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A Note on Strong Law of Large Numbers for Partial Sums of Pairwise NQD Random Variables

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Abstract In this paper, the authors study the strong law of large numbers for partial sums of pairwise negatively quadrant dependent (NQD) random variables. The results obtained improve the corresponding theorems of Hu et al. (2013), and Qiu and Yang (2006) under some weaker conditions.

Keywords pairwise NQD random variables; strong law of large numbers.

MR(2010) Subject Classification 60F15

1. Introduction

The following concept of negatively quadrant dependent (NQD) random variables was introduced by Lehmann [1].

Definition 1.1 *Two random variables X and Y are said to be NQD if*

$$P(X \le x, Y \le y) \le P(X \le x)P(Y \le y) \quad \text{for all } x \text{ and } y \in R.$$

A finite or infinite sequence of random variables is said to be pairwise NQD if every two random variables in the sequence are NQD.

Sequences of pairwise NQD random variables are a family of very wide scope, which contain sequences of negatively associated (NA, Joag and Proschan [2]) random variable, negatively orthant dependent (NOD, Ebrahimi and Ghosh [3]) random variables and linearly negative quadrant dependent (LNQD, Newman [4]) random variables. As we know, pairwise independent class is the most important and special case of pairwise NQD class, which has been investigated in many literature and attracted extensive attentions. Therefore, it is very significant to study probabilistic properties of this wider pairwise NQD class. Years after the appearance of Lehmann [1], many literature of investigation concerning the convergence properties of pairwise NQD random

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variables has emerged. Matula [5], Wan [6] and Li and Yang [7] obtained some results on the Kolmogorov strong law of large numbers, Wu [8] investigated the Kolmogorov-Chung type strong law of large numbers, Wu and Jiang [9] and Hu et al. [10] studied the Marcinkiewicz-Zygmund strong law of large number, Gan and Chen [11] discussed the strong stability of Jamison's weighted sums, Wu and Guan [12] presented the weak laws of large numbers.

The purpose of this article is to investigate the strong law of large numbers for sequences of pairwise NQD random variables.

Definition 1.2 *A sequence of random variables $\{X_n, n\geq 1\}$ is said to be stochastically dominated by a random variable X if there exists a constant $C>0$ such that*

$$\sup_{n\geq 1} P(|X_n|>x)\leq CP(|X|>x) \text{ for all } x\geq 0.$$

The above concept of stochastic domination is a generalization of the concept of identical distributions. Stochastic dominance of $\{X_n, n\geq 1\}$ by the random variable X implies $E|X_n|^p\leq CE|X|^p$ if the p-moment of $|X|$ exists, i.e., if $E|X|^p<\infty$.

Hu et al. [10] studied the strong law of large numbers for sequences of pairwise NQD random variables and obtained the following theorem.

Theorem 1.1 ([10]) *Let $\{X_n,\ n\geq 1\}$ be a pairwise NQD sequence with $EX_n=0$ for all $n\geq 1$. Suppose that $\{X_n,\ n\geq 1\}$ is stochastically dominated by a random variable X. If there exist constants $1\leq r<2$ and $\alpha>r+1$ such that*

$$E(|X|^r\log^\alpha|X|)<\infty, \tag{1.1}$$

then

$$\lim_{n\to\infty} n^{-1/r}\sum_{k=1}^{n} X_k=0 \text{ a.s.} \tag{1.2}$$

Remark 1.1 It is well known that $E|X|^r<\infty$ $(1<r<2)$ and (1.2) are equivalent in i.i.d case [13]. To some extent, Theorem 1.1 generalizes the sufficient parts of the Kolmogorov and Marcinkiewicz-Zygmund SLLN for i.i.d case to pairwise NQD case. However, (1.1) is stronger than the optimal condition.

Qiu and Yang [14] studied the strong law of large numbers for weighted sums of NA random variables and obtained the following theorem.

