Journal of Mathematical Research with Applications
Mar., 2014, Vol. 34, No. 2, pp. 240–248
DOI:10.3770/j.issn:2095-2651.2014.02.014
Http://jmre.dlut.edu.cn

Local Precise Large Deviations for Independent Sums in Multi-Risk Model

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Abstract In this paper, we study the case of independent sums in multi-risk model. Assume that there exist k types of variables. The ith are denoted by $\{X_{ij}, j \geq 1\}$, which are i.i.d. with common density function $f_i(x) \in \mathcal{OR}$ and finite mean, $i = 1, \ldots, k$. We investigate local large deviations for partial sums $\sum_{i=1}^{k} S_{n_i} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{ij}$.

Keywords multi-risk model; O-regularly varying function; local precise large deviations; regular density.

MR(2010) Subject Classification 62E20; 60F10; 60F05

1. Introduction

Mainstream research on precise large deviation probabilities has been concentrated on the study of the asymptotic relation $P(S_n - ES_n > x) \sim n\bar{F}(x)$, which holds uniformly for some x-region T_n as $n \to \infty$. Let $S_n = \sum_{i=1}^{n} X_i$, where X_i are a sequence of independent identically distributed (i.i.d.) random variables (rv's). X_i $(i \geq 1)$ have a common density function $f(x)$ of absolutely continuous distribution function (d.f.) $F(x) = 1 - \bar{F}(x)$ and a finite mean $\mu = EX_1$. See [1–5] for more details. Furthermore, Wang and Wang [6] extended the results to multi-risk model. Lu [7] studied lower and upper bounds of large deviation for sums of subexponential claims in a multi-risk model. In addition, Lu [8] extended the results to long-tailed class and studied lower bounds of large deviation for sums of long-tailed claims in a multi-risk model. Recently, more and more researchers concentrate on the local precise large deviations, which is about the large deviation probabilities $P(x < S_n - ES_n \leq x + T)$. Doney [9] investigated the probabilities of large deviations for i.i.d. integer-valued rv's. Yang et al. [10] studied the local precise large deviation for i.i.d. rv's supported on $(-\infty, \infty)$ with some regularly varying density $f(x)$, see Yang et al. [10] for more details on the local precise large deviations.

Let $A(n,x)$ and $B(n,x)$ be two positive functions $(n = 1, 2, \ldots; x \in \mathbb{R})$. We say $A(n,x) \lesssim B(n,x)$ holds uniformly for $x \in \triangle$ as $n \to \infty$, if $\limsup_{n\to\infty} \sup_{x\in\triangle} A(n,x)/B(n,x) \leqslant 1$. Furthermore, we denote $A(n,x) \sim B(n,x)$ uniformly for $x \in \triangle$ as $n \to \infty$, if $\limsup_{n\to\infty} \sup_{x\in\triangle} |A(n,x)/B(n,x) - 1| = 0$.

Received August 15, 2012; Accepted November 12, 2013
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A measurable function $f : \mathbb{R} \to [0, \infty)$ is O-regularly varying ($f \in \mathcal{OR}$), if $f(x)$ is positive for sufficiently large x and $\limsup_{x\to\infty} f(xy)/f(x) < \infty$ for every fixed $y > 0$, or equivalently, for every fixed $y \geq 1$, $0 < \liminf_{x\to\infty} f(xy)/f(x) \leq \limsup_{x\to\infty} f(xy)/f(x) < \infty$. A measurable function $f : \mathbb{R} \to [0, \infty)$ belongs to the class $\mathcal{L}$, if $f(x)$ is positive for sufficiently large x and for every fixed $y > 0$, $\lim_{x\to\infty} f(x+y)/f(x) = 1$. A measurable function $f : \mathbb{R} \to [0, \infty)$ belongs to the class $\mathcal{C}$, if $f(x)$ is positive for sufficiently large x, $\lim_{\varepsilon\downarrow 0} \liminf_{x\to\infty} \inf_{(1-\varepsilon)x\leq z\leq(1+\varepsilon)x} f(z)/f(x) = \lim_{\varepsilon\downarrow 0} \limsup_{x\to\infty} \sup_{(1-\varepsilon)x\leq z\leq(1+\varepsilon)x} f(z)/f(x) = 1$. We have the following inclusion relationship: $\mathcal{C} \subset \mathcal{L} \cap \mathcal{OR}$. A measurable function $f : \mathbb{R} \to [0, \infty)$ is almost decreasing, if $\limsup_{x\to\infty} \sup_{u\geq x} f(u)/f(x) < \infty$.

