BFI(S)$/P^t=\{[A]_{P^t}\mid A=(\mu_A^P,\mu_A^N)\in \mathrm{BFI}(S)\}$(resp., BFI($S$)$/N^s=\{[A]_{N^s}\mid A=(\mu_A^P,\mu_A^N)\in \mathrm{BFI}(S)\}$). Let $I(S)$ be the family of all ideals of S. For all $A=(\mu_A^P,\mu_A^N)\in$ BFI(S), we define two maps as follows:

$$f_t:\mathrm{BFI}(S)\to I(S)\cup\{\emptyset\},\quad A\to A_t^P,$$

$$g_s:BFI(S)\to I(S)\cup\{\emptyset\},\quad A\to A_s^N.$$

Then f_t and g_s are clearly well-defined.

Theorem 4.1 *For any $(t,s)\in(0,1)\times(-1,0)$, the maps f_t and g_s are all surjective.*

Proof Clearly, a bipolar fuzzy set $\mathbf{0}=(\mathbf{0}^P,\mathbf{0}^N)$ is a bipolar fuzzy ideal of S, where $\mathbf{0}^P(x)=\mathbf{0}^N(x)=0$ for all $x\in S$. Then we have

$$f_t(\mathbf{0})=\mathbf{0}_t^P=\{x\in S\mid \mathbf{0}^P(x)\ge t\}=\emptyset,\quad g_s(\mathbf{0})=\mathbf{0}_s^N=\{x\in S\mid \mathbf{0}^N(x)\le s\}=\emptyset.$$

For any non-empty B in $I(S)$, we consider a bipolar fuzzy set $B_\sim=(\mu_{B_\sim}^P,\mu_{B_\sim}^N)$ in S, where

$$\mu_{B_\sim}^P:S\to[0,1],\quad \mu_{B_\sim}^P(x)=\begin{cases}1, & \text{if } x\in B,\\ 0, & \text{otherwise}\end{cases}$$

and

$$\mu_{B_\sim}^N:S\to[-1,0],\quad \mu_{B_\sim}^N(x)=\begin{cases}-1, & \text{if } x\in B,\\ 0, & \text{otherwise.}\end{cases}$$

By Proposition 3.6, we have $B_\sim=(\mu_{B_\sim}^P,\mu_{B_\sim}^N)\in$ BFI(S). Consequently, we obtain

$$f_t(B_\sim)=B_{\sim t}^P=\{x\in S\mid \mu_{B_\sim}^P(x)\ge t\}=\{x\in S\mid \mu_{B_\sim}^P(x)=1\}=B,$$

and

$$g_s(B_\sim)=B_{\sim s}^N=\{x\in S\mid \mu_{B_\sim}^N(x)\le s\}=\{x\in S\mid \mu_{B_\sim}^N(x)=-1\}=B.$$

Therefore, f_t and g_s are surjective.

Theorem 4.2 *The quotient sets* BFI(S)$/P^t$ *and* BFI(S)$/N^s$ *are equipotent to $I(S)\cup\emptyset$ for all $(t,s)\in(0,1)\times(-1,0)$.*

Proof For all $(t,s)\in(0,1)\times(-1,0)$ and $A=(\mu_A^P,\mu_A^N)\in BFI(S)$, let

$$f_t^*:\mathrm{BFI}(S)/P^t\to I(S)\cup\{\emptyset\},\quad [A]_{P^t}\to f_t(A),$$

and

$$g_s^*:\mathrm{BFI}(S)/N^s\to I(S)\cup\{\emptyset\},\quad [A]_{N^s}\to g_s(A),$$

respectively. For every $A, B \in \mathrm{BFI}(S)$, if $\mu_A^P = \mu_B^P$ and $\mu_A^N = \mu_B^N$, then $(A, B) \in P^t$ and $(A, B) \in N^s$, which means $[A]_{P^t} = [B]_{P^t}$ and $[A]_{N^s} = [B]_{N^s}$. Thus f_t and g_s are injective.