Theorem 1.2 ([14]) *Suppose $1/r=1/\alpha+1/\beta$ for $1<\alpha,\beta<\infty$ and $1<r<2$. Let $\{X, X_n, n\geq 1\}$ be a sequence of NA random variables with indentical distributions, and let $\{a_{ni}, 1\leq i\leq n, n\geq 1\}$ be an array of constants satisfying*

$$A_\alpha=\lim_{n\to\infty}\sup A_{\alpha,n}<\infty,\ \ A_{\alpha,n}^{\alpha}=\sum_{i=1}^{n}|a_{ni}|^\alpha/n. \tag{1.3}$$

If $E|X|^\beta<\infty$ and $EX=0$, then

$$n^{-1/r}\sum_{i=1}^{n} a_{ni}X_i\to 0 \text{ a.s.} \tag{1.4}$$

In this work, we obtain some results on the strong law of large numbers for sequences of pairwise NQD random variables, which improve and extend Theorems 1.1 and 1.2 under some weaker conditions. We point out that the method used in this article differs from that in Hu et al. [10] or Qiu and Yang [14].

Throughout this paper, the symbol C is used to represent positive constants whose values may change from one place to another.

2. Preliminaries and main results

To prove our main results, we need some technical lemmas.

Lemma 2.1 ([1]) *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables. Let $\{f_n, n \geq 1\}$ be a sequence of increasing functions. Then $\{f_n(X_n), n \geq 1\}$ is a sequence of pairwise NQD random variables.*

Lemma 2.2 ([8]) *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables with mean zero and $EX_n^2 < \infty$, $n \geq 1$, and let $T_j(k) = \sum_{i=j+1}^{j+k} X_i$, $j \geq 0$, $k \geq 1$. Then*

$$E \max_{1 \leq k \leq n} (T_j(k))^2 \leq C \log^2 n \sum_{i=j+1}^{j+n} EX_i^2, \quad n \geq 1.$$

Lemma 2.3 ([15]) *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. Then there exists a constant C such that, for all $q > 0$ and $x > 0$,*

(i) $E(|X_k|^q I(|X_k| \leq x)) \leq C\{E(|X|^q I(|X| \leq x)) + x^q P(|X| > x)\}$,

(ii) $E(|X_k|^q I(|X_k| > x)) \leq CE(|X|^q I(|X| > x))$.

Now we state our main results and the proofs will be presented in next section.

Theorem 2.1 *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables with $EX_n = 0$ and be stochastically dominated by a random variable X. If there exist constants $1 \leq r < 2$ such that*

$$E(|X|^r \log^2 |X|) < \infty, \tag{2.1}$$

then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1 \leq j \leq n} \Big|\sum_{k=1}^{j} X_k\Big| > n^{1/r}\varepsilon \Big) < \infty. \tag{2.2}$$

Corollary 2.1 *Under the conditions of Theorem 2.1, (1.2) holds.*

Remark 2.1 Since (2.1) is weaker than (1.1) and since (2.2) implies (1.2) as will be shown in the proof of Corollary 2.1, Theorem 2.1 and Corollary 2.1 improve Theorem 1.1. Though (2.1) is weaker than (1.1), it is not desirable compared with the moment condition $E|X|^r < \infty$ $(1 < r < 2)$ for i.i.d case. It is still unknown whether Theorem 2.1 or Corollary 2.1 remains true by replacing (2.1) with $E|X|^r < \infty$. Despite our efforts to solve this problem, it is still an open problem.

Theorem 2.2 *Suppose $1/r = 1/\alpha + 1/\beta$ for $1 < \alpha, \beta < \infty$ and $1 < r < 2$. Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables with $EX_n = 0$ and be stochastically dominated by a random variable X, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying (1.3). If $E|X|^\beta < \infty$, then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\leq j\leq n} \big|\sum_{i=1}^{j} a_{ni}X_i\big| > n^{1/r}\varepsilon\Big) < \infty. \tag{2.3}$$

Corollary 2.2 *Under the conditions of Theorem 2.2, (1.4) holds.*

Remark 2.2 Since NA implies pairwise NQD and since (2.3) implies (1.4) as will be shown in the proof of Corollary 2.2, Theorem 2.2 and Corollary 2.2 improve Theorem 1.2. The proof of Corollary 2.2 is similar to that of Corollary 2.1, and we will omit the details. In addition, it should be noted that the moment condition in Theorems 1.2 and 2.2 is not so good as that of results by Chow and Lai [16].

