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On the Congruence $\sigma(n) \equiv 1 \pmod{n}$, II

Meng LI, Min TANG*
School of Mathematics and Computer Science, Anhui Normal University, Anhui 241003*, P. R. China*

Abstract Let $k \geq 2$ be an integer, and let $\sigma(n)$ denote the sum of the positive divisors of an integer n. We call n a quasi-multiperfect number if $\sigma(n) = kn + 1$. In this paper, we give some necessary properties of quasi-multiperfect numbers with four different prime divisors.

Keywords quasiperfect number; quasi-multiperfect number.

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1. Introduction

For a positive integer n, let $\phi(n)$, $\omega(n)$ and $\sigma(n)$ denote the Euler function of n, the number of distinct prime factors of n and the sum of the positive divisors of n, respectively. We call n a quasi-multiperfect(QM) number if $\sigma(n) = kn + 1$ with $k \geq 2$. In particular, we call n a quasiperfect number if $\sigma(n) = 2n + 1$, n a quasi-triperfect(QT) number if $\sigma(n) = 3n + 1$. Up to now, no quasi-multiperfect numbers are known, but necessary properties of them are described in many papers [1–8]. Recently, Anavi, Pollack and Pomerance [2] showed that the number of composite solutions $n \leq x$ to the congruence $\sigma(n) \equiv a \pmod{n}$ is at most $x^{\frac{1}{2}+o(1)}$. The authors of this paper [9] gave some necessary properties of quasi-multiperfect numbers with three different prime divisors.

In this paper, we obtain the following result.

Theorem 1.1 *If n is a QM and $\omega(n) = 4$, then n is QT and has the form $n = 2^{\alpha_1}3^{\alpha_2}p^{\alpha_3}q^{\alpha_4}$, where α_i's are even.*

Remark We can show that if n is QM and $\omega(n) = 4$, then prime factor p in Theorem 1.1 must satisfy $p \geq 23$.

2. Lemmas

In this section, we prepare several lemmas.

Lemma 2.1 ([6, Theorem 3]) *If n is a quasiperfect number, then $\omega(n) \geq 7$.*

Lemma 2.2 ([9, Theorem 1]) *If n is QM and odd, then $\omega(n) \geq 7$. If n is QM and even, then*

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* Corresponding author
E-mail address: limengde163youx@163.con (Meng LI); tmzzz2000@163.com (Min TANG)

$\omega(n) \geq 3$.

Lemma 2.3 *If n is an even QT, then n is a square number.*

Proof Let $n = 2^{\alpha_0}\prod_{i=1}^{t} p_i^{\alpha_i}$ be the standard factorization of n. Since n is a QT, we have $(\sigma(n), 3) = 1$. Thus $2^{\alpha_0+1} - 1 \equiv 1 \pmod 3$, $\alpha_0 \equiv 0 \pmod 2$.

Moreover,

$$\sigma(n) = (2^{\alpha+1} - 1)\prod_{i=1}^{t}(1 + p_i + \cdots + p_i^{\alpha_i}) \equiv \prod_{i=1}^{t}(\alpha_i + 1) \equiv 1 \pmod 2,$$

thus $\alpha_i \equiv 0 \pmod 2$, $1 \leq i \leq t$.

This completes the proof of Lemma 2.3. □

Lemma 2.4 *If $\sigma(n) = 4n+1$, then either $n = m^2$ with $(m, 2) = 1$, or $n = 2m^2$ with $(m, 6) = 1$.*

Proof Let $n = 2^{\alpha} \prod\limits_{i=1}^{t} p_i^{\alpha_i}$ be the standard factorization of n. By

$$\sigma(n) = (2^{\alpha+1} - 1)\prod_{i=1}^{t}(1 + p_i + \cdots + p_i^{\alpha_i}) \equiv \prod_{i=1}^{t}(\alpha_i + 1) \equiv 1 \pmod 2,$$

we have $\alpha_i \equiv 0 \pmod 2$. Thus $n = m^2$ with $(m, 2) = 1$, or $n = 2^{\alpha}m^2$ with $(m, 2) = 1$.

Suppose that $n = 2^{2a}m^2$, where $a \geq 0$ and m is odd. Then for any prime divisor q of $\sigma(n)$, we have

$$\sigma(n) = 4n + 1 = 2^{2a+2}m^2 + 1 \equiv 0 \pmod q.$$

Thus $(\frac{-1}{q}) = 1$, $q \equiv 1 \pmod 4$. If $a \geq 1$, then $\sigma(2^{2a}) = 2^{2a+1} - 1 \equiv 3 \pmod 4$, which is impossible. Hence $a = 0$.

