*Journal of Mathematical Research with Applications*
Mar., 2014, Vol. 34, No. 2, pp. 209–215
DOI:10.3770/j.issn:2095-2651.2014.02.010
Http://jmre.dlut.edu.cn

# Generalized Stability of Multi-Quadratic Mappings

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**Abstract** In this paper we unify the system of functional equations defining multi-quadratic mappings to a single equation, find out the general solution of it and prove its generalized Hyers-Ulam stability.

**Keywords** Hyers-Ulam stability; multi-quadratic mapping; functional equation.

**MR(2010) Subject Classification** 39B52; 39B82

## 1. Introduction

Throughout this paper, let $X$ and $Y$ be vector spaces over $\mathcal{Q}$, the field of rational numbers, and $n \geq 2$ be an integer. A mapping $g : X \longrightarrow Y$ is called quadratic if $g$ satisfies the functional equation $g(x+y)+g(x-y)=2g(x)+2g(y)$ for all $x, y \in X$.

A mapping $f : X^n \longrightarrow Y$ is called multi-quadratic or $n$-quadratic if it is quadratic in each variable; that is,

$$\begin{aligned}&f(x_1,\dots,x_{i-1},x_i+x_i',x_{i+1},\dots,x_n)+f(x_1,\dots,x_{i-1},x_i-x_i',x_{i+1},\dots,x_n)\\&\quad=2f(x_1,\dots,x_n)+2f(x_1,\dots,x_{i-1},x_i',x_{i+1},\dots,x_n)\end{aligned}\tag{1.1}$$

for all $i \in \{1,\dots,n\}$ and all $x_1,\dots,x_{i-1},x_i,x_i',x_{i+1},\dots,x_n \in X$.

For a mapping $f : X^n \longrightarrow Y$, consider the functional equation

$$\begin{aligned}&\sum_{i_1,\dots,i_n\in\{0,1\}} f(x_{11}+(-1)^{i_1}x_{12},\dots,x_{n1}+(-1)^{i_n}x_{n2})\\&\quad=2^n\sum_{j_1,\dots,j_n\in\{1,2\}} f(x_{1j_1},\dots,x_{nj_n})\end{aligned}\tag{1.2}$$

for all $x_{11},x_{12},\dots,x_{n1},x_{n2} \in X$.

In this paper we reduce system (1.1) to equation (1.2), establish the general solution of (1.2) and prove its generalized Hyers-Ulam stability.

The theory of Hyers-Ulam's type stability is a very popular subject of investigations. For the historical background on it, we refer to [1, 2] and the references therein. Recently, some

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Received April 17, 2013; Accepted November 12, 2013
Supported by the National Natural Science Foundation of China (Grant No. 10971117).
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mathematicians established the general solution and investigated the stability of some multivariable functional equations. In particular, Ciepliński [3] established the solution of multi-additive functional equation, and the stability of it was proved in [4, 5]. In [6], Prager and Schwaiger got the solution of multi-Jensen equation, and the stability of it was recently investigated in [7–12].

The stability of 2-quadratic mappings were studied in [13]. In [14], K. Ciepliński proved the stability of the system (1.1) of equations defining the multi-quadratic mappings. So the result of this paper generalizes the result of [13].

## 2. Solutions of Eq. (1.2)

We start with the following lemma.

**Lemma 2.1** *A function $f : X^n \longrightarrow Y$ satisfies (1.2) for all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$ if and only if $f$ is $n$-quadratic.*

**Proof** Assume that $f : X^n \longrightarrow Y$ satisfies (1.2). First, we use induction to show that $f(x_1, \ldots, x_n) = 0$ for any $(x_1, \ldots, x_n) \in X^n$ with at least one component which is equal to zero.

Putting $x_{11} = x_{12} = \cdots = x_{n1} = x_{n2} = 0$ in (1.2), we get

$$2^n f(0, \ldots, 0) = 2^n \times 2^n f(0, \ldots, 0),$$

and consequently $f(0, \ldots, 0) = 0$.

