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On a Generalization of Semicommutative Rings

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Abstract In this paper, a generalization of the class of semicommutative rings is investigated. A ring R is called left GWZI if for any $a \in R$, $l(a)$ is a GW-ideal of R. We prove that a ring R is left GWZI if and only if $S_3(R)$ is left GWZI if and only if $V_n(R)$ is left GWZI for any $n \geq 2$.

Keywords semicommutative rings; GWZI rings; trivial extension.

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1. Introduction

All rings considered in this paper are associated with identity, and all modules are unital. The symbols $N(R)$, $E(R)$ stand respectively for the set of all nilpotent elements, the set of all idempotent elements of R. For any nonempty subset X of R, $r(X) = r_R(X)$ and $l(X) = l_R(X)$ denote the set of right annihilators of X and the set of left annihilators of X in R, respectively. Especially, if $X = a$, we write $r(X) = r(a)$ and $l(X) = l(a)$.

According to Cohn [1], a ring R is called reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$, and R is said to be semicommutative [2] or ZI [3] if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Note that R is a semicommutative ring if and only if $l(a)$ is an ideal of R for any $a \in R$ by [4, Lemma 1.2]. Following Zhou [5], a left ideal L of R is called a generalized weak ideal (simply, GW-ideal) if for any $a \in L$, there exists a positive integer $n \geq 1$ such that $a^nR \subseteq L$. Clearly, ideals are GW-ideals, but the converse is not true, in general, by Zhou [5, Example 1.2]. A ring R is called reduced if it has no nonzero nilpotent elements. In this paper, we will define a left GWZI ring which is a generalization of semicommutative rings. For several years, the applications of semicommutative rings have been studied by many authors. In [6], Kim and Lee showed that if R is a reduced ring, then

$$S_3(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid \ a, b, c, d \in R \right\}$$

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is a semicommutative ring. But

$$S_n(R)=\left\{\begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n}\\ 0 & a & a_{23} & \cdots & a_{2n}\\ 0 & 0 & a & \cdots & a_{3n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{12}, \ldots, a_{n-1,n}\in R\right\}$$

may not be semicommutative for $n \geqslant 4$. In [7], the authors introduced a class of generalized semicommutative rings. A rings R is said to be weakly semicommutative ring, if for any $a, b\in R$, $ab = 0$ implies arb is a nilpotent element for any $r \in R$. They show that R is weakly semicommutative if and only if for any n, the $n\times n$ upper triangular matrix ring

$$T_n(R)=\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n}\\ 0 & a_{22} & a_{23} & \cdots & a_{2n}\\ 0 & 0 & a_{33} & \cdots & a_{3n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} \mid a_{11}, a_{12}, a_{13}, \ldots, a_{nn}\in R\right\}$$

is a weakly semicommutative ring by [7, Example 2.1]. In this paper, we show that a ring R is left GWZI if and only if $S_3(R)$ is left GWZI if and only if $V_n(R)$ is left GWZI for any $n \geq 2$.

2. Main results

We denote $N_2(R) = \{x \in R | x^2 = 0\}$, and suppose that $N(R) \neq 0$ for a given ring R. Now, we have the following characterization of weakly semicommutative rings.

Proposition 2.1 *The following conditions are equivalent for a ring R:*

(1) For any $0 \neq a \in N(R)$, there exists a positive integer n such that $a^n \neq 0$ and $a^nR \subseteq N(R)$;

(2) $r_1ar_2 \in N(R)$ for any $0 \neq a \in N_2(R)$ and $r_1, r_2 \in R$;

(3) $ar \in N(R)$ for any $0 \neq a \in N_2(R)$ and $r \in R$;

(4) R is weakly semicommutative.

Proof (1)⇒(2). Suppose $a^2 = 0$ and $a \neq 0$. By the hypothesis, $ar_1r_2 \in N(R)$ for any $r_1, r_2 \in R$. Then there exists $k \geq 2$ such that $(r_2ar_1)^k = r_2(ar_1r_2)^{k-1}ar_1 = 0$, that is $r_2ar_1 \in N(R)$.

(2)⇒(3). It is clear.

(3)⇒(1). For any $0 \neq a \in N(R)$, there exists a positive integer n such that $a^n \neq 0$ and $a^{n+1} = 0$. Clearly, $a^n \in N_2(R)$ and $a^nR \subseteq N(R)$ by the hypothesis.