A distribution function F with support on $(-\infty, \infty)$ belongs to $\mathcal{D}$, if $\limsup_{x\to\infty} \bar{F}(xy)/\bar{F}(x) < \infty$, for any $y \in (0, 1)$ (or equivalently, for $y = 1/2$). A distribution function F with support on $(-\infty, \infty)$ belongs to $\mathcal{C}$, if $\lim_{y\downarrow 1} \liminf_{x\to\infty} \bar{F}(xy)/\bar{F}(x) = 1$ or equivalently, $\lim_{y\uparrow 1} \limsup_{x\to\infty} \bar{F}(xy)/\bar{F}(x) = 1$.

Set $\gamma(y) := \liminf_{x\to\infty} \bar{F}(xy)/\bar{F}(x)$ and $\gamma_F := -\lim_{y\to\infty}\{\log \gamma(y)/\log y\}$. In Tang [11], γ_F is called the upper Matuszewska index of a d.f. F.

These results motivate our study. In this paper, we investigate the local large deviations for $\sum_{i=1}^{k} S_{n_i} = \sum_{i=1}^{k}\sum_{j=1}^{n_i} X_{ij}$, where $\{X_{ij}, j \geq 1\}$ are i.i.d. rv's ($i = 1, \ldots, k$). The rest of the paper is organized as follows. In Section 2, we present some useful propositions. The main results are given in Section 3. Finally, the proof of the main results are presented in Section 4.

2. Preliminaries

In this section, we introduce some useful propositions which will be used in the proof of the main results in our paper.

Proposition 2.1 ([12, Proposition 2.2.1]) *Let f be positive. If $f \in \mathcal{OR}$, then for every $\beta < \beta(f)$, there exist positive constants $C_{1,\beta}$ and $C_{2,\beta}$, such that $f(u)/f(x) \geq C_{1,\beta}(u/x)^{\beta}$ for $u \geq x \geq C_{2,\beta}$, where $\beta(f) = \lim_{y\to\infty} \log(\liminf_{x\to\infty} f(xy)/f(x))/\log y$.*

Proposition 2.2 ([5, Lemma 2.1]) *If $F \in \mathcal{D}$ is a distribution function with finite expectation, $1 \leq \gamma_F < \infty$, then for any $\rho > \gamma_F$, there exist positive constants x_0 and B, such that for all $x \geq y \geq x_0$, $\bar{F}(y)/\bar{F}(x) \leq B(x/y)^{\rho}$.*

Proposition 2.3 ([10, Lemma 4.3]) *Let the function $f(x)$ be a density function of some absolutely continuous d.f. $F(x)$. If $f \in \mathcal{OR}$, then $F \in \mathcal{C}$.*

3. Main results

In this section, we give the main results using Propositions 2.1–2.3.

Theorem 3.1 *For $i = 1, \ldots, k$, let $\{X_{ij}, j \geq 1\}$ be i.i.d. rv's with common almost decreasing density function $f_i(x)$ and finite expectations μ_i. Assume that $E(X_i^+)^{r_i} < \infty$ for some $r_i > 1$, and $f_i(x)$, $f_j(x)$ ($i \neq j$, $1 \leq i$, $j \leq k$) satisfy that $\limsup_{x\to\infty} f_j(x)/f_i(x) < \infty$. Let γ and T be* *any fixed positive constants. If $f_i \in \mathcal{OR}$, then we have*

$$\sum_{i=1}^{k} L_{f_i}^{-} n_i T f_i(x+\mu_i) \lesssim P\Big(x < \sum_{i=1}^{k} S_{n_i} - \sum_{i=1}^{k} n_i\mu_i \le x+T\Big) \lesssim \sum_{i=1}^{k} L_{f_i}^{+} n_i T f_i(x+\mu_i)$$

holds uniformly for all $x \ge \max\{\gamma n_1, \ldots, \gamma n_k\} =: \Delta(k)$ as $n_i \to \infty$ $(i = 1, \ldots, k)$, where

$$L_{f_i}^{-} = \lim_{\varepsilon\downarrow 0}\liminf_{x\to\infty} \inf_{(1-\varepsilon)x\le z\le(1+\varepsilon)x} f_i(z)/f_i(x),\ \ L_{f_i}^{+} = \lim_{\varepsilon\downarrow 0}\limsup_{x\to\infty} \sup_{(1-\varepsilon)x\le z\le(1+\varepsilon)x} f_i(z)/f_i(x).$$

With respect to Theorem 3.1, we have the following corollaries.

Corollary 3.2 *If $f_i \in \mathcal{OR} \cap \mathcal{L}$ $(i = 1, 2, \ldots, k)$ and all conditions of Theorem 3.1 are satisfied, then*

$$\sum_{i=1}^{k} L_{f_i}^{-} n_i(F_i(x+T)-F_i(x)) \lesssim P\Big(x < \sum_{i=1}^{k} S_{n_i} - \sum_{i=1}^{k} n_i\mu_i \le x+T\Big) \lesssim \sum_{i=1}^{k} L_{f_i}^{+} n_i(F_i(x+T)-F_i(x))$$

holds uniformly for all $x \ge \max\{\gamma n_1, \ldots, \gamma n_k\} =: \Delta(k)$ as $n_i \to \infty$ $(i = 1, \ldots, k)$.

