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Weighted Norm Inequalities for a Class of Multilinear Singular Integral Operators

Rongqian WANG
Basic Education and Scientific Department, Henan Polytechnic Institute, Henan 473000*, P. R. China*

Abstract In this paper, weighted estimates with general weights are established for the multilinear singular integral operator defined by

$$T_Af(x)=\mathrm{p.\,v.}\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n+1}}\big(A(x)-A(y)-\nabla A(y)(x-y)\big)f(y)\mathrm{d}y,$$

where Ω is homogeneous of degree zero, has vanishing moment of order one, and belongs to $\mathrm{Lip}_\gamma(S^{n-1})$ with $\gamma\in(0,\,1]$, A has derivatives of order one in $\mathrm{BMO}(\mathbb{R}^n)$.

Keywords multilinear singular integral operator; weighted norm inequality; sharp function estimate; BMO.

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1. Introduction

We will work on $\mathbb{R}^n$, $n\ge 1$. For a point $x\in\mathbb{R}^n$, we denote by x_j the j-th variable of x. Let Ω be homogeneous of degree zero, integrable on the unit sphere S^{n-1}, and have vanishing moment of order one which means that for each j with $1\le j\le n$,

$$\int_{S^{n-1}}\Omega(x')x'_j\mathrm{d}\sigma(x')=0.$$

Let A be a function on $\mathbb{R}^n$ having derivatives of order one in $\mathrm{BMO}(\mathbb{R}^n)$. Define the multilinear singular integral operator T_A by

$$T_Af(x)=\mathrm{p.\,v.}\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n+1}}\big(A(x)-A(y)-\nabla A(y)(x-y)\big)f(y)\mathrm{d}y. \tag{1.1}$$

This operator was first considered by Cohen [1] and is closely related to the Calderón commutator. A well known result of Cohen states that if $\Omega\in\mathrm{Lip}_\gamma(S^{n-1})$ with $\gamma\in(0,\,1]$, then for $p\in(1,\,\infty)$ and $u\in A_p(\mathbb{R}^n)$, T_A is a bounded operator on $L^p(\mathbb{R}^n,\,u)$ with bound $C(n,\,p)\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)}$, here $A_p(\mathbb{R}^n)$ denotes the weight function class of Muckenhoupt [3]. Hofmann [4] proved that $\Omega\in\cup_{1<q\le\infty}L^q(S^{n-1})$ is a sufficient condition such that T_A is a bounded operator on $L^p(\mathbb{R}^n)$ for $p\in(1,\,\infty)$. Hu and Yang [8] considered the weighted estimate with general weights for T_A, and proved that if $p\in(1,\,\infty)$, $\Omega\in\mathrm{Lip}_\gamma(S^{n-1})$ with $\gamma\in(0,\,1]$, then for any $\delta>0$, any weight w, bounded function f with compact support,

$$\|T_Af\|_{L^p(\mathbb{R}^n,\,w)}\lesssim\|f\|_{L^p(\mathbb{R}^n,\,M_{L(\log L)^{2p-1+\delta}}w)}, \tag{1.2}$$

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E-mail address: 1923189387@qq.com

and for any $\lambda > 0$,

$$w(\{x \in \mathbb{R}^n : |T_A f(x)| > \lambda\}) \lesssim \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log\big(2 + \frac{|f(x)|}{\lambda}\big) M_{L(\log L)^{1+\delta}} w(x) \mathrm{d}x, \tag{1.3}$$

here and in the following, for a weight w, we mean w is nonnegative and locally integrable in $\mathbb{R}^n$. For other works about the operator T_A, see [5–7] and the reference therein.