Now we assume that $f(x_1, \ldots, x_n) = 0$ for any $(x_1, \ldots, x_n) \in X^n$ with at most $i$ $(0 \le i \le n-2)$ components which are not equal to zero. Fix $1 \le k_1 \le \cdots \le k_{i+1} \le n$ and put $x_{j2} = 0$ for all $j \in \{1, \ldots, n\}$ and $x_{j1} = 0$ for $j \in \{1, \ldots, n\} \setminus \{k_1, \ldots, k_{i+1}\}$. By assumption, we have $f(0, \ldots, x_{k_1 j_{k_1}}, 0, \ldots, x_{k_{i+1} j_{k_{i+1}}}, 0, \ldots, 0) = 0$ if $j_{k_1}, \ldots, j_{k_{i+1}} \in \{1, 2\}$ and at least one is 2. Then, by (1.2)

$$\begin{aligned}
&2^n f(0, \ldots, x_{k_1 1}, 0, \ldots, x_{k_{i+1} 1}, 0, \ldots, 0) \\
&\quad = 2^n (2^{n-i-1} f(0, \ldots, x_{k_1 1}, 0, \ldots, x_{k_{i+1} 1}, 0, \ldots, 0) + \\
&\qquad 2^{n-i-1} \sum \{ f(0, \ldots, x_{k_1 j_{k_1}}, 0, \ldots, x_{k_{i+1} j_{k_{i+1}}}, 0, \ldots, 0) : \\
&\qquad j_{k_1}, \ldots, j_{k_{i+1}} \in \{1, 2\} \text{ and at least one is } 2\}),
\end{aligned}$$

and thus $f(0, \ldots, x_{k_1 1}, 0, \ldots, x_{k_{i+1} 1}, 0, \ldots, 0) = 0$. By induction, we have $f(x_1, \ldots, x_n) = 0$ for any $(x_1, \ldots, x_n) \in X^n$ with at least one component which is equal to zero.

Next, fix an $i \in \{1, \ldots, n\}$. Putting $x_{j2} = 0$ for $j \in \{1, \ldots, n\} \setminus \{i\}$ in (1.2), we have

$$\begin{aligned}
&2^{n-1} f(x_{11}, \ldots, x_{(i-1)1}, x_{i1} + x_{i2}, x_{(i+1)1}, \ldots, x_{n1}) + \\
&\quad 2^{n-1} f(x_{11}, \ldots, x_{(i-1)1}, x_{i1} - x_{i2}, x_{(i+1)1}, \ldots, x_{n1}) \\
&= 2^n \Big( \sum_{j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_n \in \{1,2\}} f(x_{1j_1}, \ldots, x_{(i-1)j_{i-1}}, x_{i1}, x_{(i+1)j_{i+1}}, \ldots, x_{nj_n}) + \\
&\quad \sum_{j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_n \in \{1,2\}} f(x_{1j_1}, \ldots, x_{(i-1)j_{i-1}}, x_{i2}, x_{(i+1)j_{i+1}}, \ldots, x_{nj_n}) \Big)
\end{aligned}$$

$$= 2^n(f(x_{11}, \ldots, x_{(i-1)1}, x_{i1}, x_{(i+1)1}, \ldots, x_{n1}) +$$
$$f(x_{11}, \ldots, x_{(i-1)1}, x_{i2}, x_{(i+1)1}, \ldots, x_{n1})).$$

Thus $f(x_{11}, \ldots, x_{(i-1)1}, x_{i1}+x_{i2}, x_{(i+1)1}, \ldots, x_{n1})+f(x_{11}, \ldots, x_{(i-1)1}, x_{i1}-x_{i2}, x_{(i+1)1}, \ldots, x_{n1}) = 2(f(x_{11}, \ldots, x_{(i-1)1}, x_{i1}, x_{(i+1)1}, \ldots, x_{n1}) + f(x_{11}, \ldots, x_{(i-1)1}, x_{i2}, x_{(i+1)1}, \ldots, x_{n1}))$ for all $i \in \{1, 2, \ldots, n\}$ and all $x_{11}, \ldots, x_{(i-1)1}, x_{i1}, x_{i2}, x_{(i+1)1}, \ldots, x_{n1} \in X$, which proves that $f$ is multi-quadratic. The rest of the proof is clear.

**Theorem 2.2** *A function $f : X^n \longrightarrow Y$ satisfies Eq. (1.2) for all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$ if and only if there exists a function $F : X^{2n} \longrightarrow Y$ such that $f(x_1, \ldots, x_n) = F(x_1, x_1, \ldots, x_n, x_n)$ for all $x_1, \ldots, x_n \in X$, and $F$ is additive in each variable and is symmetric about the $(2i-1)th$ variable and $2ith$ variable for $i = 1, \ldots, n$.*

**Proof** If there exists a function $F : X^{2n} \longrightarrow Y$ such that $f(x_1, \ldots, x_n) = F(x_1, x_1, \ldots, x_n, x_n)$ for all $x_1, \ldots, x_n \in X$, and $F$ is additive in each variable and is symmetric about the $(2i-1)$th variable and $2i$th variable for $i \in \{1, \ldots, n\}$, then it is obvious that $f$ satisfies Eq. (1.2).