Proof Using the relation of $f_i \in \mathcal{OR} \cap \mathcal{L}$, we can easily get that $f_i(x+\mu_i) \sim f_i(x)$. For any fixed T,

$$\limsup_{x\to\infty}(Tf_i(x)/(F_i(x+T)-F_i(x))) \le \Big\{\liminf_{x\to\infty}(\inf_{x\le u\le x+T} f_i(u)/f_i(x))\Big\}^{-1} = 1,$$

$$\liminf_{x\to\infty}(Tf_i(x)/(F_i(x+T)-F_i(x))) \ge \Big\{\limsup_{x\to\infty}(\sup_{x\le u\le x+T} f_i(u)/f_i(x))\Big\}^{-1} = 1.$$

Combining Theorem 3.1 and the two inequalities, we get the desired result. □

Corollary 3.3 *If all conditions of Theorem 3.1 are satisfied, in addition, $f_i \in \mathcal{C}$ $(i = 1, 2, \ldots, k)$, then*

$$P\Big(x < \sum_{i=1}^{k} S_{n_i} - \sum_{i=1}^{k} n_i\mu_i \le x+T\Big) \sim \sum_{i=1}^{k} n_i(F_i(x+T)-F_i(x))$$

holds uniformly for all $x \ge \max\{\gamma n_1, \ldots, \gamma n_k\} =: \Delta(k)$ as $n_i \to \infty$ $(i = 1, \ldots, k)$.

Proof By $f_i \in \mathcal{C}$, we have $L_{f_i}^{-} = L_{f_i}^{+} = 1$. Corollary 3.3 follows immediately from Corollary 3.2. □

4. Proof of Theorem 3.1

Now, we will give a proof of Theorem 3.1 in detail.

Proof Assume that $\mu_i = 0$, $i = 1, \ldots, k$. Denote $v = v(x) = -\log(\sum_{i=1}^{k} f_i(x))$. Due to estimation (4.5) in Yang et al. [10] we have that $xf_1(x), xf_2(x), \ldots, xf_k(x)$ vanish as $x \to +\infty$, implying $\lim_{x\to+\infty} v(x) = +\infty$. Thus, from the definition of $\mathcal{OR}$ and the fact that for any fixed $y > 0$, $(\sum_{i=1}^{k} f_i(xy))/(\sum_{i=1}^{k} f_i(x)) \le \sum_{i=1}^{k}(f_i(xy)/f_i(x))$, we can obtain that $v(x)$ is slowly varying.

Denote $\tilde{X}_i = X_i I_{\{X_i \le x/v^4\}}$ $(i = 1, \ldots, k)$, $\tilde{X}_{ij} = X_{ij} I_{\{X_{ij} \le x/v^4\}}$ $(j = 1, 2, \ldots, n_i)$, and $\tilde{S}_{n_i} = \sum_{j=1}^{n_i} \tilde{X}_{ij}$ $(i = 1, \ldots, k)$. Let $\eta = \eta(n_1, n_2, \ldots, n_k, x)$ be the number of summands X_{ij} $(i =$

$1,\dots,k;\ 1 \le j \le n_i)$ in the sum $\sum_{i=1}^k S_{n_i} = \sum_{i=1}^k \sum_{j=1}^{n_i} X_{ij}$, such that $X_{ij} > x/v^4$, i.e., $\eta = \sum_{i=1}^k \sum_{j=1}^{n_i} I_{\{X_{ij} > x/v^4\}}$. We have $P(x < \sum_{i=1}^k S_{n_i} \le x+T) = W_0 + W_1 + W_2$, $W_i = P(x < \sum_{i=1}^k S_{n_i} \le x+T, \eta = i)$, $i = 0, 1$ and $W_2 = P(x < \sum_{i=1}^k S_{n_i} \le x+T, \eta \ge 2)$.

Estimation of W_0. For any $h > 0$, we obtain that

$$W_0 \le P\biggl(\sum_{i=1}^k \tilde{S}_{n_i} > x\biggr) \le e^{-hx} \prod_{i=1}^k (Ee^{h\tilde{X}_i})^{n_i}. \tag{1}$$

Using the inequality $e^u - 1 \le ue^u$ $(u \in \mathbb{R})$, for $i = 1,\dots,k$, we have that

$$\begin{aligned} Ee^{h\tilde{X}_i} &= \bar{F}_i\left(\frac{x}{v^4}\right) + \int_{-\infty}^{\frac{x}{v^4}} e^{hu} f_i(u)\mathrm{d}u \\ &= 1 + \int_{-\infty}^{\frac{x}{v^4}} (e^{hu} - 1) f_i(u)\mathrm{d}u \\ &\le 1 + \int_{-\infty}^{0} (e^{hu} - 1) f_i(u)\mathrm{d}u + \int_0^{\frac{x}{v^4}} hue^{hu} f_i(u)\mathrm{d}u. \end{aligned}$$