3. The proofs of main results

In this section, we state the proofs of our main results.

Proof of Theorem 2.1 For fixed $n \geq 1$, let

$$\begin{aligned} Y_{nk} =& -n^{1/r} I(X_k < -n^{1/r}) + X_k I(|X_k| \leq n^{1/r}) + n^{1/r} I(X_k > n^{1/r}), \\ Z_{nk} =& (X_k + n^{1/r}) I(X_k < -n^{1/r}) + (X_k - n^{1/r}) I(X_k > n^{1/r}). \end{aligned}$$

Then $Y_{nk} + Z_{nk} = X_k$, and it follows by Lemma 2.1 that $\{Y_{nk}, k \geq 1\}$ is a sequence of pairwise NQD random variables. Then

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\leq j\leq n} \big|\sum_{k=1}^{j} X_k\big| > n^{1/r}\varepsilon\Big) \\ &\leq \sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\leq j\leq n} \big|\sum_{k=1}^{j} (Z_{nk} - EZ_{nk})\big| > n^{1/r}\varepsilon/2\Big) + \\ &\quad \sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\leq j\leq n} \big|\sum_{k=1}^{j} (Y_{nk} - EY_{nk})\big| > n^{1/r}\varepsilon/2\Big) \\ &=: I_1 + I_2. \end{aligned}$$

To prove (2.2), it only needs to be shown that $I_1 < \infty$ and $I_2 < \infty$. Note that $|Z_{nk}| \leq |X_k| I(|X_k| > n^{1/r})$. By the Markov inequality, Lemma 2.3, and (2.1), we have

$$\begin{aligned} I_1 &\leq C \sum_{n=1}^{\infty} n^{-1-1/r} \sum_{k=1}^{n} E|Z_{nk} - EZ_{nk}| \\ &\leq C \sum_{n=1}^{\infty} n^{-1-1/r} \sum_{k=1}^{n} E\big(|X_k| I(|X_k| > n^{1/r})\big) \end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{n=1}^{\infty} n^{-1/r}\sum_{s=n}^{\infty} E\big(|X|I(s<|X|^r\le s+1)\big)\\
&= C\sum_{s=1}^{\infty} E\big(|X|I(s<|X|^r\le s+1)\big)\sum_{n=1}^{s} n^{-1/r}\\
&\le \begin{cases} C\sum_{s=1}^{\infty}\log s\, E(|X|I(s<|X|\le s+1)) & \text{for } r=1\\ C\sum_{s=1}^{\infty} s^{1-1/r}E(|X|I(s<|X|^r\le s+1)) & \text{for } 1<r<2\end{cases}\\
&\le \begin{cases} CE|X|\log|X|<\infty & \text{for } r=1\\ CE|X|^r<\infty & \text{for } 1<r<2.\end{cases}
\end{aligned}
$$

Next we prove $I_2<\infty$. By the Markov inequality, Lemmas 2.2, and 2.3, we have

$$
\begin{aligned}
I_2 \le& C\sum_{n=1}^{\infty} n^{-1-2/r}\log^2 n\sum_{k=1}^{n} EY_{nk}^2\\
\le& C\sum_{n=1}^{\infty} n^{-1-2/r}\log^2 n\sum_{k=1}^{n} EX_k^2 I(|X_k|\le n^{1/r}) + C\sum_{n=1}^{\infty} n^{-1}\log^2 n\sum_{k=1}^{n} P(|X_k|>n^{1/r})\\
\le& C\sum_{n=1}^{\infty} n^{-2/r}\log^2 n\, EX^2 I(|X|\le n^{1/r}) + C\sum_{n=1}^{\infty}\log^2 n\, P(|X|>n^{1/r})\\
=:& I_3+I_4.
\end{aligned}
$$