As it is well known, for a standard Calderón-Zygmund T, Pérez proved that if $p \in (1, \infty)$, then for any $\delta > 0$, any bounded function f with compact support and any weight w,

$$\|Tf\|_{L^p(\mathbb{R}^n,\, w)} \lesssim \|f\|_{L^p(\mathbb{R}^n,\, M_{L(\log L)^{p-1+\delta}} w)}, \tag{1.4}$$

and for any $\lambda > 0$,

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \lesssim \lambda^{-1} \int_{\mathbb{R}^n} |f(x)| M_{L(\log L)^{\delta}} w(x) \mathrm{d}x. \tag{1.5}$$

Comparing the inequalities (1.2) and (1.4) ((1.3) and (1.5), respectively), we may ask if the weight $M_{L(\log L)^{2p-1+\delta}} w$ in the right hand side of the inequality (1.2) can be replaced by $M_{L(\log L)^{p-1+\delta}} w$, and the weight $M_{L(\log L)^{1+\delta}} w$ in the right hand side of the inequality (1.3) can be replaced by $M_{L(\log L)^{\delta}} w$. The purpose of this paper is to consider this question. Our main result can be stated as follows.

Theorem 1.1 *Let Ω be homogeneous of degree zero, have vanishing moment of order one and belong to $\mathrm{Lip}_\gamma(S^{n-1})$ with $\gamma \in (0, 1]$. Let A be a function on $\mathbb{R}^n$ with derivatives of order one in $\mathrm{BMO}(\mathbb{R}^n)$ and T_A be the operator defined by (1.1). Then*

(i) For $p \in (1, \infty)$, any weight w and bounded function f with compact support,

$$\|T_A f\|_{L^p(\mathbb{R}^n,\, w)} \lesssim \|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n,\, M_{L(\log L)^{p-1+\delta}} w)}; \tag{1.6}$$

(ii) For any weight w and bounded function f with compact support,

$$\begin{aligned} &w(\{x \in \mathbb{R}^n : |T_A f(x)| > \lambda\}) \\ &\quad \lesssim \Phi(\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)}) \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log\Big(\mathrm{e} + \frac{|f(x)|}{\lambda}\Big) M_{L(\log L)^{\delta}} w(x) \mathrm{d}x \end{aligned}$$

where $\Phi(t) = t\log(\mathrm{e} + t)$ for $t > 0$.

Remark 1.2 In [8], to obtain the estimate (1.2), the authors first established a variant sharp function estimate for T_A, from which they deduced that for $0 < p < \infty$ and bounded function f,

$$\|T_A f\|_{L^p(\mathbb{R}^n,\, u)} \lesssim \|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)} \|M^2 f\|_{L^p(\mathbb{R}^n,\, u)}$$

provided that $u \in A_\infty(\mathbb{R}^n)$, where and in the following, $A_\infty(\mathbb{R}^n) = \cup_{p \geq 1} A_p(\mathbb{R}^n)$. This, via a duality argument and the weighted estimates with general weights for the commutator of the Calderón-Zygmund operator [10], leads to (1.2). The argument in this paper is somewhat different from that used in [8]. We will first establish a variant estimate for the operator T_A^*, defined by

$$T_A^* f(x) = \mathrm{p.\,v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}} \big(A(x) - A(y) - \nabla A(x)(x-y)\big) f(y) \mathrm{d}y. \tag{1.7}$$

This, together with the relation of the operator T_A and some Calderón-Zygmund operators, and the ideas used in [9], leads to our result (1.6). For details, see Section 2.

Remark 1.3 Theorem 1.1 is of interest since it implies that the singularity of T_A is the same as that of the classical Calderón-Zygmund operator.

We now make some conventions. Throughout this paper, we denote by C a positive constant which is independent of the main parameters, but may vary from line to line. The symbol $f \lesssim g$ means that there exists a positive constant C such that $f \le Cg$. For any subset $E \subset \mathbb{R}^n$, χ_E denotes the characteristic function of E.

2. Proof of Theorem 1.1

We begin with a preliminary lemma.

Lemma 2.1 *Let b be a function on $\mathbb{R}^n$ with derivatives of order one in $L^q(\mathbb{R}^n)$ for some q with $n < q \le \infty$. Then*

$$|b(x) - b(y)| \lesssim |x - y| \sum_{k=1}^{n} \Big(\frac{1}{|\widetilde{Q}(x,\, y)|} \int_{\tilde{Q}(x,\, y)} |D_k b(z)|^q \mathrm{d}z \Big)^{1/q},$$

where $\widetilde{Q}(x,\, y)$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

For the proof of Lemma 2.1, see [1].