(3)⇒(4). Suppose that $ab = 0$ for any $a, b \in R$. If $ba = 0$, then it is easy to get $(arb)^2 = 0$. Now, let $ba \neq 0$. Then we have $0 \neq ba \in N_2(R)$ and $bar \in N(R)$ for any $r \in R$ by the hypothesis. That is, $arb \in N(R)$ for any $r \in R$, which implies that R is weakly semicommutative.

(4)⇒(3). Suppose that $a^2 = 0$ and $a \neq 0$. Then $a^2r = 0$ for any $r \in R$, that is $a(ar) = 0$. Therefore $aR(ar) \subseteq N(R)$ by the hypothesis. Then $(ar)^2 \in N(R)$ and $ar \in N(R)$. □

By Proposition 2.1, it is easy to get the following corollary.

Corollary 2.2 ([7, Claim 2.1]) *R is a weakly semicommutative ring if and only if $T_n(R)$ is a weakly semicommutative ring.*

Proof It is enough to show the sufficiency since the class of weakly semicommutative rings is closed under the subrings. Assume that R is weakly semicommutative. For any $0 \neq A \in N_2(T_n(R))$, we have $a_{ii} \in N_2(R)$ for all i. Since R is weakly semicommutative, $a_{ii}r \in N(R)$ for any $r \in R$ and all i by Proposition 2.1. So, we have $AB \in N(T_n(R))$ for any $B \in T_n(R)$, and so $T_n(R)$ is weakly semicommutative by Proposition 2.1. □

Note that R is a ZI ring if and only if $l(a)$ is an ideal of R for any $a \in R$ by [4, Lemma 1.2]. This will give a new class of generalized semicommutative rings using the GW-ideal.

Definition 2.3 *A ring R is called a left GWZI ring if for any $a \in R$, $l(a)$ is a GW-ideal of R.*

We do not know whether the property of GWZI is left-right symmetric. Obviously, semicommutative rings are left GWZI, but the converse is not true, in general.

Set $R = \left\{ \begin{pmatrix} a & a_1 & a_2 & a_3 \\ 0 & a & a_4 & a_5 \\ 0 & 0 & a & a_6 \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, a_1, a_2, \ldots, a_6 \in \mathbb{Z}_2 \right\}$, where Z_2 is the ring of integers modulo 2. Then every non-unit of R is nilpotent and hence it follows that every left ideal of R is a GW-ideal. But R is not semicommutative by [6].

Theorem 2.4 *Let R be a left GWZI ring. Then $ar \in N(R)$ and $ra \in N(R)$ for any $a \in N(R)$ and $r \in R$.*

Proof Let R be a left GWZI ring. For any $0 \neq a \in N(R)$, there exists a positive integer n such that $a^n = 0$, then $raa^{n-1} = 0$ for any $r \in R$. Since R is left GWZI, there exists a positive integer m such that $(ra)^m Ra^{n-1} = 0$, especially, $(ra)^{m+1}a^{n-2} = 0$, then there exists a positive integer t such that $(ra)^{(m+1)t}Ra^{n-2} = 0$, especially, $(ra)^{(m+1)t+1}a^{n-3} = 0$. Using the induction, we can attain $(ra)^k = 0$ for some positive integer k, then $ra \in N(R)$. It is obvious that $ar \in N(R)$ if and only if $ra \in N(R)$ for any $r \in R$. □

As a corollary, we have the following result, combining Theorem 2.4 and Proposition 2.1.

Corollary 2.5 *Left GWZI rings are weakly semicommutative.*

Remark 2.6 By Corollary 2.2, when R is left GWZI, $T_n(R)$ is weakly semicommutative. However, in the following we will have $T_n(R)$ is not GWZI by Remark 2.19, in general. So the converse of Corollary 2.5 is not true.

Motivated by these, in the following we will consider if R is a GWZI ring, whether the $n \times n$ upper triangular matrix ring $V_n(R)$ is also GWZI.

For a ring R and $n \geq 2$, we denote

$$V_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \mid a_1, a_2, a_3, \ldots, a_n \in R \right\}.$$

Given a ring R and an (R, R)-bimodule M, the trivial extension of R by M is the ring $T(R, M) = R \bigoplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used. Note that $T(R, R) = V_2(R)$.

Lemma 2.7 *If R is a left GWZI ring, then the trivial extension $T(R, R)$ is also left GWZI.*

Proof Let $0 \neq A = \begin{pmatrix} u & v \\ 0 & u \end{pmatrix} \in T(R, R)$ and $0 \neq B = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in l_{T(R,R)}(A)$. Then $BA = \begin{pmatrix} au & av + bu \\ 0 & au \end{pmatrix} = 0$ and so $au = 0$ and $av + bu = 0$.