Conversely, we define a function $F : X^{2n} \longrightarrow Y$ by

$$F(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) = \frac{1}{4^n} \sum_{j_1, \ldots, j_n \in \{0,1\}} (-1)^{j_1+\cdots+j_n} f(x_{11} + (-1)^{j_1} x_{12}, \ldots, x_{n1} + (-1)^{j_n} x_{n2}) \tag{2.1}$$

for all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$. Fix $i \in \{1, \ldots, n\}$ and $x_{kj_k}$, $k \in \{1, \ldots, n\} \setminus \{i\}$, $j_k \in \{1, 2\}$. Since $f$ is quadratic in the $i$th variable, we see from [15, pp. 165–178] that the mapping

$$g_{x_{11}+(-1)^{j_1}x_{12}, \ldots, x_{(i-1)1}+(-1)^{j_{i-1}}x_{(i-1)2}, x_{(i+1)1}+(-1)^{j_{i+1}}x_{(i+1)2}, \ldots, x_{n1}+(-1)^{j_n}x_{n2}}(x_{i1}, x_{i2})$$
$$= \frac{1}{4}(f(x_{11} + (-1)^{j_1} x_{12}, \ldots, x_{(i-1)1} + (-1)^{j_{i-1}} x_{(i-1)2},$$
$$x_{i1} + x_{i2}, x_{(i+1)1} + (-1)^{j_{i+1}} x_{(i+1)2}, \ldots, x_{n1} + (-1)^{j_n} x_{n2}) -$$
$$f(x_{11} + (-1)^{j_1} x_{12}, \ldots, x_{(i-1)1} + (-1)^{j_{i-1}} x_{(i-1)2},$$
$$x_{i1} - x_{i2}, x_{(i+1)1} + (-1)^{j_{i+1}} x_{(i+1)2}, \ldots, x_{n1} + (-1)^{j_n} x_{n2}))$$

is additive in each variable and symmetric about $x_{i1}$ and $x_{i2}$. Thus the function $F : X^{2n} \longrightarrow Y$, defined by

$$F(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) = \frac{1}{4^{n-1}} \sum_{j_1, \ldots, j_{i-1}, j_{i+1}, j_n \in \{0,1\}} (-1)^{j_1+\cdots+j_{i-1}+j_{i+1}+\cdots+j_n}$$
$$g_{x_{11}+(-1)^{j_1}x_{12}, \ldots, x_{(i-1)1}+(-1)^{j_{i-1}}x_{(i-1)2}, x_{(i+1)1}+(-1)^{j_{i+1}}x_{(i+1)2}, \ldots, x_{n1}+(-1)^{j_n}x_{n2}}(x_{i1}, x_{i2}),$$

is additive in each variable and symmetric about $x_{i1}$ and $x_{i2}$, i.e., $F(x_{11}, x_{12}, \ldots, x_{i1}, x_{i2}, \ldots, x_{n1}, x_{n2}) = F(x_{11}, x_{12}, \ldots, x_{i2}, x_{i1}, \ldots, x_{n1}, x_{n2})$, for all $i \in \{1, \ldots, n\}$.

Since $f$ is quadratic in each variable, we have $f(x_1, \ldots, x_n) = 0$ for any $(x_1, \ldots, x_n) \in X^n$ with at least one component which is equal to zero, and $f(2x_1, \ldots, 2x_n) = 4^n f(x_1, \ldots, x_n)$. Choosing $x_{i1} = x_{i2}$ for all $i \in \{1, \ldots, n\}$ in Eq. (2.1), we get

$$F(x_{11}, x_{11}, \ldots, x_{n1}, x_{n1}) = \frac{1}{4^n} f(2x_{11}, \ldots, 2x_{n1}) +$$

$$\frac{1}{4^n}\sum\big\{f(x_{11}+(-1)^{j_1}x_{11},\ldots,x_{n1}+(-1)^{j_n}x_{n1}):j_1,\ldots,j_n\in\{0,1\},$$
$$\text{and at least one is }1\big\}=\frac{1}{4^n}f(2x_{11},\ldots,2x_{n1})=f(x_{11},\ldots,x_{n1}).$$

## 3. Stability of Eq. (1.2): The direct method

From now on, let $X$ and $Y$ be vector space and Banach space, respectively.

