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# Exact Multiplicity of One-Sign Solutions for a Class of Quasilinear Eigenvalue Problems

**Guowei DAI***, **Xiaoling HAN**
*Department of Mathematics, Northwest Normal University, Gansu 730070, P. R. China*

**Abstract** This paper concerns the exact multiplicity of one-sign solutions of a class of quasilinear elliptic eigenvalue problems with asymptotical nonlinearity at 0 and $\infty$. The proofs of our main results are based upon bifurcation techniques and stability analysis.

**Keywords** bifurcation; one-sign solution; exact multiplicity.

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## 1. Introduction

In [1], Shi and Wang studied the precise global bifurcation diagrams of one-sign and sign-changing solutions for a semilinear elliptic equation. In [2], Ma and Thompson studied the global bifurcation structure of nodal solutions for a one-dimensional weighted semilinear elliptic equation. In [3], Dai and Ma studied the global bifurcation structure of one-sign solutions for $p$-Laplacian 0-Dirichlet problem with asymptotical nonlinearities at 0 and $\infty$.

Motivated by the above papers, we shall investigate the problems of the type

$$\begin{cases} -\mathrm{div}\,(\varphi_p(\nabla u)) = \lambda a(x) f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ is a bounded smooth domain, $\varphi_p(s) = |s|^{p-2}s$, $1 < p < +\infty$, $\lambda$ is a positive parameter, $a$ and $f$ satisfy the assumptions:

(H1) $a \in C(\Omega, [0, +\infty))$ with $a \not\equiv 0$;

(H2) There exist $f_0$, $f_\infty \in (0, +\infty)$ such that $f_0 \neq f_\infty$ and

$$f_0 = \lim_{s \to 0} \frac{f(s)}{\varphi_p(s)}, \quad f_\infty = \lim_{|s| \to +\infty} \frac{f(s)}{\varphi_p(s)};$$

(H3) $f \in C^1(\mathbb{R}, \mathbb{R})$ such that $f(s)/\varphi_p(s)$ is decreasing in $(0, +\infty)$ and is increasing in $(-\infty, 0)$.

**Remark 1.1** From (H2) and (H3), we can see that $f_0 \geq f(s)/\varphi_p(s) \geq f_\infty > 0$ for any $s \neq 0$, $f(0) = 0$ and $f_0 > f_\infty$.

**Remark 1.2** Note that if $f_0 = f_\infty$, Remark 1.1 shows that $f(s)/\varphi_p(s) \equiv f_0$ for any $s \neq 0$. In

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* Corresponding author
E-mail address: daiguowei@nwnu.edu.cn (Guowei DAI); hanxiaoling@nwnu.edu.cn (Xiaoling HAN)

this case, it is well known (see [4] or [5]) that problem (1.1) has one-sign solutions if and only if $\lambda$ is the principal eigenvalue of the following problem

$$\begin{cases} -\mathrm{div}\,(\varphi_p(\nabla u)) = \lambda a(x)\varphi_p(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Let $E := W_0^{1,p}(\Omega)$ with the norm $\|u\| = \left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{1/p}$. Set

$$\mathbb{S}^\nu = \left\{u \in C^{1,\alpha}(\overline{\Omega}) \middle| \nu u(x) > 0 \text{ for all } x \in \Omega\right\},$$

where $\nu \in \{+,-\}$. In [3], we have showed that there exists an unbounded continuum $\mathcal{C}^\nu$ of solutions to problem (1.1) emanating from $(\lambda_1/f_0, 0)$ and joining to $(\lambda_1/f_\infty, \infty)$, such that $\mathcal{C}^\nu \subseteq (\{(\lambda_1/f_0, 0)\} \cup (\mathbb{R} \times \mathbb{S}^\nu))$, where $\lambda_1$ is the principal eigenvalue of problem (1.2). However, we do not know the whole one-sign solution set of problem (1.1). The purpose of this work is to obtain the full information of the one-sign solutions for problem (1.1) under the assumptions of (H1), (H2) and (H3). More precisely, we shall show that the unbounded continuum $\mathcal{C}^\nu$ is smooth curve under the conditions of (H1), (H2) and (H3). Hence, we can give the optimal intervals for the parameter $\lambda$ so as to ensure the existence of exact zero or two one-sign solutions for problem (1.1).