Case 1 Set $u = 0$. Then $v \neq 0$ and $av = 0$.

If $a = 0$, it is easy to get $BCA = 0$ for any $C = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \in T(R, R)$. Thus we get $BT(R, R) \subseteq l_{T(R,R)}(A)$.

If $a \neq 0$, then by the hypothesis there exists a positive integer n such that $a^n Rv = 0$ from $av = 0$. So

$$B^n CA = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^n \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a^n & * \\ 0 & a^n \end{pmatrix} \begin{pmatrix} 0 & xv \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a^n xv \\ 0 & 0 \end{pmatrix} = 0$$

for any $C = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \in T(R, R)$. That shows $B^n T(R, R) \subseteq l_{T(R,R)}(A)$.

Case 2 Set $u \neq 0$.

If $a = 0$, then $B^2 CA = 0$ for any $C \in T(R, R)$. Hence $B^2 T(R, R) \subseteq l_{T(R,R)}(A)$.

If $a \neq 0$, then by the hypothesis, there exists a positive integer m such that $a^m Ru = 0$ from $au = 0$. Then

$$B^{m+1} CA = \begin{pmatrix} a^{m+1} & a^m b + a^{m-1} ba + \cdots + ba^m \\ 0 & a^{m+1} \end{pmatrix} \begin{pmatrix} xu & xv + yu \\ 0 & xu \end{pmatrix}$$

$$= \begin{pmatrix} 0 & a^{m+1} xv + a^{m-1} baxu + \cdots + aba^{m-1} xu \\ 0 & 0 \end{pmatrix}.$$

It follows that $B^{m-1}B^{m+1}CA = \begin{pmatrix} 0 & (a^m)^2xv \\ 0 & 0 \end{pmatrix}$. Since $av + bu = 0$, we have $a^m av + a^m bu = 0$ and $a^m av = 0$, and so $a^{m+1}v = 0$. By the hypothesis, there exists a positive integer n such that $(a^{m+1})^n Rv = 0$. If $2m \geq mn + n$, then $B^{2m}CA = 0$. If $2m < mn + n$, then $B^{mn+n-2m}B^m B^m CA = 0$. Hence $B^{mn+n}T(R, R) \subseteq l_{T(R,R)}(A)$.

This implies that $T(R, R)$ is left GWZI. $\square$

Corollary 2.8 *If R is a left GWZI ring, then $R[x]/(x^2)$ is also a left GWZI ring.*

Clearly, $WT_4(R) = \left\{ \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\} \cong T(T(R, R), T(R, R))$ via

$$\begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix} \longmapsto \begin{pmatrix} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} & \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \end{pmatrix}.$$

Corollary 2.9 *If R is a left GWZI ring, then $WT_4(R)$ is also a left GWZI ring.*

Lemma 2.10 *Let R be a left GWZI ring and $n \geq 2$. If $A = \begin{pmatrix} A_1 & \alpha \\ 0 & a_1 \end{pmatrix}$ and $B = \begin{pmatrix} B_1 & \beta \\ 0 & b_1 \end{pmatrix}$ with $AB = 0$, then there exists a positive integer t such that $A^t\gamma b_1 = 0$, for any $\gamma \in R^n = \left\{ \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \mid c_1, c_2, \ldots, c_n \in R \right\}$ and $A_1, B_1 \in V_{n-1}(R)$.*

Proof We will show by induction on n.

If $n = 2$, then the proof is trivial. Suppose that the result is also true for $n = k - 1$.

Now let $A = \begin{pmatrix} A_1 & \alpha \\ 0 & a_1 \end{pmatrix}$, $B = \begin{pmatrix} B_1 & \beta \\ 0 & b_1 \end{pmatrix} \in V_k(R)$ and $AB = 0$. Then $A_1B_1 = 0$, $A_1\beta + \alpha b_1 = 0$ and $a_1b_1 = 0$. Since R is a left GWZI ring, there exists a positive integer m_1 such that $a_1^{m_1}Rb_1 = 0$ from $a_1b_1 = 0$. Since $A_1B_1 = 0$, there exists a positive integer m_2 such that $A_1^{m_2}R^{k-1}b_1 = 0$ by induction hypothesis. Write $m = \max\{m_1, m_2\}$. Then $a_1^m Rb_1 = 0$ and $A_1^m R^{k-1}b_1 = 0$. For any $\gamma = \begin{pmatrix} \delta \\ c_k \end{pmatrix} \in R^k$, where $\delta = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k-1} \end{pmatrix} \in R^{k-1}$, then $A^{m+1}\gamma b_1 = \begin{pmatrix} A_1^{m+1} & A_1^m\alpha + A_1^{m-1}\alpha a_1 + \cdots + \alpha a_1^m \\ 0 & a_1^{m+1} \end{pmatrix} \begin{pmatrix} \delta b_1 \\ c_k b_1 \end{pmatrix} = \begin{pmatrix} A_1^{m-1}\alpha a_1 c_k b_1 + \cdots + \alpha a_1^m c_k b_1 \\ 0 \end{pmatrix}$. It is easy to know that $A^{2m+1}\gamma b_1 = 0$, for any $\gamma \in R^k$. $\square$