**Theorem 3.1** *Let $\phi:X^{2n}\longrightarrow[0,\infty)$ be a function such that*

$$\tilde{\phi}(x_{11},x_{12},\ldots,x_{n1},x_{n2})=\sum_{k=0}^{\infty}\frac{1}{4^{n(k+1)}}\phi(2^kx_{11},2^kx_{12},\ldots,2^kx_{n1},2^kx_{n2})<\infty \tag{3.1}$$

*for all $x_{11},x_{12},\ldots,x_{n1},x_{n2}\in X$. Suppose that a function $f:X^n\longrightarrow Y$ satisfies the inequality*

$$\begin{aligned}\|&\sum_{i_1,\ldots,i_n\in\{0,1\}}f(x_{11}+(-1)^{i_1}x_{12},\ldots,x_{n1}+(-1)^{i_n}x_{n2})-\\&2^n\sum_{j_1,\ldots,j_n\in\{1,2\}}f(x_{1j_1},\ldots,x_{nj_n})\|\le\phi(x_{11},x_{12},\ldots,x_{n1},x_{n2})\end{aligned} \tag{3.2}$$

*for all $x_{11},x_{12},\ldots,x_{n1},x_{n2}\in X$ and $f(x_1,\ldots,x_n)=0$ for any $(x_1,\ldots,x_n)\in X^n$ with at least one component which is equal to zero. Then there exists a unique multi-quadratic function $Q:X^n\longrightarrow Y$ such that*

$$\|f(x_1,\ldots,x_n)-Q(x_1,\ldots,x_n)\|\le\tilde{\phi}(x_1,x_1,\ldots,x_n,x_n) \tag{3.3}$$

*for all $x_1,\ldots,x_n\in X$.*

**Proof** Choosing $x_{i1}=x_{i2}=x_i$ for all $i\in\{1,\ldots,n\}$ and dividing by $4^n$ in Eq. (3.2), we have

$$\|\frac{1}{4^n}f(2x_1,\ldots,2x_n)-f(x_1,\ldots,x_n)\|\le\frac{1}{4^n}\phi(x_1,x_1,\ldots,x_n,x_n)$$

for all $x_1,\ldots,x_n\in X$, using the assumption $f(x_1,\ldots,x_n)=0$ for any $(x_1,\ldots,x_n)\in X^n$ with at least one component which is equal to zero. Replacing $x_i$ by $2^kx_i$ for all $i\in\{1,\ldots,n\}$, respectively, and dividing by $4^{nk}$ in the above inequality, we get

$$\|\frac{1}{4^{n(k+1)}}f(2^{k+1}x_1,\ldots,2^{k+1}x_n)-\frac{1}{4^{nk}}f(2^kx_1,\ldots,2^kx_n)\|\le\frac{1}{4^{n(k+1)}}\phi(2^kx_1,2^kx_1,\ldots,2^kx_n,2^kx_n)$$

for all $x_1,\ldots,x_n\in X$. Hence

$$\begin{aligned}&\|\frac{1}{4^{nm}}f(2^mx_1,\ldots,2^mx_n)-\frac{1}{4^{nk}}f(2^kx_1,\ldots,2^kx_n)\|\\&\le\sum_{i=k}^{m-1}\frac{1}{4^{n(i+1)}}\phi(2^ix_1,2^ix_1,\ldots,2^ix_n,2^ix_n)\end{aligned} \tag{3.4}$$

for all nonnegative integers $k$ and $m$ with $k<m$ and all $x_1,\ldots,x_n\in X$. Therefore we conclude from (3.1) and (3.4) that $\{\frac{1}{4^{nk}}f(2^kx_1,\ldots,2^kx_n)\}$ is a Cauchy sequence in $Y$. Since $Y$ is a Banach space, this sequence is convergent. We define $Q:X^n\longrightarrow Y$ by

$$Q(x_1,\ldots,x_n)=\lim_{k\longrightarrow\infty}\frac{1}{4^{nk}}f(2^kx_1,\ldots,2^kx_n)$$