It is well known that the spectral structure of high-dimensional $p$-Laplacian is not so clear as the case $p = 2$ or one-dimensional $p$-Laplacian. Our methods used in this paper cannot be extended to the high eigenvalue by now. Hence, instead of all eigenvalues which are considered in [1, 2], we only consider the principal eigenvalue $\lambda_1$. Our results generalize and improve the corresponding results to [1–3] in some sense.

The rest of this paper is arranged as follows. In Section 2, we shall give the main results and their proofs of this work. In Section 3, we shall give some applications of the main results.

## 2. Main results

Firstly, we propose the definition of linearly stable solution. For any $\phi \in E$ and nontrivial solution $u$ of problem (1.1), Afrouzi and Rasouli [6] have shown that the linearized problem of problem (1.1) about $u$ at the direction $\phi$ is

$$\begin{cases} -(p-1)\mathrm{div}\left(|\nabla u|^{p-2}\nabla\phi\right) - \lambda a(x) f'(u)\phi = \mu\phi, & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

A solution $u$ of problem (1.1) is stable if all eigenvalues of problem (2.1) are positive, otherwise it is unstable. We define the Morse index $M(u)$ of a solution $u$ to problem (1.1) to be the number of negative eigenvalues of problem (2.1). A solution $u$ of problem (1.1) is degenerate if 0 is an eigenvalue of problem (2.1), otherwise it is non-degenerate.

The main results of this paper are the following:

**Theorem 2.1** *Let (H1), (H2) and (H3) hold. Then*

*(1) Problem (1.1) has exactly two solutions $u^+(\lambda, \cdot)$ and $u^-(\lambda, \cdot)$ for $\lambda \in (\lambda_1/f_0, \lambda_1/f_\infty)$, such that $u^+(\lambda, \cdot)$ is positive in $\Omega$, and $u^-(\lambda, \cdot)$ is negative in $\Omega$;*

*(2) All one-sign solutions of problem (1.1) lie on two smooth curves*

$$\Sigma^{\pm}=\left\{\left(\lambda,u^{\pm}(\lambda,\cdot)\right)\big|\lambda\in(\lambda_1/f_0,\lambda_1/f_\infty)\right\},$$

*$\Sigma^+$ and $\Sigma^-$ join at $(\lambda_1/f_0,0)$, and $\lim_{\lambda\to\lambda_1/f_\infty}\|u^{\pm}(\lambda,\cdot)\|=+\infty$;*

*(3) For a solution $(\lambda,u)\in\Sigma^+\cup\Sigma^-$, $u$ is non-degenerate and the Morse index $M(u)=0$;*

*(4) $u^+(\lambda,\cdot)$ $(u^-(\lambda,\cdot))$ is increasing (decreasing) with respect to $\lambda$.*

**Remark 2.2** From Lemma 3.6 of [7], we can see that $\mathcal{C}=\mathcal{C}^+\cup\mathcal{C}^-$ near $(\lambda_1,0)$ is given by a curve $(\lambda(s),u(s))=(\lambda_1+o(1),s\varphi_1+o(s))$ for $s$ near 0, where $\varphi_1$ is the positive eigenfunction corresponding to $\lambda_1$ with $\|\varphi_1\|=1$. Moreover, we can distinguish between two portions of this curve by $s\geq 0$ and $s\leq 0$.

The following lemma is our main stability result for the positive solution.

**Lemma 2.3** *Let (H1) and (H3) hold. Then any positive solution $u$ of problem (1.1) is stable, hence, non-degenerate and Morse index $M(u)=0$.*

**Proof** Let $u$ be a positive solution of problem (1.1), and let $(\mu_1,\phi_1)$ be the corresponding principal eigen-pairs of problem (2.1) with $\phi_1>0$ in $\Omega$. We notice that $u$ and $\phi_1$ satisfy the problems

$$\begin{cases}-\mathrm{div}\left(\varphi_p(\nabla u)\right)-\lambda a(x)f(u)=0, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega\end{cases}\tag{2.2}$$

and

$$\begin{cases}-(p-1)\mathrm{div}\left(|\nabla u|^{p-2}\nabla\phi\right)-\lambda a(x)f'(u)\phi=\mu\phi, & \text{in } \Omega,\\ \phi=0, & \text{on } \partial\Omega.\end{cases}\tag{2.3}$$