Since the function $\log^2 x$ is slowly varying at ∞, by applying Lemma 2.4(ii) of Zhou [17], we have

$$
\begin{aligned}
I_3 =& C\sum_{n=1}^{\infty} n^{-2/r}\log^2 n\sum_{s=1}^{n} EX^2 I(s-1<|X|^r\le s)\\
=& C\sum_{s=1}^{\infty} EX^2 I(s-1<|X|^r\le s)\sum_{n=s}^{\infty} n^{-2/r}\log^2 n\\
\le& C\sum_{s=1}^{\infty} s^{1-2/r}\log^2 s\, EX^2 I(s-1<|X|^r\le s)\\
\le& CE\big(|X|^r\log^2|X|\big)<\infty.
\end{aligned}
$$

Finally we prove $I_4<\infty$. For $1<r<2$, we have

$$
\begin{aligned}
I_4 \le& C\sum_{n=1}^{\infty} n^{-1/r}\log^2 n\, E|X|I(|X|>n^{1/r})\\
=& C\sum_{n=1}^{\infty} n^{-1/r}\log^2 n\sum_{s=n}^{\infty} E|X|I(s<|X|^r\le s+1)\\
=& C\sum_{s=1}^{\infty} E|X|I(s<|X|^r\le s+1)\sum_{n=1}^{s} n^{-1/r}\log^2 n\\
\le& C\sum_{s=1}^{\infty} E|X|I(s<|X|^r\le s+1)(\log^2 s)\sum_{n=1}^{s} n^{-1/r}
\end{aligned}
$$

$$\begin{aligned}&\le C\sum_{s=1}^{\infty} s^{1-1/r}\log^2 s\, E|X|I(s<|X|^r\le s+1)\\&\le CE\big(|X|^r\log^2|X|\big)<\infty.\end{aligned}$$

For $r=1$, we have

$$\begin{aligned}I_4 =&C\sum_{n=1}^{\infty}\log^2 n\sum_{m=n}^{\infty}P(m<|X|\le m+1)\\=&C\sum_{m=1}^{\infty}P(m<|X|\le m+1)\sum_{n=1}^{m}\log^2 n\\\le&C\sum_{m=1}^{\infty}m\log^2 m\, P(m<|X|\le m+1)\\\le&CE\big(|X|\log^2|X|\big)<\infty.\end{aligned}$$

The proof is completed. □

Proof of Corollary 2.1 Let $S_j=\sum_{k=1}^{j}X_k$, $j\ge 1$. From (2.2), we have for arbitrary $\varepsilon>0$,

$$\begin{aligned}\infty>&\sum_{n=1}^{\infty}n^{-1}P\Big(\max_{1\le j\le n}\big|S_j\big|>n^{\frac{1}{r}}\varepsilon\Big)\\=&\sum_{i=0}^{\infty}\sum_{n=2^i}^{2^{i+1}-1}n^{-1}P\Big(\max_{1\le j\le n}\big|S_j\big|>n^{\frac{1}{r}}\varepsilon\Big)\\\ge&\frac{1}{2}\sum_{i=1}^{\infty}P\Big(\max_{1\le j\le 2^i}\big|S_j\big|>2^{\frac{i+1}{r}}\varepsilon\Big).\end{aligned}$$

Then by the Borel-Cantelli Lemma and the arbitrariness of $\varepsilon>0$,

$$\lim_{i\to\infty}2^{-\frac{i+1}{r}}\max_{1\le j\le 2^i}\big|S_j\big|=0\ \text{ a.s.}$$