For any non-empty B in $I(S)$, consider the bipolar fuzzy ideal $B_\sim = (\mu_{B_\sim}^P, \mu_{B_\sim}^N)$ which is given in the proof of Theorem 4.1, then we have

$$f_t^*([B_\sim]_{P^t}) = f_t(B_\sim) = B_{\sim t}^P = B \text{ and } g_s^*([B_\sim]_{N^s}) = g_s(B_\sim) = B_{\sim s}^N = B.$$

For the bipolar fuzzy ideal $\mathbf{0} = (\mathbf{0}^P, \mathbf{0}^N)$ of S, we have

$$f_t^*([\mathbf{0}]_{P^t}) = f_t(\mathbf{0}) = \mathbf{0}_t^P = \{x \in S \mid \mathbf{0}^P(x) \geq t\} = \emptyset$$

and

$$g_s^*([\mathbf{0}]_{N^s}) = g_s(\mathbf{0}) = \mathbf{0}_s^N = \{x \in S \mid \mathbf{0}^N(x) \leq s\} = \emptyset.$$

Hence f_t and g_s are surjective. This completes the proof. □

For any $0 < k < 1$, we define another relation S^k on $\mathrm{BFI}(S)$ as follows:

$$(A, B) \in S^k \Leftrightarrow A_k = B_k,$$

where $A_k = A_k^P \cap A_{-k}^N$. Then the relation S^k is also an equivalence relation on $\mathrm{BFI}(S)$.

Theorem 4.3 *Let $0 < k < 1$. Then the map $\varphi_k : \mathrm{BFI}(S) \to I(S) \cup \emptyset$ defined by $\varphi_k(A) = A_k$ is surjective.*

Proof *Suppose $0 < k < 1$. We have $\varphi_k(\mathbf{0}) = \mathbf{0}_k^P \cap \mathbf{0}_{-k}^N = \emptyset$. For any non-empty B in $\mathrm{BFI}(S)$, considering a bipolar fuzzy ideal $B_\sim = (\mu_{B_\sim}^P, \mu_{B_\sim}^N)$ which is given in the proof of Theorem 4.1, we can obtain*

$$\varphi_k(B_\sim) = B_{\sim k} = B_{\sim k}^P \cap B_{\sim(-k)}^N = \{x \in S \mid \mu_{B_\sim}^P(x) \geq k\} \cap \{x \in S \mid \mu_{B_\sim}^N(x) \leq (-k)\} = B.$$

Therefore, φ_k is surjective.

Theorem 4.4 *Let $0 < k < 1$. Then the quotient set $\mathrm{BFI}(S)/S^k$ is equipotent to $I(S) \cup \emptyset$.*

Proof Suppose that $0 < k < 1$, then $\varphi_k^* : \mathrm{BFI}(S)/S^k \to I(S) \cup \emptyset$ is a map defined by $\varphi_k^*([A]_{S^k}) = \varphi_k(A)$ for all $[A]_{S^k} \in \mathrm{BFI}(S)/S^k$. For every $[A]_{S^k}, [B]_{S^k} \in BFI(S)/S^k$, let $\varphi_k^*([A]_{S^k}) = \varphi_k^*([B]_{S^k})$. Then $\varphi_k(A) = \varphi_k(B)$. Namely, $A_k = B_k$, which implies that $(A, B) \in S^k$. Thus $[A]_{S^k} = [B]_{S^k}$ and φ_k^* is injective. Moreover, for any non-empty B in $I(S)$, consider the bipolar fuzzy ideal $B_\sim = (\mu_{B_\sim}^P, \mu_{B_\sim}^N)$ which is given in the proof of Theorem 4.1. Then similarly to the proof of Theorem 4.1, we can prove that φ_k^* is surjective. This completes the proof. □

5. Normal bipolar fuzzy ideals

In this section, we introduce and characterize normal bipolar fuzzy ideals of semirings.