For positive h and real u, $\frac{|e^{hu}-1-hu|}{h} \le |u|$ and, since μ_i is finite, $\mu_+^i = EX_i I_{\{X_i \ge 0\}}$, $\mu_-^i = EX_i I_{\{X_i < 0\}}$ $(i = 1,\dots,k)$ are finite. Therefore, using the dominated convergence theorem, we get

$$\lim_{h\downarrow 0} \frac{1}{h} \int_{-\infty}^0 (e^{hu} - 1) f_i(u)\mathrm{d}u = \lim_{h\downarrow 0} \int_{-\infty}^0 \frac{e^{hu} - 1 - hu}{h} f_i(u)\mathrm{d}u + \mu_-^i = \mu_-^i, \quad i = 1,\dots,k.$$

Hence,

$$\int_{-\infty}^0 (e^{hu} - 1) f_i(u)\mathrm{d}u = (1 + \tau_i(h)) h\mu_-^i,$$

where $\tau_i(h) \to 0$ as $h \to 0$, $i = 1,\dots,k$.

By the fact

$$\int_0^{\frac{x}{v^4}} hue^{hu} f_i(u)\mathrm{d}u \le he^{\frac{hx}{v^4}} \mu_+^i, \quad i = 1,\dots,k$$

and using $1 + x \le e^x$, we obtain that for $h > 0$,

$$\begin{aligned} \frac{W_0}{\sum\limits_{i=1}^k f_i(x)} &\le e^{-hx+v} \prod_{i=1}^k (1 + (1 + \tau_i(h)) h\mu_-^i + he^{\frac{hx}{v^4}} \mu_+^i)^{n_i} \\ &\le \exp\Bigl\{ -hx + v + \sum_{i=1}^k n_i h\bigl[(1 + \tau_i(h))\mu_-^i + e^{\frac{hx}{v^4}} \mu_+^i\bigr]\Bigr\}. \end{aligned}$$

The function $v(x)$ is slowly varying, so $v(x)/x$ vanishes as $x \to 0$. Setting $h = h(x) = 2v(x)/x$, we get

$$\frac{W_0}{\sum\limits_{i=1}^k f_i(x)} \le \exp\Bigl\{ -v(x) + \frac{2v(x)}{\gamma} \sum_{i=1}^k \bigl[(1 + \tau_i(h))\mu_-^i + e^{\frac{2}{v^3}} \mu_+^i\bigr]\Bigr\}$$

holds uniformly for $x \ge \Delta(k)$.

For $i = 1, \ldots, k$, we have

$$\lim_{x\to\infty}(1+\tau_i(h))\mu_-^i + e^{\frac{2}{v^3}}\mu_+^i = \mu_-^i + \mu_+^i = 0,$$

and we get

$$\lim_{x\to\infty}\frac{W_0}{\sum\limits_{i=1}^{k} f_i(x)} = 0. \tag{2}$$

By the fact that $\frac{W_0}{\sum_{i=1}^{k} L_{f_i}^+ n_i T f_i(x)} \le \frac{1}{\min\{n_1,n_2,\ldots,n_k\}T}\cdot\frac{W_0}{\sum_{i=1}^{k} f_i(x)}\cdot(\sum_{i=1}^{k}\frac{1}{L_{f_i}^+})$, we have

$$\lim_{n_1,n_2,\ldots,n_k\to\infty}\sup_{x\ge\Delta(k)}\frac{W_0}{\sum\limits_{i=1}^{k} L_{f_i}^+ n_i T f_i(x)} = 0. \tag{3}$$

Next, we estimate W_2. We show that $W_2 \le \sum_{m=1}^{k} W_{2mm} + \sum_{1\le m<l\le k} W_{2ml}$, where

$$W_{2mm} = P\Big(x < \sum_{i=1}^{k} S_{n_i} \le x+T, \max_{1\le j\le n_m} X_{mj} > \frac{x}{v^4}, \max_{l\ne m, 1\le l\le k}\max_{1\le j\le n_l} X_{lj} \le \frac{x}{v^4}, \eta\ge 2\Big),$$

$$W_{2ml} = P\Big(x < \sum_{i=1}^{k} S_{n_i} \le x+T, \max_{1\le j\le n_m} X_{mj} > \frac{x}{v^4}, \max_{1\le j\le n_l} X_{lj} > \frac{x}{v^4}\Big).$$