For all positive integers n, there exists a positive integer i_0 such that $2^{i_0-1}\le n<2^{i_0}$. Then $i_0\to\infty$ as $n\to\infty$ and

$$\begin{aligned}n^{-\frac{1}{r}}\big|S_n\big|\le&\max_{2^{i_0-1}\le j<2^{i_0}}j^{-\frac{1}{r}}\big|S_j\big|\\\le&2^{\frac{2}{r}}2^{-\frac{i_0+1}{r}}\max_{1\le j<2^{i_0}}\big|S_j\big|\to 0\ \text{ a.s. as } n\to\infty.\end{aligned}$$

The proof is completed. □

Proof of Theorem 2.2 Since $a_{ni}=a_{ni}^{+}-a_{ni}^{-}$, without loss of generality, we may assume that $a_{ni}\ge 0$. From (1.3), we may assume that $\sum_{i=1}^{n}|a_{ni}|^{\alpha}\le n$. Then by the Hölder inequality, for $\forall\ 1\le\gamma<\alpha$,

$$\sum_{i=1}^{n}|a_{ni}|^{\gamma}\le\Big(\sum_{i=1}^{n}|a_{ni}|^{\gamma\frac{\alpha}{\gamma}}\Big)^{\frac{\gamma}{\alpha}}\Big(\sum_{i=1}^{n}1\Big)^{\frac{\alpha-\gamma}{\alpha}}\le n. \tag{3.1}$$

For $\forall\ \gamma \geq \alpha$,

$$\sum_{i=1}^{n} |a_{ni}|^{\gamma} = \sum_{i=1}^{n} |a_{ni}|^{\alpha} |a_{ni}|^{\gamma-\alpha} \leq \sum_{i=1}^{n} |a_{ni}|^{\alpha} \Big(\sum_{i=1}^{n} |a_{ni}|^{\alpha}\Big)^{\frac{\gamma-\alpha}{\alpha}} \leq n^{\frac{\gamma}{\alpha}}. \tag{3.2}$$

We let

$$\begin{aligned} Y_{ni} =& -n^{1/r} I(a_{ni}X_i < -n^{1/r}) + a_{ni}X_i I(a_{ni}|X_i| \leq n^{1/r}) + n^{1/r} I(a_{ni}X_i > n^{1/r}), \\ Z_{ni} =& (a_{ni}X_i + n^{1/r}) I(a_{ni}X_i < -n^{1/r}) + (a_{ni}X_i - n^{1/r}) I(a_{ni}X_i > n^{1/r}). \end{aligned}$$

Then it follows by Lemma 2.1 that $\{Y_{ni}, i \geq 1, n \geq 1\}$ and $\{Z_{ni}, i \geq 1, n \geq 1\}$ are both pairwise NQD. Hence we have

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\leq j\leq n} \big|\sum_{i=1}^{j} a_{ni}X_i\big| > n^{1/r}\varepsilon\Big) \\ &\leq \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P\big(a_{ni}|X_i| > n^{1/r}\big) + \sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\leq j\leq n} \big|\sum_{i=1}^{j} Y_{ni}\big| > n^{1/r}\varepsilon\Big) \\ &=: I_5 + I_6. \end{aligned}$$

By Definition 1.2, $\sum_{i=1}^{n} |a_{ni}|^{\alpha} \leq n$ and $1/r = 1/\alpha + 1/\beta$, we have

$$\begin{aligned} I_5 &\leq \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P\big(|X|^{\alpha} > n^{\alpha/r} a_{ni}^{-\alpha}\big) \\ &\leq \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P\Big(|X|^{\alpha} > n^{\alpha/r} \Big(\sum_{i=1}^{n} |a_{ni}|^{\alpha}\Big)^{-1}\Big) \\ &\leq \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P\big(|X| > n^{1/r-1/\alpha}\big) \\ &= \sum_{n=1}^{\infty} P\big(|X| > n^{1/\beta}\big) \ \leq\ E|X|^{\beta} < \infty. \end{aligned}$$