Definition 5.1 *A bipolar fuzzy ideal $A = (\mu_A^P, \mu_A^N)$ of S is said normal if there exists an element $x \in S$ such that $A(x) = (1, -1)$, i.e., $\mu_A^P(x) = 1$ and $\mu_A^N(x) = -1$.*

Definition 5.2 *An element $x_0 \in S$ is called an extremal element of a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ if $\mu_A^P(x_0) \geq \mu_A^P(x)$ and $\mu_A^N(x_0) \leq \mu_A^N(x)$ for all $x \in S$.*

From the above definitions, we can easily obtain the following proposition.

Proposition 5.3 *A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ of S is a normal bipolar fuzzy ideal if and only if $A(x) = (1, -1)$ for its all extremal elements.*

Theorem 5.4 *If x_0 is an extremal element of a bipolar fuzzy ideal $A = (\mu_A^P, \mu_A^N)$ of S, then a bipolar fuzzy set $\bar{A} = (\bar{\mu}_A^P, \bar{\mu}_A^N)$ of S, defined by $\bar{\mu}_A^P(x) = \mu_A^P(x) + 1 - \mu_A^P(x_0)$ and $\bar{\mu}_A^N(x) = \mu_A^N(x) - 1 - \mu_A^N(x_0)$ for all $x \in S$, is a normal bipolar fuzzy ideal of S containing A.*

Proof First, we claim that $\bar{A}$ is normal. In fact, since $\bar{\mu}_A^P(x) = \mu_A^P(x) + 1 - \mu_A^P(x_0)$, $\bar{\mu}_A^N(x) = \mu_A^N(x) - 1 - \mu_A^N(x_0)$ and x_0 is an extremal element of A, we have $\bar{\mu}_A^P(x_0) = 1$, $\bar{\mu}_A^N(x_0) = -1$, $\bar{\mu}_A^P(x) \in [0, 1]$ and $\bar{\mu}_A^N(x) \in [-1, 0]$ for all $x \in S$. Thus $\bar{A} = (\bar{\mu}_A^P, \bar{\mu}_A^N)$ is normal.

Next we show $\bar{A}$ is a bipolar fuzzy ideal. For all $x, y \in S$, we have

$$\begin{aligned}\bar{\mu}_A^P(x+y) &= \mu_A^P(x+y) + 1 - \mu_A^P(x_0) \\ &\geq \min\{\mu_A^P(x), \mu_A^P(y)\} + 1 - \mu_A^P(x_0) \\ &= \min\{\mu_A^P(x) + 1 - \mu_A^P(x_0), \mu_A^P(y) + 1 - \mu_A^P(x_0)\} \\ &= \min\{\bar{\mu}_A^P(x), \bar{\mu}_A^P(y)\}\end{aligned}$$

and

$$\begin{aligned}\bar{\mu}_A^N(x+y) &= \mu_A^N(x+y) - 1 - \mu_A^N(x_0) \\ &\leq \max\{\mu_A^N(x), \mu_A^N(y)\} - 1 - \mu_A^N(x_0) \\ &= \max\{\mu_A^N(x) - 1 - \mu_A^N(x_0), \mu_A^N(y) - 1 - \mu_A^N(x_0)\} \\ &= \max\{\bar{\mu}_A^N(x), \bar{\mu}_A^N(y)\}.\end{aligned}$$

Thus, ($1a$) is valid. Similarly, we can prove that $\bar{A} = (\bar{\mu}_A^P, \bar{\mu}_A^N)$ satisfies (3c). Hence $\bar{A} = (\bar{\mu}_A^P, \bar{\mu}_A^N)$ is a normal bipolar fuzzy ideal of S. Clearly, $A \subseteq \bar{A}$. $\square$

Corollary 5.5 *From the definition of $\bar{A}$ in Theorem 5.4, we get that $\bar{\bar{A}} = \bar{A}$ for all $A \in \mathrm{BFI}(S)$. In particular, if A is normal, then $\bar{A} = A$.*

Let $\mathcal{N}(S)$ be the set consisting of all normal bipolar fuzzy ideals of S. Then $\mathcal{N}(S)$ is a poset under the set inclusion obviously.