Since $\{X_{mj}, j\ge 1\}$ are i.i.d., applying the similar arguments in (5.8) in Yang et al. [10] gives

$$W_{2mm} \le \sum_{1\le s<t\le n_m} P(x < \sum_{i=1}^{k} S_{n_i} \le x+T, X_{ms} > \frac{x}{v^4}, X_{mt} > \frac{x}{v^4}) \le n_m^2 T\bar{F}_m(\frac{x}{v^4})\sup_{u\ge x/v^4} f_m(u).$$

Using Proposition 2.3, from $f_i \in \mathcal{OR}$, we get $F_i \in \mathcal{C} \subset \mathcal{D}$ or equivalently, $\gamma_{F_i} < \infty$. Denote $q := \max\{-\beta_i(f_i), \gamma_{F_i}, i = 1,\ldots,k\} + 1 < \infty$, by Proposition 2.2, for large x, we have that $\bar{F}_m(x/v^4) = O(v^{4q}\bar{F}_m(x))$, $m = 1,\ldots,k$. Since f_m is almost decreasing and $f_m \in \mathcal{OR}$, we obtain that $\sup_{u\ge x/v^4} f_m(u) = O(f_m(x/v^4)) = O(v^{4q} f_m(x))$, $m = 1,\ldots,k$.

By the arguments above, we obtain that for some positive constant C and large x,

$$\frac{W_{2mm}}{\sum_{i=1}^{k} n_i T f_i(x)} \le W_{2mm}/n_m T f_m(x) \le C n_m v^{8q}\bar{F}_m(x).$$

Thus, we have that

$$\sup_{x\ge\Delta(k)}\frac{W_{2mm}}{\sum\limits_{i=1}^{k} n_i T f_i(x)} \le \frac{C}{\gamma}\sup_{x\ge\Delta(k)}\frac{v^{8q}}{x^{r_m-1}}\sup_{x\ge\Delta(k)} x^{r_m}\bar{F}_m(x). \tag{4}$$

As $v = v(x)$ is slowly varying and $r_m > 1$, we have $v^{8q}/x^{r_m-1} \to 0$. On the other hand, $E(X_m^+)^{r_m} < \infty$ implies $\lim_{x\to\infty} x^{r_m}\bar{F}_m(x) = 0$. Hence, both supremums in (4) tend to zero as $n_1, n_2, \ldots, n_k \to \infty$. Then we have $\lim_{n_1,n_2,\ldots,n_k\to\infty}\sup_{x\ge\Delta(k)}\frac{W_{2mm}}{\sum_{i=1}^{k} n_i T f_i(x)} = 0$. Similarly, we get that $\lim_{n_1,n_2,\ldots,n_k\to\infty}\sup_{x\ge\Delta(k)}\frac{W_{2ml}}{\sum_{i=1}^{k} n_i T f_i(x)} = 0$. Thus,

$$\lim_{n_1,n_2,\ldots,n_k\to\infty}\sup_{x\ge\Delta(k)}\frac{W_2}{\sum_{i=1}^{k} n_i T f_i(x)} = 0.$$

From the fact that $\frac{W_2}{\sum_{i=1}^k L_{f_i}^+ n_i T f_i(x)} \leq \frac{1}{\min\{L_{f_1}^+, L_{f_2}^+, \ldots, L_{f_k}^+\}} \cdot \frac{W_2}{\sum_{i=1}^k n_i T f_i(x)}$, we obtain that

$$\lim_{n_1,n_2,\ldots,n_k\to\infty} \sup_{x\geq\Delta(k)} \frac{W_2}{\sum\limits_{i=1}^k L_{f_i}^+ n_i T f_i(x)} = 0. \tag{5}$$

Next, we consider the estimation of W_1.

By the Strong Law of Large Number for i.i.d. rv's, there exist k sequences of positive numbers a_{mn_m}, $1 \leq m \leq k$ such that $a_{mn_m} \uparrow \infty$, $a_{mn_m}/n_m \to 0$, $P(|S_{n_m}| > a_{mn_m}) \to 0$ as $n_m \to \infty$.

Since $\{X_{mj}, j \geq 1\}$ are i.i.d. rv's, for any fixed $\varepsilon \in (0,1)$ we obtain that

$$\begin{aligned} W_1 &= \sum_{m=1}^k n_m P\Big(x < \sum_{i=1}^k S_{n_i} \leq x+T, X_{mn_m} > \frac{x}{v^4}, \max_{1\leq j\leq n_m-1} X_{mj} \leq \frac{x}{v^4}, \\ &\max_{1\leq l\leq k, l\neq m} \max_{1\leq j\leq n_l} X_{lj} \leq \frac{x}{v^4}\Big) \\ &= \sum_{m=1}^k (W_{1m1} + W_{1m2} + W_{1m3}), \end{aligned}$$

where $W_{1m1} = n_m \int_{-\infty}^{-a_{mn_m-1}-\sum_{1\leq t\leq k, t\neq m} a_{tn_t}} M_P \mathrm{d}Q$, $W_{1m2} = n_m \int_{-a_{mn_m-1}-\sum_{1\leq t\leq k,t\neq m} a_{tn_t}}^{\varepsilon x} M_P \mathrm{d}Q$, $W_{1m3} = n_m \int_{\varepsilon x}^{\infty} M_P \mathrm{d}Q$.