Theorem 2.11 *R is a left GWZI ring if and only if $V_n(R)$ is also a left GWZI ring for any $n \geq 2$.*

Proof Let R be a left GWZI ring. We will show by induction on n.

If $n=2$, then $V_2(R)=T(R,R)$. Thus the proof is similar to Lemma 2.7.

Now suppose that $2<n\leq k-1$ is such that $V_n(R)$ is left GWZI. We will show that $V_k(R)$ is left GWZI.

Let $A=\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_k \\ 0 & a_1 & a_2 & \cdots & a_{k-1} \\ 0 & 0 & a_1 & \cdots & a_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix}$ and $B=\begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_k \\ 0 & b_1 & b_2 & \cdots & b_{k-1} \\ 0 & 0 & b_1 & \cdots & b_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_1 \end{pmatrix}$ be such that $AB=0$.

Then $A=\begin{pmatrix} A_1 & \alpha \\ 0 & a_1 \end{pmatrix}$, $B=\begin{pmatrix} B_1 & \beta \\ 0 & b_1 \end{pmatrix}$ with $A_1B_1=0$, $A_1\beta+\alpha b_1=0$ and $a_1b_1=0$, where $\alpha=\begin{pmatrix} a_k \\ \vdots \\ a_2 \end{pmatrix}$ and $\beta=\begin{pmatrix} b_k \\ \vdots \\ b_2 \end{pmatrix}$. By induction hypothesis, we have $T_1=V_{k-1}(R)$ is left GWZI, then there exists a positive integer m such that $A_1^mT_1B_1=0$ and $a_1^mRb_1=0$. Set $C=\begin{pmatrix} C_1 & \gamma \\ 0 & c_1 \end{pmatrix}\in V_k(R)$ where $C_1\in T_1$ and $\gamma=\begin{pmatrix} c_k \\ \vdots \\ c_2 \end{pmatrix}$. Then

$$A^mCB=\begin{pmatrix} A_1^m & A_1^{m-1}\alpha+A_1^{m-2}\alpha a_1+A_1^{m-3}\alpha a_1^2+\cdots+\alpha a_1^{m-1} \\ 0 & a_1^m \end{pmatrix}\begin{pmatrix} C_1B_1 & C_1\beta+\gamma b_1 \\ 0 & c_1b_1 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & A_1^mC_1\beta+A_1^m\gamma b_1+A_1^{m-1}\alpha c_1b_1+A_1^{m-2}\alpha a_1c_1b_1+A_1^{m-3}\alpha a_1^2c_1b_1+\cdots+\alpha a_1^{m-1}c_1b_1 \\ 0 & 0 \end{pmatrix}.$$

By Lemma 2.10, we have $A_1^m\gamma b_1=0$. Then $A^mA^mCB=\begin{pmatrix} A_1^m & * \\ 0 & a_1^m \end{pmatrix}A^mCB=\begin{pmatrix} 0 & A_1^{2m}C_1\beta \\ 0 & 0 \end{pmatrix}$.

Since $A_1\beta+\alpha b_1=0$ and $A_1^m\gamma b_1=0$, we have $A_1^m(A_1\beta+\alpha b_1)=0$ and $A_1^{m+1}\beta=0$.