Theorem 5.6 *A non-constant maximal element of $(\mathcal{N}(S), \subseteq)$ only takes a value among $(0, 0)$, $(1, -1)$ and $(1, 0)$.*

Proof Assume that $A = (\mu_A^P, \mu_A^N) \in \mathcal{N}(S)$ is a non-constant maximal element of $(\mathcal{N}(S), \subseteq)$. Then $\mu_A^P(x_0) = 1$ and $\mu_A^N(x_0) = -1$ for some $x_0 \in S$. Let $x \in S$ and $\mu_A^P(x) \neq 1$. Then $\mu_A^P(x) = 0$. Otherwise, there exists $m \in S$ such that $0 < \mu_A^P(m) < 1$. On the other hand, let $A_m = (\alpha_A^P, \alpha_A^N)$ be a bipolar fuzzy set of S defined by $\alpha_A^P(x) = \frac{1}{2}(\mu_A^P(x) + \mu_A^P(m))$ and $\alpha_A^N(x) = \frac{1}{2}(\mu_A^N(x) + \mu_A^N(m))$

for all $x \in S$. Apparently, A_m is well defined. So, for all $x \in S$, we have

$$\alpha_A^P(x_0) = \frac{1}{2}(\mu_A^P(x_0) + \mu_A^P(m)) \geq \frac{1}{2}(\mu_A^P(x) + \mu_A^P(m)) = \alpha_A^P(x)$$

and

$$\alpha_A^N(x_0) = \frac{1}{2}(\mu_A^N(x_0) + \mu_A^N(m)) \leq \frac{1}{2}(\mu_A^N(x) + \mu_A^N(m)) = \alpha_A^N(x).$$

Further, for all $x, y \in S$, we have

$$\begin{aligned}\alpha_A^P(x+y) &= \frac{1}{2}(\mu_A^P(x+y) + \mu_A^P(m)) \\ &\geq \frac{1}{2}(\min\{\mu_A^P(x), \mu_A^P(y)\} + \mu_A^P(m)) \\ &= \min\{\frac{1}{2}(\mu_A^P(x) + \mu_A^P(m)), \frac{1}{2}(\mu_A^P(y) + \mu_A^P(m))\} \\ &= \min\{\alpha_A^P(x), \alpha_A^P(y)\}\end{aligned}$$

and

$$\begin{aligned}\alpha_A^P(xy) &= \frac{1}{2}(\mu_A^P(xy) + \mu_A^P(m)) \\ &\geq \frac{1}{2}(\max\{\mu_A^P(x), \mu_A^P(y)\} + \mu_A^P(m)) \\ &= \max\{\frac{1}{2}(\mu_A^P(x) + \mu_A^P(m)), \frac{1}{2}(\mu_A^P(y) + \mu_A^P(m))\} \\ &= \max\{\alpha_A^P(x), \alpha_A^P(y)\}.\end{aligned}$$

By the same argument, we can prove

$$\alpha_A^N(x+y) \leq \max\{\alpha_A^N(x), \alpha_A^N(y)\} \text{ and } \alpha_A^N(xy) \leq \min\{\alpha_A^N(x), \alpha_A^N(y)\}.$$

This means A_m is a bipolar fuzzy ideal of S with the same extremal elements as A. So, by Theorem 5.4, a bipolar fuzzy set $\bar{A}_m = (\bar{\alpha}_A^P, \bar{\alpha}_A^N)$ belongs to $\mathcal{N}(S)$, where

$$\bar{\alpha}_A^P(x) = \alpha_A^P(x) + 1 - \alpha_A^P(x_0) = \frac{1}{2}(1 + \mu_A^P(x))$$

and

$$\bar{\alpha}_A^N(x) = \alpha_A^N(x) - 1 - \alpha_A^N(x_0) = \frac{1}{2}(\mu_A^N(x) - 1).$$