Here

$$Q := P(S_{n_m-1} + \sum_{1\leq t\leq k, t\neq m} S_{n_t} \leq u, \max_{1\leq j\leq n_m-1} X_{mj} \leq \frac{x}{v^4}, \max_{1\leq l\leq k, l\neq m} \max_{1\leq j\leq n_l} X_{lj} \leq \frac{x}{v^4})$$

$$M_P := P(x-u < X_{mn_m} \leq x-u+T, X_{mn_m} > \frac{x}{v^4}).$$

We start to consider W_{1m1}. Obviously, as $u \leq -a_{mn_m-1} - \sum\limits_{1\leq t\leq k, t\neq m} a_{tn_t} < 0$, for sufficiently large x, we have $x - u > x > \frac{x}{v^4}$. Thus, for large enough x, we get

$$W_{1m1} \leq n_m \sup_{z>x} P(z < X_{mn_m} \leq z+T) Q_1 \leq n_m T \sup_{z>x} f_m(u) Q_1 \leq n_m T C f_m(x) Q_1.$$

here $Q_1 := P(S_{n_m-1} + \sum_{1\leq t\leq k, t\neq m} S_{n_t} \leq -a_{mn_m-1} - \sum_{1\leq t\leq k, t\neq m} a_{tn_t})$.

The last inequality and C follow from the definition of almost decreasing function. By construction of the sequences a_{mn_m}, we can obtain that $W_1 \to 0$ as $n_i \to \infty$, $i = 1, 2, \ldots, k$. So, we have

$$\lim_{n_1,n_2,\ldots,n_k\to\infty} \sup_{x\geq\Delta(k)} \frac{\sum\limits_{m=1}^k W_{1m1}}{\sum\limits_{m=1}^k L_{f_m}^+ n_m T f_m(x)} = 0. \tag{6}$$

Now, we consider W_{1m3}. $W_{1m3} \leq n_m Q = n_m P(\tilde{S}_{n_m-1} + \sum_{1\leq t\leq k, t\neq m} \tilde{S}_{n_t} > \varepsilon x)$. Using (1) and (2), we have

$$\lim_{n_1,n_2,\ldots,n_k\to\infty} \sup_{x\geq\Delta(k)} \frac{P(\tilde{S}_{n_m-1} + \sum\limits_{1\leq t\leq k, t\neq m} \tilde{S}_{n_t} > \varepsilon x)}{\sum\limits_{m=1}^k f_m(\varepsilon x)} = 0. \tag{7}$$

By the fact that

$$\sup_{x\ge\Delta(k)}\frac{W_{1m3}}{\sum\limits_{m=1}^{k}n_mTf_m(x)}$$
$$\le\frac{1}{T}\sup_{x\ge\Delta(k)}\frac{P(\tilde{S}_{n_m-1}+\sum\limits_{1\le t\le k,t\ne m}\tilde{S}_{n_t}>\varepsilon x)}{\sum\limits_{m=1}^{k}f_m(\varepsilon x)}\sup_{x\ge\Delta(k)}\Big(\frac{f_m(\varepsilon x)}{f_m(x)}+\sum_{1\le s\le k,s\ne m}\frac{f_s(\varepsilon x)}{f_s(x)}\frac{f_s(x)}{f_m(x)}\Big),$$

According to Proposition 2.1, the fact $\limsup_{x\to\infty}f_j(x)/f_i(x)<\infty,\ 1\le i,j\le k$ and (7), we get that

$$\lim_{n_1,n_2,\ldots,n_k\to\infty}\sup_{x\ge\Delta(k)}\frac{\sum\limits_{m=1}^{k}W_{1m3}}{\sum\limits_{m=1}^{k}L^{+}_{f_m}n_mTf_m(x)}=0. \tag{8}$$

Next, we show

$$\liminf_{n_1,\ldots,n_k\to\infty}\inf_{x\ge\Delta(k)}\frac{\sum\limits_{m=1}^{k}W_{1m2}}{\sum\limits_{m=1}^{k}L^{-}_{f_m}n_mTf_m(x)}\ge 1, \tag{9}$$

$$\limsup_{n_1,\ldots,n_k\to\infty}\sup_{x\ge\Delta(k)}\frac{\sum\limits_{m=1}^{k}W_{1m2}}{\sum\limits_{m=1}^{k}L^{+}_{f_m}n_mTf_m(x)}\le 1. \tag{10}$$