Since

$$A_1^{m+1}\begin{pmatrix} b_2 & b_3 & \dots & b_k \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_3 \\ 0 & \cdots & 0 & b_2 \end{pmatrix}=\begin{pmatrix} (*) & (A_1^{m+1}\beta) \end{pmatrix}\in V_{k-1}(R)$$

and $A_1^{m+1}\beta = 0$, it follows that

$$A_1^{m+1}\begin{pmatrix} b_2 & b_3 & \dots & b_k \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_3 \\ 0 & \cdots & 0 & b_2 \end{pmatrix} = 0.$$

By the hypothesis, there exists a positive integer t such that

$$A_1^{t}C_1\begin{pmatrix} b_2 & b_3 & \dots & b_k \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_3 \\ 0 & \cdots & 0 & b_2 \end{pmatrix} = 0.$$

Hence $A_1^tC_1\beta = 0$. It is easy to obtain that $V_k(R)$ is left GWZI. Therefore, $V_n(R)$ is a left GWZI ring for $n \geq 2$.

Conversely, if $V_n(R)$ is a left GWZI ring for $n \geq 2$, we clearly have R is a left GWZI ring. □

Since $R[x]/(x^n) \cong V_n(R)$ via $a_0+a_1\overline{x}+a_2\overline{x}^2+\cdots+a_{n-1}\overline{x}^{n-1} \longmapsto \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix}$, $n \geq 2$. We have the following corollary.

Corollary 2.12 *If R is a left GWZI ring, then $R[x]/(x^n)$ is also left GWZI.*

Note that the homomorphic image of a GWZI ring need not be GWZI. Consider the following example.

Example 2.13 Let Z_2 denote the field of integers modulo 2 and $Z_2(y)$ rational functions field of polynomial ring $Z_2[y]$ and $R = Z_2(y)[x]$ the ring of polynomials in x over $Z_2(y)$ subject to the relation $xy + yx = 1$. By [8, Example 2.11], R is symmetric. Then R is also GWZI. Let $I = x_2R$. Then I is a maximal ideal of R. Consider the ring $S = R/I$. We write $\bar{x}$ and $\bar{y}$ for the images of x and y respectively under the natural epimorphism from R onto S. For $\bar{x}, \bar{y} \in S$, we have $\bar{x}^2 = \bar{0}$ and $\bar{x}\bar{y} + \bar{y}\bar{x} = \bar{1}$. Then we have $(\bar{y}\bar{x})^2 = \bar{y}\bar{x}$. If S is GWZI, then $\bar{y}\bar{x}S\bar{x} = 0$ and $\bar{y}\bar{x}\bar{y}\bar{x} = 0$ by $\bar{y}\bar{x}\bar{x} = 0$. This implies that $\bar{y}\bar{x} = 0$ and $\bar{x}\bar{y} = \bar{1}$. Hence we get $\bar{x} = \bar{0}$, leading to a contradiction.

Proposition 2.14 *Suppose that R/I is a left GWZI ring for some ideal I of a ring R. If I is reduced as a ring without identity, then R is left GWZI.*

Proof Let $ab = 0$ with $a, b \in R$. Then we have $\bar{a}\bar{b} = \bar{0}$ in $\bar{R} = R/I$. Since $\bar{R}$ is left GWZI, hence there exists a positive integer m such that $\bar{a}^m\bar{R}\bar{b} = \bar{0}$ and so $a^mrb \in I$ for any $r \in R$. Since $(b(a^mrb)a)^2 = b(a^mrb)ab(a^mrb)a = 0$ and I is a reduced ideal of R, we get $b(a^mrb)a = 0$. Since $(a^mrb)^3 = a^mr(ba^mrba)a^{m-1}rb = 0$, it follows that $a^mrb = 0$. Thus this implies that R is left

GWZI. $\square$

Set

$$I=\left\{\begin{pmatrix} 0 & c_{12} & c_{13} & \cdots & c_{1n} \\ 0 & 0 & c_{23} & \cdots & c_{2n} \\ 0 & 0 & 0 & \cdots & c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \mid 0\neq c_{12}, c_{13}, \ldots, c_{n-1,n}\in R\right\}$$

and R is a left GWZI ring, then $T_n(R)$ is a weakly semicommutative ring and I is an idea of $T_n(R)$. It is easy to get $T_n(R)/I$ is left GWZI. But $T_n(R)$ is not left GWZI for I is not reduced. So, this implies that the condition that I is reduced cannot be dropped in Proposition 2.14.

Proposition 2.15 *The following conditions are equivalent for a ring R:*

(1) R is Abelian.

(2) $l(e)$ is a GW-ideal of R for every $e\in E(R)$.

(3) $r(e)$ is a GW-ideal of R for every $e\in E(R)$.

Proof (1) $\Rightarrow$ (2). It is clear.