Clearly, $A \subseteq \bar{A}_m$. Since $\bar{\alpha}_A^P(x) = \frac{1}{2}(1 + \mu_A^P(x)) > \mu_A^P(x)$, A is a proper subset of $\bar{A}_m$. By the definition, we have $\bar{\alpha}_A^P(m) = \frac{1}{2}\{1 + \mu_A^P(m)\} < 1 = \bar{\alpha}_A^P(x_0)$. Therefore, $\bar{A}_m$ is non-constant, and A is not a maximal element of $\mathcal{N}(S)$. This is a contradiction. Thus, μ_A^P only takes two possible values 0 and 1. Likewise, we can prove that μ_A^N just takes a value between 0 and -1. This implies that all the possible values of A are $(0,0), (0,-1), (1,-1)$ and $(1,0)$. Further, if A takes a value from above four values, then

$$\begin{aligned}&S^{(0,0)} = \{x \in S \mid \mu_A^P(x) \geq 0\} \cap \{x \in S \mid \mu_A^N(x) \leq 0\} = S, \\ &S^{(0,-1)} = \{x \in S \mid \mu_A^P(x) \geq 0\} \cap \{x \in S \mid \mu_A^N(x) \leq -1\} = \{x \in S \mid \mu_A^N(x) = -1\}, \\ &S^{(1,-1)} = \{x \in S \mid \mu_A^P(x) \geq 1\} \cap \{x \in S \mid \mu_A^N(x) \leq -1\} = \{x \in S \mid \mu_A^P(x) = 1, \mu_A^N(x) = -1\}, \\ &S^{(1,0)} = \{x \in S \mid \mu_A^P(x) \geq 1\} \cap \{x \in S \mid \mu_A^N(x) \leq 0\} = \{x \in S \mid \mu_A^P(x) = 1\}\end{aligned}$$

are all non-empty ideals satisfying

(i) $S^{(1,-1)} \subseteq S^{(0,-1)} \subseteq S^{(0,0)}$ and (ii) $S^{(1,-1)} \subseteq S^{(1,0)} \subseteq S^{(0,0)}$.

For case (i), according to Proposition 3.6, a bipolar fuzzy set $B = (\mu_B^P, \mu_B^N)$ defined by

$$\mu_B^P(x) = \begin{cases} 1, & \text{if } x \in S^{(0,-1)}; \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \mu_B^N(x) = \begin{cases} -1, & \text{if } x \in S^{(0,-1)}; \\ 0, & \text{otherwise,} \end{cases}$$

is a bipolar fuzzy ideal of S. Moreover, it is normal. Now, for all $x \in S^{(0,-1)}$, we have $\mu_B^P(x) = 1 \geq \mu_A^P(x)$ and $\mu_B^N(x) = -1 = \mu_A^N(x)$, that is $A \subseteq B$. For all $x \in S^{(0,0)} - S^{(0,-1)}$, we have $\mu_B^P(x) = 0 = \mu_B^N(x)$. Since μ_A^P only takes two possible values 0 and 1, if $\mu_A^P(x) = 0$, then $\mu_A^P(x) = \mu_B^P(x) = 0, \mu_A^N(x) \leq 0 = \mu_B^N(x)$, hence $B \subseteq A$. Otherwise, if $\mu_A^P(x) = 1$, then $\mu_A^P(x) \geq \mu_B^P(x), \mu_A^N(x) \leq 0 = \mu_B^N(x)$. Whence, $B \subseteq A$. In addition, for all $x \in S^{(0,-1)} - S^{(1,-1)}$, we have $\mu_A^P(x) = 0 < 1 = \mu_B^P(x)$ and $\mu_A^N(x) = -1 = \mu_B^N(x)$. Then $A \subset B$, which contradicts the fact that A is a non-constant maximal element of $(\mathcal{N}(S), \subseteq)$. Therefore, $A \neq (0,-1)$. For case (ii), we can show that $A \neq (0,-1)$ similarly. Hence A only takes a value among $(0,0), (1,-1)$ and $(1,0)$. $\Box$