for all $x_1, \ldots, x_n \in X$. It follows from (3.2) and (3.1) that

$$\|\sum_{i_1,\ldots,i_n\in\{0,1\}} Q(x_{11}+(-1)^{i_1}x_{12},\ldots,x_{n1}+(-1)^{i_n}x_{n2}) - 2^n \sum_{j_1,\ldots,j_n\in\{1,2\}} Q(x_{1j_1},\ldots,x_{nj_n})\|$$
$$= \lim_{k\longrightarrow\infty} \frac{1}{4^{nk}} \Big| \sum_{i_1,\ldots,i_n\in\{0,1\}} f(2^k x_{11}+(-1)^{i_1}2^k x_{12},\ldots,2^k x_{n1}+(-1)^{i_n}2^k x_{n2}) -$$
$$2^n \sum_{j_1,\ldots,j_n\in\{1,2\}} f(2^k x_{1j_1},\ldots,2^k x_{nj_n}) \Big|$$
$$\le \lim_{k\longrightarrow\infty} \frac{1}{4^{nk}} \phi(2^k x_{11}, 2^k x_{12},\ldots,2^k x_{n1},2^k x_{n2}) = 0$$

for all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$. Hence, by Lemma 2.1, $Q$ is multi-quadratic.

Choosing $k = 0$ and letting $m \longrightarrow \infty$ in (3.4), we obtain

$$\|Q(x_1,\ldots,x_n) - f(x_1,\ldots,x_n)\| \le \sum_{i=0}^{\infty} \frac{1}{4^{n(i+1)}} \phi(2^i x_1, 2^i x_1,\ldots,2^i x_n, 2^i x_n) = \tilde{\phi}(x_1,x_1,\ldots,x_n,x_n)$$

for all $x_1, \ldots, x_n \in X$.

It remains to show that $Q$ is unique. Suppose that there eixsts another multi-quadratic function $\tilde{Q} : X^n \longrightarrow Y$ which satisfies (3.3). Since $Q(2^k x_1,\ldots,2^k x_n) = 4^{nk} Q(x_1,\ldots,x_n)$ and $\tilde{Q}(2^k x_1,\ldots,2^k x_n) = 4^{nk}\tilde{Q}(x_1,\ldots,x_n)$ for all $x_1,\ldots,x_n \in X$, we conclude that

$$\|\tilde{Q}(x_1,\ldots,x_n) - Q(x_1,\ldots,x_n)\| = \frac{1}{4^{nk}} \|\tilde{Q}(2^k x_1,\ldots,2^k x_n) - Q(2^k x_1,\ldots,2^k x_n)\|$$
$$\le \frac{1}{4^{nk}} (\|\tilde{Q}(2^k x_1,\ldots,2^k x_n) - f(2^k x_1,\ldots,2^k x_n)\| + \|f(2^k x_1,\ldots,2^k x_n) - Q(2^k x_1,\ldots,2^k x_n)\|)$$
$$\le \frac{2}{4^{nk}} \tilde{\phi}(2^k x_1, 2^k x_1,\ldots,2^k x_n, 2^k x_n)$$
$$\le 2\sum_{i=0}^{\infty} \frac{1}{4^{n(k+i+1)}} \phi(2^{k+i} x_1, 2^{k+i} x_1,\ldots,2^{k+i} x_n, 2^{k+i} x_n)$$
$$\le 2\sum_{i=k}^{\infty} \frac{1}{4^{n(i+1)}} \phi(2^i x_1, 2^i x_1,\ldots,2^i x_n, 2^i x_n)$$

for every nonnegative integer $k$ and all $x_1,\ldots,x_n \in X$. Letting $k \longrightarrow \infty$ in this inequality, we have $\tilde{Q}(x_1,\ldots,x_n) = Q(x_1,\ldots,x_n)$ for all $x_1,\ldots,x_n \in X$, which gives the conclusion.

Similarly, one can prove the following theorem.