For each fixed k with $1 \le k \le n$, let T_k be the operator defined by

$$T_k f(x) = \mathrm{p.\,v.} \int_{\mathbb{R}^n} \frac{\Omega(x - y)}{|x - y|^{n+1}} (x_k - y_k) f(y) \mathrm{d}y. \tag{2.1}$$

Under the hypothesis of Theorem 1.1, T_k is a Calderón-Zygmund operator. For a function $b \in \mathrm{BMO}(\mathbb{R}^n)$, define the commutator $[b,\, T_k]$ as

$$[b,\, T_k] f(x) = b(x) T f(x) - T(bf)(x).$$

It is well known that

$$\|[b,\, T_k] f\|_{L^p(\mathbb{R}^n,\, u)} \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n,\, u)}, \quad p \in (1,\, \infty), \ u \in A_p(\mathbb{R}^n).$$

Note that

$$T_A^* f(x) = T_A f(x) - \sum_{j=1}^{n} [D_j A,\, T_j] f(x),$$

where T_A^* is the operator defined by (1.7). Thus, under the hypothesis of Theorem 1.1,

$$\|T_A^* f\|_{L^p(\mathbb{R}^n,\, u)} \lesssim \|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n,\, u)}, \quad p \in (1,\, \infty), \ u \in A_p(\mathbb{R}^n). \tag{2.2}$$

Also, we have that

Lemma 2.2 *Let $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$ for some $\gamma \in (0,\, 1)$. Then T_A^* is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\,\infty}(\mathbb{R}^n)$, namely, for any $\lambda > 0$,*

$$|\{x \in \mathbb{R}^n :\, T_A^* f(x)| > \lambda\}| \lesssim \lambda^{-1} \|f\|_{L^1(\mathbb{R}^n)}.$$

Proof Without loss of generality, we assume that $\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$. For each fixed $\lambda > 0$ and $f \in L^1(\mathbb{R}^n)$, applying the Calderón-Zygmund decomposition to f at level λ, we then obtain a sequence of cubes $\{Q_j\}_j$ with disjoint interiors, such that

$$\lambda < \frac{1}{|Q_j|}\int_{Q_j} |f(y)|\mathrm{d}y \le 2^n\lambda,$$

and

$$|f(x)| \le \lambda, \text{ a.e. } x \in \mathbb{R}^n\backslash\big(\cup_j Q_j\big).$$

Set

$$g(x) = f(x)\chi_{\mathbb{R}^n\backslash\cup_j Q_j}(x) + \sum_j m_{Q_j}(f)\chi_{Q_j}(x),$$

$$h(x) = \sum_j (f - m_{Q_j}(f))\chi_{Q_j}(x) = \sum_j h_j(x).$$

By the $L^2(\mathbb{R}^n)$ boundedness of T_A^*, we know that

$$|\{x \in \mathbb{R}^n : |T_A^* g(x)| > \lambda\}| \lesssim \lambda^{-2}\|T_A^* g\|_{L^2(\mathbb{R}^n)} \lesssim \lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}.$$

Set $E_\lambda = \cup_j 2\sqrt{n}Q_j$. It is obvious that $|E_\lambda| \lesssim \lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}$. Thus, the proof of Lemma 2.2 is now reduced to proving that

$$|\{x \in \mathbb{R}^n\backslash E_\lambda : |T_A^* h(x)| > \lambda\}| \lesssim \lambda^{-1}. \tag{2.3}$$