Multiplying the first equation of problem (2.3) by $u$ and the first equation of problem (2.2) by $(p-1)\phi_1$, subtracting and integrating, we obtain

$$\mu_1\int_\Omega\phi_1 u\mathrm{d}x=\lambda\int_\Omega a(x)\phi_1\left((p-1)f(u)-f'(u)u\right)\mathrm{d}x.$$

By some simple computations, we can show that it follows from (H3) that $(p-1)f(s)-f'(s)s\geq 0$ for any $s\geq 0$. Since $u>0$ and $\phi_1>0$ in $\Omega$, we have $\mu_1>0$ and the positive solution $u$ must be stable. □

Similarly, we also have:

**Lemma 2.4** *Under the assumptions of Lemma 2.3, any negative solution $u$ of problem (1.1) is stable, hence, non-degenerate and Morse index $M(u)=0$.*

**Proof of Theorem 2.1** Define $F:\mathbb{R}\times E\to\mathbb{R}$ by

$$F(\lambda,u)=-\mathrm{div}\left(\varphi_p(\nabla u)\right)-\lambda a(x)f(u).$$

From Lemmas 2.3 and 2.4, we know that any one-sign solution $(\lambda,u)$ of problem (1.1) is stable. Therefore, at any one-sign solution $(\lambda^*,u^*)$, we can apply Implicit Function Theorem to $F(\lambda,u)=0$, and all the solutions of $F(\lambda,u)=0$ near $(\lambda^*,u^*)$ are on a curve $(\lambda,u(\lambda))$ with $|\lambda-\lambda^*|\leq\varepsilon$ for some small $\varepsilon > 0$. Furthermore, by virtue of Remark 2.2, the unbounded continua $\mathcal{C}^+$ and $\mathcal{C}^-$ are all curves.

To complete the proof, it suffices to show that $u^+(\lambda,\cdot)$ $(u^-(\lambda,\cdot))$ is increasing (decreasing) with respect to $\lambda$. We only prove the case of $u^+(\lambda,\cdot)$. The proof of $u^-(\lambda,\cdot)$ can be given similarly. Since $u^+(\lambda,\cdot)$ is differentiable with respect to $\lambda$ (as a consequence of Implicit Function Theorem), $\frac{\mathrm{d}u^+(\lambda,\cdot)}{\mathrm{d}\lambda}$ satisfies

$$-(p-1)\mathrm{div}\big(\left|\nabla u^+\right|^{p-2}\nabla\frac{\mathrm{d}u^+}{\mathrm{d}\lambda}\big)=\lambda a(x)f'\left(u^+\right)\frac{\mathrm{d}u^+}{\mathrm{d}\lambda}+a(x)f\left(u^+\right).$$

By an argument similar to that of Lemma 2.3, we can show that

$$\int_\Omega\Big(\lambda a(x)\left(f'\left(u^+\right)u^+-(p-1)f\left(u^+\right)\right)\frac{\mathrm{d}u^+}{\mathrm{d}\lambda}+f\left(u^+\right)u^+\Big)\mathrm{d}x=0.$$

Remak 2.1 implies $f(s)s \geq 0$ for any $s \in \mathbb{R}$. So we get $\left(f'\left(u^+\right)u^+-(p-1)f\left(u^+\right)\right)\frac{\mathrm{d}u^+}{\mathrm{d}\lambda} \leq 0$ by (H1). While (H3) shows that $f'\left(u^+\right)u^+-(p-1)f\left(u^+\right) \leq 0$. Therefore, we have $\frac{\mathrm{d}u^+}{\mathrm{d}\lambda} \geq 0$. □

**Remark 2.5** From Theorem 2.1, we can also get that problem (1.1) has no one-sign nontrivial solution for all $\lambda \in (0, \lambda_1/f_0] \cup [\lambda_1/f_\infty, +\infty)$ under the assumptions of Theorem 2.1.

Moreover, under more strict condition, we may have the following uniqueness result.