Suppose that $n = 2^{2a+1}m^2$, where $a \geq 0$ and m is odd. Then for any prime divisor q of $\sigma(n)$, we have

$$\sigma(n) = 4n + 1 = 2(2^{a+1}m)^2 + 1 \equiv 0 \pmod q.$$

Thus $(\frac{-2}{q}) = 1$, $q \equiv 1, 3 \pmod 8$. If $a \geq 1$, then $\sigma(2^{2a+1}) = 2^{2a+2} - 1 \equiv 7 \pmod 8$, which is also impossible. Hence $a = 0$, $n = 2m^2$, $\sigma(n) = 3\sigma(m^2) = 8m^2 + 1$, we have $(m, 3) = 1$.

This completes the proof of Lemma 2.4. □

Lemma 2.5 *If n is a QM and $\omega(n) = 4$, then we know that n is a QT and n must be one of the following forms:*

(i) $n = 2^{\alpha_1}3^{\alpha_2}p^{\alpha_3}q^{\alpha_4}$, *where p, q are odd primes;*

(ii) $n = 2^{\alpha_1}5^{\alpha_2}7^{\alpha_3}p^{\alpha_4}$, *where $p \in \{11, 13, 17, 19, 23, 29, 31\}$.*

Proof Since $\omega(n) = 4$, by Lemma 2.2 we have n is even. Assume that $n = 2^{\alpha_1}p_1^{\alpha_2}p_2^{\alpha_3}p_3^{\alpha_4}$. If $p_1 \geq 7$, then

$$2 \leq k = \frac{\sigma(n)}{n} - \frac{1}{n} < \frac{n}{\phi(n)} \leq \frac{2}{1} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{12} < 3,$$

thus $k = 2$, n is a quasiperfect number. By Lemma 2.1, this is impossible. Hence $p_1 = 3$ or $p_1 = 5$.

Case 1 $p_1 = 3$. Then

$$2 \le k = \frac{\sigma(n)}{n} - \frac{1}{n} < \frac{n}{\phi(n)} \le \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} = 4.375.$$

Thus $k = 2, 3$ or 4. By Lemmas 2.1 and 2.4, we have $\sigma(n) = 3n + 1$.

Case 2 $p_1 = 5$. If $p_2 \ge 11$, then

$$2 \le k = \frac{\sigma(n)}{n} - \frac{1}{n} < \frac{n}{\phi(n)} \le \frac{2}{1} \cdot \frac{5}{4} \cdot \frac{11}{10} \cdot \frac{13}{12} < 3,$$

thus $k = 2$, n is a quasiperfect number. By Lemma 2.1, this is impossible. Hence $p_2 = 7$. If $p_3 \ge 37$, we have

$$2 \le k = \frac{\sigma(n)}{n} - \frac{1}{n} < \frac{n}{\phi(n)} \le \frac{2}{1} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{37}{36} < 3,$$

thus $k = 2$, n is a quasiperfect number. By Lemma 2.1, this is impossible. Thus $n = 2^{\alpha_1} 5^{\alpha_2} 7^{\alpha_3} p^{\alpha_4}$ where $p \in \{11, 13, 17, 19, 23, 29, 31\}$. In these cases, we have

$$2 \le k = \frac{\sigma(n)}{n} - \frac{1}{n} < \frac{n}{\phi(n)} \le \frac{2}{1} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} = 3.20\cdots,$$

thus $k = 2$ or 3. By Lemma 2.1, we have $\sigma(n) = 3n + 1$.