We now prove (2.3). For each fixed j, set $A_j(y) = A(y) - m_{Q_j}(\nabla A)y$. It is obvious that for $x, y \in \mathbb{R}^n$,

$$A_j(x) - A_j(y) - \nabla A_j(x)(x-y) = A(x) - A(y) - \nabla A(y)(x-y)$$

and so

$$\begin{aligned} T_A^* h(x) =& \sum_j \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}}(A_j(x) - A_j(y))h_j(y)\mathrm{d}y + \\ & \sum_j \sum_{k=1}^n \big(D_k A(x) - m_{Q_j}(D_k A)\big)\int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}}(x_k - y_k)h_j(y)\mathrm{d}y \\ :=& T_A^{*,\mathrm{I}} h(x) + \sum_j T_A^{*,\mathrm{II}} h_j(x). \end{aligned}$$

As pointed out in [8, p. 765],

$$|\{x \in \mathbb{R}^n\backslash E_\lambda : |T_A^{*,\mathrm{I}} h(x)| > \lambda/2\}| \lesssim \lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}.$$

For each fixed j and k with $1 \le k \le n$, and each fixed $x \in \mathbb{R}^n\backslash E_\lambda$, by the vanishing moment of h_j and the regularity condition of Ω, we have

$$|T_A^{*,\mathrm{II}} h_j(x)| \lesssim \sum_{k=1}^n |D_k A(x) - m_{Q_j}(D_k A)|\frac{|y - y_j^0|^\gamma}{|x - y_j^0|^{n+\gamma}}\|h_j\|_{L^1(\mathbb{R}^n)},$$

where y_j^0 is the center of Q_j. On the other hand, a straightforward computation leads to that

$$\int_{\mathbb{R}^n\backslash 2\sqrt{n}Q_j} |D_k A(x) - m_{Q_j}(D_k A)|\frac{\{\ell(Q_j)\}^\gamma}{|x - y_j^0|^{n+\gamma}}\mathrm{d}x$$

$$
\begin{aligned}
&\lesssim \sum_{l=1}^{\infty}\frac{\{\ell(Q_j)\}^{\gamma}}{\{2^l\ell(Q_j)\}^{n+\gamma}}\int_{2^{l+1}\sqrt{n}Q_j}|D_kA(x)-m_{2^{l+1}Q_j}(D_kA)|\mathrm{d}x+\\
&\quad \sum_{l=1}^{\infty}\frac{\{\ell(Q_j)\}^{\gamma}}{\{2^l\ell(Q_j)\}^{\gamma}}|m_{Q_j}(D_kA)-m_{2^{l+1}Q_j}(D_kA)|\\
&\lesssim 1,
\end{aligned}
$$

where $\ell(Q_j)$ denotes the side length of Q_j. Therefore,

$$
\begin{aligned}
&|\{x\in\mathbb{R}^n\backslash E_\lambda:\ \sum_j|T_A^{*,\,\mathrm{I}}h_j(x)|>\lambda/2\}|\\
&\lesssim \lambda^{-1}\sum_j\|h_j\|_{L^1(\mathbb{R}^n)}\sum_{k=1}^{n}\int_{\mathbb{R}^n\backslash 2\sqrt{n}Q_j}|D_kA(x)-m_{Q_j}(D_kA)|\frac{\{\ell(Q_j)\}^{\gamma}}{|x-y_j^0|^{n+\gamma}}\mathrm{d}x\\
&\lesssim \lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}.
\end{aligned}
$$

This lead to (2.3) and then completes the proof of Lemma 2.2. □

To prove Theorem 1.1, we will also use a sharp function estimate for T_A^*. Let $r\in(0,\,\infty)$. Define the operator $M_r^\sharp$ for by

$$
M_r^\sharp f(x)=\sup_{Q\ni x}\inf_{c\in\mathbb{C}}\Big(\frac{1}{|Q|}\int_Q|f(y)-c|^r\mathrm{d}y\Big)^{1/r},
$$

where the sup is taken over all cube containing x. Obviously, for the case of $r\in(0,\,1]$,

$$
\{M^\sharp(|f|^r)(x)\}^{1/r}\lesssim M_r^\sharp f(x).
$$

Also, $M_1^\sharp$, which will be denoted by $M^\sharp$ for simplicity, is just the sharp maximal operator of Fefferman-Stein.