**Theorem 2.6** *Besides the assumptions of Theorem 2.1, we also assume that $f \geq 0$. Then for any $\lambda \in (\lambda_1/f_0, \lambda_1/f_\infty)$, problem (1.1) has exactly one positive solution $u^+$.*

**Proof** Define

$$\widetilde{f}(s)=\begin{cases} f(s), & \text{if } s>0,\\ 0, & \text{if } s=0,\\ -f(-s), & \text{if } s<0.\end{cases}$$

We consider the following problem

$$\begin{cases} -\mathrm{div}\left(\varphi_p(\nabla u)\right)=\lambda a(x)\widetilde{f}(u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega.\end{cases} \tag{2.4}$$

Applying Theorem 2.1 to problem (2.4), we obtain that for any $\lambda \in (\lambda_1/f_0, \lambda_1/f_\infty)$, problem (2.4) has exactly two solutions $u^+(\lambda,\cdot)$ and $u^-(\lambda,\cdot)$ such that $u^+(\lambda,\cdot)$ is positive in $\Omega$ and increasing with respect to $\lambda$, and $u^-(\lambda,\cdot)$ is negative in $\Omega$ and decreasing with respect to $\lambda$. Clearly, $u^+(\lambda,\cdot)$ is also the solution of problem (1.1). On the other hand, $f(s) \geq 0$ implies that any solution of problem (1.1) is nonnegative. The proof is completed. □

## 3. Applications

Consider the following semilinear elliptic eigenvalue problem

$$\begin{cases} -\Delta u=\lambda a(x)f(u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega,\end{cases} \tag{3.1}$$

where $a$ satisfies (H1), and $f$ satisfies the assumptions:

(F1) There exist $f_0,\ f_\infty \in (0,+\infty)$ such that $f_0 \neq f_\infty$ and

$$f_0 = \lim_{s\to 0} \frac{f(s)}{s} \quad \text{and} \quad f_\infty = \lim_{|s|\to+\infty} \frac{f(s)}{s};$$

(F2) $f \in C^1(\mathbb{R},\mathbb{R})$ such that $f(s)/s$ is decreasing in $(0,+\infty)$ and is increasing in $(-\infty,0)$.

Applying Theorem 2.1, we have the following results.

**Corollary 3.1** *Let (H1), (F1) and (F2) hold. Then*

*(1) Problem (3.1) has exactly two solutions $u^+(\lambda,\cdot)$ and $u^-(\lambda,\cdot)$ for $\lambda \in (\lambda_1/f_0, \lambda_1/f_\infty)$, such that $u^+(\lambda,\cdot)$ is positive in $\Omega$, and $u^-(\lambda,\cdot)$ is negative in $\Omega$;*

*(2) All one-sign solutions of problem (3.1) lie on two smooth curves*

$$\Sigma^\pm = \left\{ \left(\lambda, u^\pm(\lambda,\cdot)\right) \middle| \lambda \in (\lambda_1/f_0, \lambda_1/f_\infty) \right\},$$

*$\Sigma^+$ and $\Sigma^-$ join at $(\lambda_1/f_0, 0)$, and $\lim_{\lambda\to\lambda_1/f_\infty} \|u^\pm(\lambda,\cdot)\| = +\infty$;*

*(3) For a solution $(\lambda,u)\in\Sigma^+\cup\Sigma^-$, $u$ is non-degenerate and the Morse index $M(u)=0$;*

*(4) $u^+(\lambda,\cdot)$ ($u^-(\lambda,\cdot)$) is increasing (decreasing) with respect to $\lambda$.*

**Remark 1.10** It is not difficult to verify that the results of [1, Theorem 1.3] are also valid for problem (3.1). Obviously, the results of Corollary 3.1 are better than the corresponding ones of [1, Theorem 1.3] in the case of $\mu_k = \lambda_1$.

**Remark 1.11** Note that the results of Theorem 2.1 or Corollary 3.1 extend the corresponding ones of [8].

**Example 1.12** Take $f(s) = 2s + 1 - \sqrt{s^2+1}$ for $s \in [0,+\infty)$, then it is easy to verify that $f_0 = 2$, $f_\infty = 1$ and $f$ satisfies (F2). Hence, from Theorem 2.6 it follows that problem (3.1) has a unique positive solution for any $\lambda \in (\lambda_1/2, \lambda_1)$.

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