Definition 5.7 *A non-constant bipolar fuzzy ideal A of S is called a maximal element of S when $\bar{A}$ defined in Theorem 5.4 is a maximal element of the poset $(\mathcal{N}(S), \subseteq)$.*

Proposition 5.8 *A maximal bipolar fuzzy ideal of S is normal, and it takes a value among $(0,0), (1,-1)$ and $(1,0)$.*

Proof Let $A = (\mu_A^P, \mu_A^N)$ be a maximal bipolar fuzzy ideal of S. Then $\bar{A}$ is a maximal element of the poset $(\mathcal{N}(S), \subseteq)$. By Theorem 5.6, $\bar{A}$ only takes a value among $(0,0), (1,-1)$ and $(1,0)$. In addition, $A \subseteq \bar{A}$ from Theorem 5.4. Hence, A also takes a value among $(0,0), (1,-1)$ and $(1,0)$. Next we show A is normal. Since $\bar{\mu}_A^P(x)$ only takes a value among 0 and 1, and $\bar{\mu}_A^P(x) = \mu_A^P(x) + 1 - \mu_A^P(x_0)$, $\bar{\mu}_A^P(x) = 1$ if and only if $\mu_A^P(x) = \mu_A^P(x_0)$, and $\bar{\mu}_A^P(x) = 0$ if and only if $\mu_A^P(x) = \mu_A^P(x_0) - 1$, where x_0 is an extremal element of A. From $A \subseteq \bar{A}$ we get $\mu_A^P(x) \leq \bar{\mu}_A^P(x)$ for all $x \in S$. Thus $\bar{\mu}_A^P(x) = 0$, which implies $\mu_A^P(x) = 0$. That is, $\mu_A^P(x_0) = 1$. Similarly, by $\bar{\mu}_A^N(x) = \mu_A^N(x) - 1 - \mu_A^N(x_0)$, we have $\mu_A^N(x_0) = -1$. Therefore, A is normal. $\Box$

Proposition 5.9 *Let $A = (\mu_A^P, \mu_A^N)$ be a maximal bipolar fuzzy ideal of S. Then $S^{(1,-1)}$ is a maximal ideal of S.*

Proof According to Corollary 3.11, $S^{(1,-1)}$ is an ideal of S. Next, we show that it is a maximal ideal of S. Let $T = S^{(1,-1)} = \{x \in S \mid \mu_A^P(x) = 1, \mu_A^N(x) = -1\}$. From Theorem 5.6, $\mu_A^P(x)$ only takes a value among 0 and 1. Thus $T \neq S$. If M is an ideal of S containing T, then $\mu_T^P \subseteq \mu_M^P$. Because $\mu_A^P = \mu_T^P$ and $\mu_A^P(x)$ only takes a value between 0 and 1, μ_M^P also takes them. While, by the assumption, A is a maximal bipolar fuzzy ideal of S. Thus $\mu_A^P = \mu_T^P = \mu_M^P$ or $\mu_M^P(x) = 1$ for all $x \in S$. In the last case, $T = S$, which is a contradiction. Hence $\mu_A^P = \mu_T^P = \mu_M^P$, i.e., $M = T$. This implies that $S^{(1,-1)}$ is a maximal ideal of S. $\Box$

Definition 5.10 *A non-empty bipolar fuzzy ideal of S is called completely normal if there exists $x \in S$ such that $A(x) = (0,0)$.*

Let $\mathcal{C}(S)$ be the family consisting of all completely normal bipolar fuzzy ideals of S. Clearly, $\mathcal{C}(S) \subseteq \mathcal{N}(S)$. So we can obtain the following result.