This completes the proof of Lemma 2.5. □

3. Proof of Theorem 1.1

By Lemma 2.5, it suffices to show that if $\alpha_i's$ are even and $p \in \{11, 13, 17, 19, 23, 29, 31\}$, then $n = 2^{\alpha_1} 5^{\alpha_2} 7^{\alpha_3} p^{\alpha_4}$ is not a QT. Assume that $n = 2^{\alpha_1} 5^{\alpha_2} 7^{\alpha_3} p^{\alpha_4}$ is a QT. Then

$$\sigma(n) = (2^{\alpha_1+1} - 1) \frac{5^{\alpha_2+1} - 1}{4} \frac{7^{\alpha_3+1} - 1}{6} \frac{p^{\alpha_4+1} - 1}{p - 1} = 2^{\alpha_1} \cdot 3 \cdot 5^{\alpha_2} 7^{\alpha_3} p^{\alpha_4} + 1. \quad (1)$$

Let

$$f(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1 - \frac{1}{2^{\alpha_1+1}})(1 - \frac{1}{5^{\alpha_2+1}})(1 - \frac{1}{7^{\alpha_3+1}})(1 - \frac{1}{p^{\alpha_4+1}}),$$

$$g(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{36(p-1)}{35p} + \frac{2^3 3(p-1)}{2^{\alpha_1+1} 5^{\alpha_2+1} 7^{\alpha_3+1} p^{\alpha_4+1}}.$$

Thus $f(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = g(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Noting that

$$f(2, \alpha_2, \alpha_3, \alpha_4) < 1 - \frac{1}{2^3} = 0.875,$$

$$g(2, \alpha_2, \alpha_3, \alpha_4) > \frac{36 \times 10}{35 \times 11} = 0.935\cdots,$$

we know that $\alpha_1 \ge 4$.

Case 1 $p = 11$. Then

$$f(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \ge f(4, 2, 2, 2) = 0.957\cdots,$$

$$g(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \le g(4, 2, 2, 2) = 0.935\cdots,$$

a contradiction.

Case 2 $p=13$. Then

$$f(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\ge f(4,2,2,2)=0.957\cdots,$$

$$g(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\le g(4,2,2,2)=0.949\cdots,$$

a contradiction.

Case 3 $p=17$. By (1), we have

$$(2^{\alpha_1+1}-1)(5^{\alpha_2+1}-1)(7^{\alpha_3+1}-1)(17^{\alpha_4+1}-1)\equiv -1 \pmod 7. \tag{2}$$

Since $\alpha_i's$ are even for $1\le i\le 4$, we have $2^{\alpha_1+1}-1\equiv 0,1,3 \pmod 7$, $5^{\alpha_2+1}-1\equiv 2,4,5 \pmod 7$, $17^{\alpha_4+1}-1\equiv 2,4,5 \pmod 7$. By (2), we have the following cases.

(a) $2^{\alpha_1+1}-1\equiv 1 \pmod 7$, $5^{\alpha_2+1}-1\equiv 2 \pmod 7$, $17^{\alpha_4+1}-1\equiv 4 \pmod 7$. Thus $\alpha_1\equiv 0 \pmod 6$, $\alpha_2\equiv 4 \pmod 6$, $\alpha_4\equiv 4 \pmod 6$.

(b) $2^{\alpha_1+1}-1\equiv 1 \pmod 7$, $5^{\alpha_2+1}-1\equiv 4 \pmod 7$, $17^{\alpha_4+1}-1\equiv 2 \pmod 7$. Thus $\alpha_1\equiv 0 \pmod 6$, $\alpha_2\equiv 0 \pmod 6$, $\alpha_4\equiv 0 \pmod 6$.

Hence

$$f(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\ge f(6,4,2,4)=0.988\cdots,$$

$$g(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\le g(6,4,2,4)=0.968\cdots,$$

a contradiction.

Case 4 $p=19$. Noting that

$$f(4,\alpha_2,\alpha_3,\alpha_4)<1-\frac{1}{2^5}=0.968\cdots,$$

$$g(4,\alpha_2,\alpha_3,\alpha_4)>\frac{36\times 18}{35\times 19}=0.974\cdots,$$

we know that $\alpha_1\ge 6$. Hence

$$f(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\ge f(6,2,2,2)=0.981\cdots,$$

$$g(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\le g(6,2,2,2)=0.974\cdots,$$

a contradiction.

Case 5 $p=23$. Noting that

$$f(4,\alpha_2,\alpha_3,\alpha_4)<1-\frac{1}{2^5}=0.968\cdots,$$

$$g(4,\alpha_2,\alpha_3,\alpha_4)>\frac{36\times 22}{35\times 23}=0.983\cdots,$$

we know that $\alpha_1\ge 6$. By (1), we have

$$(2^{\alpha_1+1}-1)(5^{\alpha_2+1}-1)(7^{\alpha_3+1}-1)(23^{\alpha_4+1}-1)\equiv 3 \pmod 7. \tag{3}$$

Since $\alpha_i's$ are even for $1\le i\le 4$, we have $2^{\alpha_1+1}-1\equiv 0,1,3 \pmod 7$, $5^{\alpha_2+1}-1\equiv 2,4,5 \pmod 7$, $23^{\alpha_4+1}-1\equiv 0,1,3 \pmod 7$. By (3), we have the following cases.