Then we prove $I_6 < \infty$. From (3.1) and (3.2), we get $\sum_{i=1}^{n} a_{ni}^{\beta} \leq n^{\max\{1,\,\beta/\alpha\}}$. Noting that $|Z_{ni}| \leq a_{ni}|X_i| I(a_{ni}|X_i| > n^{1/r})$. Then by $EX_i = 0$, Lemma 2.3, $\beta > r$ and $-\beta/r + \beta/\alpha = -1$, we get

$$\begin{aligned} &n^{-1/r} \max_{1\leq j\leq n} \big|\sum_{i=1}^{j} EY_{ni}\big| = n^{-1/r} \max_{1\leq j\leq n} \big|\sum_{i=1}^{j} EZ_{ni}\big| \\ &\leq n^{-1/r} \sum_{i=1}^{n} a_{ni} E|X_i| I(a_{ni}|X_i| > n^{1/r}) \\ &\leq C\, n^{-\beta/r} \sum_{i=1}^{n} a_{ni}^{\beta} E|X|^{\beta} I(a_{ni}|X| > n^{1/r}) \\ &\leq C\, n^{-\beta/r+\max\{1,\,\beta/\alpha\}} E|X|^{\beta} \to 0 \quad \text{as } \ n \to \infty. \end{aligned} \tag{3.3}$$

From (3.3), we know that while n is sufficiently large, $\max_{1\leq j\leq n}\big|\sum_{i=1}^{j} EY_{ni}\big| < n^{1/r}\varepsilon/2$. Then by the Markov inequality, Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
I_6 \leq & \sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\Big| > n^{1/r}\varepsilon/2\Big)\\
\leq & C\sum_{n=1}^{\infty} n^{-1-2/r} E\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\Big|^2\\
\leq & C\sum_{n=1}^{\infty} n^{-1-2/r}\log^2 n\sum_{i=1}^{n} E|Y_{ni}|^2\\
= & C\sum_{n=1}^{\infty} n^{-1-2/r}\log^2 n\sum_{i=1}^{n} a_{ni}^2 E|X_i|^2 I(a_{ni}|X_i|\leq n^{1/r})+\\
& C\sum_{n=1}^{\infty} n^{-1}\log^2 n\sum_{i=1}^{n} P(a_{ni}|X_i|> n^{1/r})\\
\leq & C\sum_{n=1}^{\infty} n^{-1-2/r}\log^2 n\sum_{i=1}^{n} a_{ni}^2 E|X|^2 I(a_{ni}|X|\leq n^{1/r})+\\
& C\sum_{n=1}^{\infty} n^{-1}\log^2 n\sum_{i=1}^{n} P(a_{ni}|X|> n^{1/r})\\
=: & I_7+I_8.
\end{aligned}$$

By $\sum_{i=1}^n a_{ni}^\beta \leq n^{\max\{1,\beta/\alpha\}}$, $\beta > r$ and $-\beta/r+\beta/\alpha=-1$, we have

$$\begin{aligned}
I_8 \leq & C\sum_{n=1}^{\infty} n^{-1-\beta/r}\log^2 n\sum_{i=1}^{n} a_{ni}^\beta E|X|^\beta I(a_{ni}|X|> n^{1/r})\\
\leq & C\sum_{n=1}^{\infty} n^{-1-\beta/r+\max\{1,\beta/\alpha\}}\log^2 n\, E|X|^\beta<\infty.
\end{aligned}$$

Finally we prove $I_7<\infty$. From $1/r=1/\alpha+1/\beta$ and $1<r<2$, we know that $\alpha\leq 2$ and $\beta\leq 2$ will not hold simultaneously. Hence we need only to consider the following three cases.