Proposition 5.11 *A non-constant maximal element of $(\mathcal{N}(S), \subseteq)$ is also a maximal element of $(\mathcal{C}(S), \subseteq)$.*

Proof Let A be a non-constant maximal element of $(\mathcal{N}(S), \subseteq)$. By Theorem 5.6, A only takes a value among $(0,0),(1,-1)$ and $(1,0)$. Then there exists $x_0, x_1, x_2 \in S$ such that $A(x_0) = (0,0), A(x_1) = (1,-1)$ and $A(x_2) = (1,0)$. Hence $A \in \mathcal{C}(S)$. Further, assume that $B \in \mathcal{C}(S)$ and $A \subseteq B$, then $A \subseteq B$ in $\mathcal{N}(S)$. Since A is a maximal element of $(\mathcal{N}(S), \subseteq)$ and B is non-constant, $A = B$. Consequently, A is a maximal element of $(\mathcal{C}(S), \subseteq)$. □

From the above results, we can easily obtain the following proposition.

Proposition 5.12 *Every maximal bipolar fuzzy ideal of S is completely normal.*

Theorem 5.13 *Let $f : [0,1] \to [0,1]$ and $g : [-1,0] \to [-1,0]$ be two increasing functions, and $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set of S. Then $A_{(f,g)} = (\mu_{A_f}^P, \mu_{A_g}^N)$ defined by $\mu_{A_f}^P(x) = f(\mu_A^P(x))$ and $\mu_{A_g}^N(x) = g(\mu_A^N(x))$ for all $x \in S$ is a bipolar fuzzy ideal of S if and only if $A = (\mu_A^P, \mu_A^N) \in \mathrm{BFI}(S)$. In particular, if $f(\mu_A^P(0)) = 1$ and $g(\mu_A^N(0)) = -1$, then $A_{(f,g)}$ is normal.*

Proof Let $A_{(f,g)} = (\mu_{A_f}^P, \mu_{A_g}^N) \in \mathrm{BFI}(S)$. Then for all $x, y \in S$, we have

$$\begin{aligned} f(\mu_A^P(x+y)) &= \mu_{A_f}^P(x+y) \\ &\geq \min\{\mu_{A_f}^P(x), \mu_{A_f}^P(y)\} = \min\{f(\mu_A^P(x)), f(\mu_A^P(y))\} \\ &= f(\min\{\mu_A^P(x), \mu_A^P(y)\}). \end{aligned}$$

Since f is increasing, it follows that $\mu_A^P(x+y) \geq \min\{\mu_A^P(x), \mu_A^P(y)\}$. Conversely, if $A = (\mu_A^P, \mu_A^N) \in \mathrm{BFI}(S)$, then for all $x, y \in S$, we have

$$\begin{aligned} \mu_{A_f}^P(x+y) &= f(\mu_A^P(x+y)) \\ &\geq f(\min\{\mu_A^P(x), \mu_A^P(y)\}) = \min\{f(\mu_A^P(x)), f(\mu_A^P(y))\} \\ &= \min\{\mu_{A_f}^P(x), \mu_{A_f}^P(y)\}. \end{aligned}$$

Similarly, we can obtain

$$\mu_{A_g}^N(x+y) \leq \max\{\mu_{A_g}^N(x), \mu_{A_g}^N(y)\} \Longleftrightarrow \mu_A^N(x+y) \leq \max\{\mu_A^N(x), \mu_A^N(y)\}.$$

Thus $A_{(f,g)} = (\mu_{A_f}^P, \mu_{A_g}^N)$ satisfies (1a) if and only if $A = (\mu_A^P, \mu_A^N)$ satisfies (1a). Analogously, $A_{(f,g)} = (\mu_{A_f}^P, \mu_{A_g}^N)$ satisfies (3c) if and only if $A = (\mu_A^P, \mu_A^N)$ satisfies the same condition. This completes the proof. □

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