(a) $2^{\alpha_1+1} - 1 \equiv 1 \pmod 7$, $5^{\alpha_2+1} - 1 \equiv 4 \pmod 7$, $23^{\alpha_4+1} - 1 \equiv 1 \pmod 7$. Thus $\alpha_1 \equiv 0 \pmod 6$, $\alpha_2 \equiv 0 \pmod 6$, $\alpha_4 \equiv 0 \pmod 6$.

(b) $2^{\alpha_1+1} - 1 \equiv 3 \pmod 7$; $5^{\alpha_2+1} - 1 \equiv 2 \pmod 7$; $23^{\alpha_4+1} - 1 \equiv 3 \pmod 7$. Thus $\alpha_1 \equiv 4 \pmod 6$; $\alpha_2 \equiv 4 \pmod 6$; $\alpha_4 \equiv 4 \pmod 6$.

Hence

$$f(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq f(6, 6, 2, 6) = 0.989\cdots,$$

$$g(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq g(6, 6, 2, 6) = 0.983\cdots,$$

a contradiction.

Case 6 $p = 29$. Noting that

$$f(4, \alpha_2, \alpha_3, \alpha_4) < f(6, \alpha_2, \alpha_3, \alpha_4) < 1 - \frac{1}{2^7} = 0.992\cdots,$$

$$g(4, \alpha_2, \alpha_3, \alpha_4) > g(6, \alpha_2, \alpha_3, \alpha_4) > \frac{36 \times 28}{35 \times 29} = 0.993\cdots,$$

we know that $\alpha_1 \geq 8$. Similarly, we have $\alpha_2 \geq 4$. Hence

$$f(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq f(8, 4, 2, 2) = 0.994\cdots,$$

$$g(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq g(8, 4, 2, 2) = 0.993\cdots,$$

a contradiction.

Case 7 $p = 31$. Noting that

$$f(4, \alpha_2, \alpha_3, \alpha_4) < f(6, \alpha_2, \alpha_3, \alpha_4) < 1 - \frac{1}{2^7} = 0.992\cdots,$$

$$g(4, \alpha_2, \alpha_3, \alpha_4) > g(6, \alpha_2, \alpha_3, \alpha_4) > \frac{36 \times 30}{35 \times 31} = 0.995\cdots,$$

we know that $\alpha_1 \geq 8$. By (1), we have

$$(2^{\alpha_1+1} - 1)(5^{\alpha_2+1} - 1)(7^{\alpha_3+1} - 1)(31^{\alpha_4+1} - 1) \equiv -1 \pmod 7. \tag{4}$$

Since $\alpha_i' s$ are even for $1 \leq i \leq 4$, we have $2^{\alpha_1+1} - 1 \equiv 0, 1, 3 \pmod 7$, $5^{\alpha_2+1} - 1 \equiv 2, 4, 5 \pmod 7$, $31^{\alpha_4+1} - 1 \equiv 2, 4, 5 \pmod 7$. By (4), we have the following cases.

(a) $2^{\alpha_1+1} - 1 \equiv 1 \pmod 7$, $5^{\alpha_2+1} - 1 \equiv 2 \pmod 7$, $31^{\alpha_4+1} - 1 \equiv 4 \pmod 7$. Thus $\alpha_1 \equiv 0 \pmod 6$, $\alpha_2 \equiv 4 \pmod 6$, $\alpha_4 \equiv 4 \pmod 6$.

(b) $2^{\alpha_1+1} - 1 \equiv 1 \pmod 7$, $5^{\alpha_2+1} - 1 \equiv 4 \pmod 7$, $31^{\alpha_4+1} - 1 \equiv 2 \pmod 7$. Thus $\alpha_1 \equiv 0 \pmod 6$, $\alpha_2 \equiv 0 \pmod 6$, $\alpha_4 \equiv 0 \pmod 6$.

Hence

$$f(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq f(12, 4, 2, 4) = 0.996\cdots,$$

$$g(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq g(12, 4, 2, 4) = 0.995\cdots,$$

a contradiction.

This completes the proof of Theorem 1.1 $\square$

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