Case 1 $\alpha<2<\beta$. By (3.2) and $1/r=1/\alpha+1/\beta$, we have

$$I_7\leq C\sum_{n=1}^{\infty} n^{-1-2/r+2/\alpha}\log^2 n\, E|X|^2 = C\sum_{n=1}^{\infty} n^{-1-2/\beta}\log^2 n\, E|X|^2<\infty.$$

Case 2 $\beta<2<\alpha$. By (3.1) and $\beta>r$, we have

$$\begin{aligned}
I_7 \leq & C\sum_{n=1}^{\infty} n^{-1-\beta/r}\log^2 n\sum_{i=1}^{n} a_{ni}^\beta E|X|^\beta I(a_{ni}|X|\leq n^{1/r})\\
\leq & C\sum_{n=1}^{\infty} n^{-\beta/r}\log^2 n\, E|X|^\beta<\infty.
\end{aligned}$$

Case 3 $\beta\geq 2$, $\alpha\geq 2$ and $\alpha\beta\neq 4$. By (3.1) and $r<2$, we have

$$I_7\leq C\sum_{n=1}^{\infty} n^{-2/r}\log^2 n\, E|X|^2<\infty.$$

The proof is completed. $\square$

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References

[1] E. L. LEHMANN. *Some concepts of dependence.* Ann. Math. Statist., 1966, **37**(5): 1137–1153.

[2] K. JOAG-DEV, F. PROSCHAN. *Negative association of random variables.* Ann. Statist., 1983, **11**(1): 286–295.

[3] N. EBRAHIMI, M. GHOSH. *Multivariate negative dependence.* Commun. Stat., Theory Methods, 1981, **10**(4): 307–337.

[4] C. M. NEWMAN. *Asymptotic independence and limit theorems for positively and negatively dependent random variables.* Inst. Math. Statist., Hayward, CA, 1984.

[5] P. MATULA. *A note on the almost sure convergence of sums of negatively dependent random variables.* Statist. Probab. Lett., 1992, **15**(3): 209–213.

[6] Chenggao WAN. *Law of large numbers and complete convergence for pairwise NQD random sequences.* Acta Math. Appl. Sin., 2005, **28**(2): 253–261. (in Chinese)

[7] Rui LI, Weiguo YANG. *Strong convergence of pairwise NQD random sequences.* J. Math. Anal. Appl., 2008, **344**(2): 741–747.

[8] Qunying WU. *Convergence properties of pairwise NQD random sequences.* Acta Math. Sinica (Chin. Ser.), 2002, **45**(3): 617–624. (in Chinese)

[9] Qunying WU, Yuanying JIANG. *The strong law of large numbers for pairwise NQD random variables.* J. Syst. Sci. Complex., 2011, **24**(2): 347–357.

[10] Shuhe HU, Xiaotao LIU, Xinghui WANG, et al. *Strong law of large numbers of partial sums for pairwise NQD sequences.* J. Math. Res. Appl., 2013, **33**(1): 111–116.

[11] Shixin GAN, Pingyan CHEN. *Some limit theorems for sequences of pairwise NQD random variables.* Acta Math. Sci. Ser. B Engl. Ed., 2008, **28**(2): 269–281.

[12] Yongfeng WU, Mei GUAN. *Mean convergence theorems and weak laws of large numbers for weighted sums of dependent random variables.* J. Math. Anal. Appl., 2011, **377**(2): 613–623.

[13] M. LOÈVE. *Probability Theory (I).* Springer-Verlag, New York-Heidelberg, 1977.

[14] Dehua QIU, Xiangqun YANG. *Strong laws of large numbers for weighted sums of identically distributed NA random variables.* J. Math. Res. Exposition, 2006, **26**(4): 778–784. (in Chinese)

[15] Qunying WU. *Limit Theorems of Probability Theory for Mixing Sequences.* Science Press, Beijing, 2006. (in Chinese)

[16] Y. S. CHOW, T. L. LAI. *Limiting behavior of weighted sums of independent random variables.* Ann. Probability, 1973, **1**: 810–824.

[17] Xingcai ZHOU. *Complete moment convergence of moving average processes under φ-mixing assumptions.* Statist. Probab. Lett., 2010, **80**(5-6): 285–292.