Now we deal with the lower bound. Denote

$$A:=\Big\{-a_{mn_m-1}-\sum_{1\le t\le k,t\ne m}a_{tn_t}<S_{n_m-1}+\sum_{1\le t\le k,t\ne m}S_{n_t}\le\varepsilon x,$$
$$\max_{1\le j\le n_m-1}X_{mj}\le\frac{x}{v^4},\max_{1\le l\le k,l\ne m}\max_{1\le j\le n_l}X_{lj}\le\frac{x}{v^4}\Big\}.$$

Since $a_{mn_m}/n_m\to 0$ as $n_m\to\infty$, for any fixed $0<\varepsilon<1$, and sufficiently large $n_1,\ldots,n_k$, we get $a_{mn_m-1}\le\varepsilon n_m<\frac{\varepsilon x}{\gamma}$, $a_{mn_m}\le\varepsilon n_m<\frac{\varepsilon x}{\gamma}$. Thus, for large $n_1,\ldots,n_k$ and $x\ge\Delta(k)$, we have that in $B=((1-\varepsilon)x,x+a_{mn_m-1}+\sum_{1\le t\le k,t\ne m}a_{tn_t})$, W_{1m2} satisfies

$$W_{1m2}\ge n_mP(A)\inf_{u\in B}P(u<X_{mn_m}\le u+T), \tag{11}$$

where

$$\inf_{u\in B}P(u<X_{mn_m}\le u+T)\ge\inf_{(1-\varepsilon)x<u\le(1+\frac{k\varepsilon}{\gamma})x}\int_u^{u+T}f_m(z)\mathrm{d}z\ge T\inf_{(1-\bar{\varepsilon})x\le u\le(1+\bar{\varepsilon})x}f_m(u) \tag{12}$$

with $\bar{\varepsilon}=\max\{2\varepsilon,\frac{2k\varepsilon}{\gamma}\}$.

Besides that

$$P(A)\ge P\Big(S_{n_m-1}+\sum_{1\le t\le k,t\ne m}S_{n_t}>-a_{mn_m-1}-\sum_{1\le t\le k,t\ne m}a_{tn_t}\Big)-1+$$
$$P\Big(\max_{1\le j\le n_m-1}X_{mj}\le\frac{x}{v^4},\max_{1\le l\le k,l\ne m}\max_{1\le j\le n_l}X_{lj}\le\frac{x}{v^4}\Big)-$$

$$P\Big(\tilde{S}_{n_m-1}+\sum_{1\le t\le k,t\ne m}\tilde{S}_{n_t}>\varepsilon x\Big).$$

Here, by construction of the sequences a_{mn_m}, we obtain that

$$\lim_{n_1,n_2,\ldots,n_k\to\infty}P\Big(S_{n_m-1}+\sum_{1\le t\le k,t\ne m}S_{n_t}>-a_{mn_m-1}-\sum_{1\le t\le k,t\ne m}a_{tn_t}\Big)=1.$$

For $x\ge\Delta(k)$ and sufficiently large $n_1,n_2,\ldots,n_k$, we have

$$P\Big(\max_{1\le j\le n_m-1}X_{mj}\le\frac{x}{v^4},\max_{1\le l\le k,l\ne m}\max_{1\le j\le n_l}X_{lj}\le\frac{x}{v^4}\Big)\ge\prod_{m=1}^{k}\Big(1-\bar{F}_m(\frac{x}{v^4})\Big)^{n_m}$$
$$\ge\prod_{m=1}^{k}\Big(1-\frac{x}{\gamma}\bar{F}_m(\frac{x}{v^4})\Big)\ge\prod_{m=1}^{k}\Big(1-\frac{C_m}{\gamma}\frac{v^{4q}}{x^{r_m-1}}\Big).$$

Hence, we obtain that

$$\lim_{n_1,n_2,\ldots,n_k\to\infty}P\Big(\max_{1\le j\le n_m-1}X_{mj}\le\frac{x}{v^4},\max_{1\le l\le k,l\ne m}\max_{1\le j\le n_l}X_{lj}\le\frac{x}{v^4}\Big)=1.$$

Obviously, using (7), we get that

$$\lim_{n_1,n_2,\ldots,n_k\to\infty}\sup_{x\ge\Delta(k)}P(\tilde{S}_{n_m-1}+\sum_{1\le t\le k,t\ne m}\tilde{S}_{n_t}>\varepsilon x)=0.$$

Then,

$$\liminf_{n_1,n_2,\ldots,n_k\to\infty}\inf_{x\ge\Delta(k)}P(A)=1. \tag{13}$$

Combining (11), (12) and (13), we get that for every $\varepsilon\in(0,1)$,

$$\liminf_{n_1,\ldots,n_k\to\infty}\inf_{x\ge\Delta(k)}\frac{\sum_{m=1}^{k}W_{1m2}}{\sum_{m=1}^{k}L^-_{f_m}n_mTf_m(x)}\ge\liminf_{x\to\infty}\frac{\sum_{m=1}^{k}n_mT\inf_{(1-\bar{\varepsilon})x\le u\le(1+\bar{\varepsilon})x}f_m(u)}{\sum_{m=1}^{k}L^-_{f_m}n_mTf_m(x)}.$$

According to the definition of $L^-_{f_m}$, we obtain (9).