**Theorem 3.2** *Let $\phi : X^{2n} \longrightarrow [0,\infty)$ be a function such that*

$$\tilde{\phi}(x_{11}, x_{12},\ldots,x_{n1},x_{n2}) = \sum_{k=0}^{\infty} 4^{nk} \phi(\frac{x_{11}}{2^{k+1}}, \frac{x_{12}}{2^{k+1}},\ldots,\frac{x_{n1}}{2^{k+1}},\frac{x_{n2}}{2^{k+1}}) < \infty$$

*for all $x_{11}, x_{12},\ldots,x_{n1},x_{n2} \in X$. Suppose that a function $f : X^n \longrightarrow Y$ satisfies the inequality*

$$\|\sum_{i_1,\ldots,i_n\in\{0,1\}} f(x_{11}+(-1)^{i_1}x_{12},\ldots,x_{n1}+(-1)^{i_n}x_{n2}) -$$
$$2^n \sum_{j_1,\ldots,j_n\in\{1,2\}} f(x_{1j_1},\ldots,x_{nj_n})\| \le \phi(x_{11},x_{12},\ldots,x_{n1},x_{n2})$$

*for all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$ and $f(x_1, \ldots, x_n) = 0$ for any $(x_1, \ldots, x_n) \in X^n$ with at least one component which is equal to zero. Then there exists a unique multi-quadratic function $Q : X^n \longrightarrow Y$ such that*

$$\|f(x_1, \ldots, x_n) - Q(x_1, \ldots, x_n)\| \leq \tilde{\phi}(x_1, x_1, \ldots, x_n, x_n)$$

*for all $x_1, \ldots, x_n \in X$.*

## 4. Stability of Eq.(1.2): The fixed point method

Apart from the direct method applied by Hyers, the fixed point method introduced by Radu [16] is effective in the investigations of the stability of functional equations. Some further applications of fixed point theorems to the Hyers-Ulam stability of functional equations can be found in [17]. Applying Radu's method, one can prove the following two results.

**Theorem 4.1** *Let $\phi : X^{2n} \longrightarrow [0, \infty)$ be a function such that*

$$\phi(2x_{11}, 2x_{12}, \ldots, 2x_{n1}, 2x_{n2}) \leq 4^n L\phi(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) \tag{4.1}$$

*for an $L \in (0, 1)$ and all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$. Suppose that a function $f : X^n \longrightarrow Y$ satisfies the inequality*

$$\begin{aligned}&\Big\| \sum_{i_1,\ldots,i_n \in \{0,1\}} f(x_{11} + (-1)^{i_1} x_{12}, \ldots, x_{n1} + (-1)^{i_n} x_{n2}) - \\ &\quad 2^n \sum_{j_1,\ldots,j_n \in \{1,2\}} f(x_{1j_1}, \ldots, x_{nj_n}) \Big\| \\ &\leq \phi(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2})\end{aligned} \tag{4.2}$$

*for all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$ and $f(x_1, \ldots, x_n) = 0$ for any $(x_1, \ldots, x_n) \in X^n$ with at least one component which is equal to zero. Then there exists a unique multi-quadratic function $Q : X^n \longrightarrow Y$ such that*

$$\|f(x_1, \ldots, x_n) - Q(x_1, \ldots, x_n)\| \leq \frac{1}{4^n(1 - L)} \phi(x_1, x_1, \ldots, x_n, x_n) \tag{4.3}$$

*for all $x_1, \ldots, x_n \in X$.*

**Theorem 4.2** *Let $\phi : X^{2n} \longrightarrow [0, \infty)$ be a function such that*

$$\phi(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) \leq \frac{L}{4^n} \phi(2x_{11}, 2x_{12}, \ldots, 2x_{n1}, 2x_{n2})$$

*for an $L \in (0, 1)$ and all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$. Suppose that a function $f : X^n \longrightarrow Y$ satisfies the inequality*

$$\begin{aligned}&\Big\| \sum_{i_1,\ldots,i_n \in \{0,1\}} f(x_{11} + (-1)^{i_1} x_{12}, \ldots, x_{n1} + (-1)^{i_n} x_{n2}) - \\ &\quad 2^n \sum_{j_1,\ldots,j_n \in \{1,2\}} f(x_{1j_1}, \ldots, x_{nj_n}) \Big\| \leq \phi(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2})\end{aligned}$$

*for all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$ and $f(x_1, \ldots, x_n) = 0$ for any $(x_1, \ldots, x_n) \in X^n$ with at least* *one component which is equal to zero. Then there exists a a unique multi-quadratic function* $Q : X^n \longrightarrow Y$ *such that*

$$\|f(x_1, \ldots, x_n) - Q(x_1, \ldots, x_n)\| \leq \frac{L}{4^n(1-L)}\phi(x_1, x_1, \ldots, x_n, x_n)$$

*for all* $x_1, \ldots, x_n \in X$.

**Acknowledgments** The authors would like to thank the anonymous referees for the constructive comments and suggestions which helped to improve the quality of this paper.

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