Lemma 2.3 *Let $0<r<s<1$. Under the hypothesis of Theorem 2.1, for any bounded function f with compact support,*

$$
M_r^\sharp(T_A^*f)(x)\lesssim\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)}\Big(\sum_{k=1}^{n}M_s(T_kf)(x)+Mf(x)\Big),
$$

where T_k is the operator defined by (2.1).

Proof Let f be a bounded function with compact support. For each fixed $x\in\mathbb{R}^n$ and each cube Q containing x, decompose f as

$$
f(y)=f(y)\chi_{3Q}(y)+f(y)\chi_{\mathbb{R}^n\backslash 3Q}(y):=f_1(y)+f_2(y).
$$

Let

$$
A_Q(y)=A(y)-m_Q(\nabla A)y
$$

and for $y\in Q$,

$$
R_Q(y)=\int_{\mathbb{R}^n}\frac{\Omega(y-z)}{|y-z|^{n+1}}\big(A_Q(y)-A_Q(z)\big)f_2(z)\mathrm{d}z.
$$

Observe that for any k with $1\le k\le n$,

$$
\int_{3Q}|D_kA(y)-m_Q(D_kA)||T_kf_2(y)|\mathrm{d}y
$$

$$\lesssim \Big(\int_{3Q}|D_kA(y)-m_Q(D_kA)|^2\mathrm{d}y\Big)^{1/2}\|T_kf_2\|_{L^2(\mathbb{R}^n)}.$$

Thus, by the fact that $T_A^*f_2(y)$ is finite for a. e. $y\in 3Q$, we can choose $y_Q\in 3Q\backslash 2Q$ such that $R_Q(y_Q)$ is finite. Write

$$\begin{aligned}&\Big(\frac{1}{|Q|}\int_Q|T_A^*f(y)-R_Q(y_Q)|^r\mathrm{d}y\Big)^{1/r}\\&\lesssim \Big(\frac{1}{|Q|}\int_Q|T_A^*f_1(y)|^r\mathrm{d}y\Big)^{1/r}+\Big(\frac{1}{|Q|}\int_Q|R_Q(y)-R_Q(y_Q)|^r\mathrm{d}y\Big)^{1/r}+\\&\quad\sum_{k=1}^n\Big(\frac{1}{|Q|}\int_Q|D_kA(y)-m_Q(D_kA)|^r|T_kf_2(y)|^r\mathrm{d}y\Big)^{1/r}\\&:=\sum_{l=1}^3\mathrm{I}_l.\end{aligned}$$

By Lemma 2.2 and the Kolmogrov inequality, we can verify that

$$\mathrm{I}_1\lesssim\frac{1}{|3Q|}\int_{3Q}|f(y)|\mathrm{d}y\lesssim Mf(x).$$

Recall that T_k is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. The Hölder inequality together with the Kolmogrov inequality, tells us that

$$\begin{aligned}\mathrm{I}_3&\lesssim\sum_{k=1}^n\Big\{\Big(\frac{1}{|Q|}\int_Q|T_kf(y)|^s\mathrm{d}y\Big)^{1/s}+\Big(\frac{1}{|Q|}\int_Q|T_kf_1(y)|^s\mathrm{d}y\Big)^{1/s}\Big\}\\&\lesssim\sum_{k=1}^nM_s(T_kf)(x)+\frac{1}{|3Q|}\int_{3Q}|f(y)|\mathrm{d}y\\&\lesssim\sum_{k=1}^nM_s(T_kf)(x)+Mf(x).\end{aligned}$$

It remains to consider the term I_2. Let $\ell(Q)$ be the side length of Q. By the regularity of Ω, it follows that for each fixed $y\in Q$ and $z\in\mathbb{R}^n\backslash 2\sqrt{n}Q$,

$$\Big|\frac{\Omega(y-z)}{|y-z|^{n+1}}-\frac{\Omega(y_Q-z)}{|y_Q-z|^{n+1}}\Big|\lesssim\frac{\{\ell(Q)\}^\gamma}{|y-z|^{n+1+\gamma}}.$$