Next, for the upper bound, we have that

$$W_{1m2}\le n_m\sup_{u\in B}P(u<X_{mn_m}\le u+T)\le n_mT\sup_{(1-\bar{\varepsilon})x\le z\le(1+\bar{\varepsilon})x}f_m(z).$$

From the argument above and the fact that f is almost decreasing, we have (10).

Combining (6), (8), (9) and (10), we get that

$$\liminf_{n_1,\ldots,n_k\to\infty}\inf_{x\ge\Delta(k)}\frac{W_1}{\sum_{m=1}^{k}L^-_{f_m}n_mTf_m(x)}\ge1,\quad\limsup_{n_1,\ldots,n_k\to\infty}\sup_{x\ge\Delta(k)}\frac{W_1}{\sum_{m=1}^{k}L^+_{f_m}n_mTf_m(x)}\le1. \tag{14}$$

Combining (3), (5) and (14) completes the proof in the case of $\mu_i=0,\ i=1,\ldots,k$.

Next, we deal with the case that $\mu_i\ne0,\ i=1,\ldots,k$. For any $\varepsilon\in(0,1)$, we have

$$\limsup_{x\to\infty}\frac{\sup_{(1-\varepsilon)x\le z\le(1+\varepsilon)x}f_i(z+\mu_i)}{f_i(x+\mu_i)}\le\limsup_{y\to\infty}\frac{\sup_{(1-2\varepsilon)y\le z'\le(1+2\varepsilon)y}f_i(z')}{f_i(y)},$$

$$\liminf_{x\to\infty}\frac{\inf\limits_{(1-\varepsilon)x\le z\le(1+\varepsilon)x} f_i(z+\mu_i)}{f_i(x+\mu_i)}\ge\liminf_{y\to\infty}\frac{\inf\limits_{(1-2\varepsilon)y\le z'\le(1+2\varepsilon)y} f_i(z')}{f_i(y)}.$$

Obviously, using the relations above, we obtain the desired result. □

Acknowledgements We thank the referees for their valuable comments which help the authors improve the exposition of this paper significantly.

References

[1] A. V. NAGAEV. *Integral limit theorems with regard to large deviations when Cramér's condition is not satisfied (I).* Teor. Verojatnost. i Primenen, 1969, **14**: 51–63. (in Russian)

[2] A. V. NAGAEV. *Integral limit theorems with regard to large deviations when Cramér's condition is not satisfied (II).* Teor. Verojatnost. i Primenen, 1969, **14**: 203–216. (in Russian)

[3] D. B. H. CLINE, G. SAMORODNITSKY. *Subexponentiality of the product of independent random variables.* Stochastic Process. Appl., 1994, **49**(1): 75–98.

[4] K. W. NG, Qihe TANG, Jiaan YAN, et al. *Precise large deviations for sums of random variables with consistently varying tails.* J. Appl. Probab., 2004, **41**(1): 93–107.

[5] Qihe TANG. *Insensitivity to negative dependence of the asymptotic behavior of precise large deviations.* Electron. J. Probab., 2006, **11**(4): 107–120.

[6] Shijie WANG, Wensheng WANG. *Precise large deviations for sums of random variables with consistently varying tails in multi-risk models.* J. Appl. Probab., 2007, **44**(4): 889–900.

[7] Dawei LU. *Lower and upper bounds of large deviation for sums of subexponential claims in a multi-risk model.* Statist. Probab. Lett., 2011, **81**(12): 1911–1919.

[8] Dawei LU. *Lower bounds of large deviation for sums of long-tailed claims in a multi-risk model.* Statist. Probab. Lett., 2012, **82**(7): 1242–1250.

[9] R. A. DONEY. *A large deviation local limit theorem.* Math. Proc. Cambridge Philos. Soc., 1989, **105**(3): 575–577.

[10] Yang YANG, R. LEIPUS, J. ŠIAULYS. *Local precise large deviations for sums of random variables with O-regularly varying densities.* Statist. Probab. Lett., 2010, **80**(19-20): 1559–1567.

[11] Qihe TANG. *Asymptotics for the finite time ruin probability in the renewal model with consistent variation.* Stoch. Models, 2004, **20**(3): 281–297.

[12] N. H. BINGHAM, C. M. GOLDIE, J. L. TEUGELS. *Regular Variation.* Cambridge University Press, Cambridge, 1987.