On the other hand, for $y\in Q$ and $z\in 2^{j+1}\sqrt{n}Q\backslash 2^j\sqrt{n}Q$ with j a positive integer, it follows from Lemma 2.1 that

$$\begin{aligned}|A_Q(y)-A_Q(z)|&\lesssim|y-z|\sum_{k=1}^n\Big(\frac{1}{|\widetilde{Q}(y,\,z)|}\int_{\tilde{Q}(y,\,z)}|D_kA(z)-m_Q(D_kA)|^q\mathrm{d}z\Big)^{1/q}\\&\lesssim j|y-z|.\end{aligned}$$

Also, we deduce from Lemma 2.1 that for $y\in Q$,

$$|A_Q(y)-A_Q(y_Q)|\lesssim\ell(Q).$$

Therefore, for each $y\in Q$,

$$|R_Q(y)-R_Q(y_Q)|\lesssim\int_{\mathbb{R}^n}\Big|\frac{\Omega(y-z)}{|y-z|^{n+1}}-\frac{\Omega(y_Q-z)}{|y_Q-z|^{n+1}}\Big|\big|A_Q(y)-A_Q(z)\big||f_2(z)|\mathrm{d}z$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^n}\frac{1}{|y-z|^{n+1}}\big|A_Q(y)-A_Q(z)\big||f_2(z)|\mathrm{d}z\\
\lesssim&\{\ell(Q)\}^{\gamma}\sum_{j=1}^{\infty}j\int_{2^{j+1}\sqrt{n}Q\backslash 2^j\sqrt{n}Q}\frac{1}{|y-z|^{n+\gamma}}|f_2(z)|\mathrm{d}z\\
&+\ell(Q)\int_{\mathbb{R}^n\backslash 2\sqrt{n}Q}\frac{1}{|y-z|^{n+1}}f(y)|\mathrm{d}y\\
\lesssim&Mf(x),
\end{aligned}
$$

and so $\mathrm{I}_2\lesssim Mf(x)$. Combining the estimates for the terms I_1, I_2 and I_3 leads to our desired conclusion.

Proof of Theorem 1.1 Employing the argument used in [8], conclusion (ii) can be deduced from conclusion (i). We omit the details. Note that we may view T_A^* as the dual operator of T_A. As pointed out in [9, p. 300], it suffices to prove that for $p\in(1,\infty)$ and $u\in A_\infty(\mathbb{R}^n)$,

$$\|T_A^*f\|_{p,\,u}\lesssim\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)}\|Mf\|_{p,\,u}.$$

For a fixed $u\in A_\infty(\mathbb{R}^n)$ and $p\in(1,\infty)$, we can choose r, s such that $0<r<s<1$, and $u\in A_{p/r}(\mathbb{R}^n)$. For any bounded function f with compact support, we know from the inequality (2.2) that $M_r(T_A^*f)\in L^p(\mathbb{R}^n,u)$. It then follows from Lemma 2.3, the Coifman-Fefferman inequality [2] and the inequality (2.2), that

$$
\begin{aligned}
\|T_A^*f\|_{L^p(\mathbb{R}^n,\,u)}&\lesssim\|M_r^{\sharp}(T_A^*f)\|_{L^p(\mathbb{R}^n,\,u)}\\
&\lesssim\|M_s(T_kf)\|_{L^p(\mathbb{R}^n,\,u)}+\|Mf\|_{L^p(\mathbb{R}^n,\,u)}\\
&\lesssim\|Mf\|_{L^p(\mathbb{R}^n,\,u)},
\end{aligned}
$$

where the last inequality follows from the fact that

$$\|T_kf\|_{L^p(\mathbb{R}^n,\,u)}\lesssim\|f\|_{L^p(\mathbb{R}^n,\,u)},\ 1<p<\infty,\,u\in A_\infty(\mathbb{R}^n),$$

which is just the inequality (16) in [9].

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