(2) $\Rightarrow$ (1). Let $e\in E(R)$ and $x\in R$. Since $1-e\in l(e)$ and $l(e)$ is a GW-ideal, there exists a positive integer n such that $(1-e)^n x\in l(e)$ which implies that $xe=exe$. Similarly, we have $ex=exe$. This implies that R is Abelian.

(3) $\Rightarrow$ (1). It is similar to (2) $\Rightarrow$ (1). $\square$

Corollary 2.16 *A left GWZI ring is Abelian.*

Theorem 2.17 *A ring R is a left GWZI ring if and only if*

$$S_3(R)=\left\{\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a,b,c,d\in R\right\}$$

is a left GWZI ring.

Proof If a ring R is left GWZI, let

$$0\neq B=\begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix}\in S_3(R) \text{ and } 0\neq A=\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix}\in l_{S_3(R)}(B).$$

Then $AB=\begin{pmatrix} a_1a_2 & a_1b_2+b_1a_2 & a_1c_2+b_1d_2+c_1a_2 \\ 0 & a_1a_2 & a_1d_2+d_1a_2 \\ 0 & 0 & a_1a_2 \end{pmatrix}=0.$

Case 1 Set $a_1=0$. Then $A=\begin{pmatrix} 0 & b_1 & c_1 \\ 0 & 0 & d_1 \\ 0 & 0 & 0 \end{pmatrix}\in N(S_3(R))$, and there exists a positive integer

3 such that $A^3S_3(R)B=0$.

Case 2 Set $a_1\neq 0$ and $b_1=c_1=d_1=0$. Then $A=\begin{pmatrix} a_1 & 0 & 0\\ 0 & a_1 & 0\\ 0 & 0 & a_1\end{pmatrix}$ and $AB=\begin{pmatrix} a_1a_2 & a_1b_2 & a_1c_2\\ 0 & a_1a_2 & a_1d_2\\ 0 & 0 & a_1a_2\end{pmatrix}=0$.

That is, $a_1a_2=0\cdots(1)$, $a_1b_2=0\cdots(2)$, $a_1c_2=0\cdots(3)$, $a_1d_2=0\cdots(4)$.

By the hypothesis and (1)–(4), there exists a positive integer N such that

$$a_1^NRa_2=0,\ a_1^NRb_2=0,\ a_1^NRc_2=0,\ a_1^NRd_2=0.$$

For any $C=\begin{pmatrix} a & b & c\\ 0 & a & d\\ 0 & 0 & a\end{pmatrix}\in S_3(R)$, then

$$A^NCB=\begin{pmatrix} a_1^N & 0 & 0\\ 0 & a_1^N & 0\\ 0 & 0 & a_1^N\end{pmatrix}\begin{pmatrix} aa_2 & ab_2+ba_2 & ac_2+bd_2+ca_2\\ 0 & aa_2 & ad_2+da_2\\ 0 & 0 & aa_2\end{pmatrix}$$

$$=\begin{pmatrix} a_1^Naa_2 & a_1^Nab_2+a_1^Nba_2 & a_1^Nac_2+a_1^Nbd_2+a_1^Nca_2\\ 0 & a_1^Naa_2 & a_1^Nad_2+a_1^Nda_2\\ 0 & 0 & a_1^Naa_2\end{pmatrix}=0.$$

There exists a positive integer N such that $A^NS_3(R)B=0$.

Case 3 Set $a_1\neq 0$, $c_1=0$, $b_1\neq 0$ and $d_1\neq 0$. Then we have

$$A=\begin{pmatrix} a_1 & b_1 & 0\\ 0 & a_1 & d_1\\ 0 & 0 & a_1\end{pmatrix},\quad A^2=\begin{pmatrix} a_1^2 & a_1b_1+b_1a_1 & b_1d_1\\ 0 & a_1^2 & a_1d_1+d_1a_1\\ 0 & 0 & a_1^2\end{pmatrix},$$

$$A^3=\begin{pmatrix} a_1^3 & a_1^2b_1+a_1b_1a_1+b_1a_1^2 & a_1b_1d_1+b_1a_1d_1+b_1d_1a_1\\ 0 & a_1^3 & a_1^2d_1+a_1d_1a_1+d_1a_1^2\\ 0 & 0 & a_1^3\end{pmatrix},\ldots,$$

$$A^n=\begin{pmatrix} a_1^n & a_1^{n-1}b_1+a_1^{n-2}b_1a_1+\cdots+b_1a_1^{n-1} & *\\ 0 & a_1^n & a_1^{n-1}d_1+a_1^{n-2}d_1a_1+\cdots+d_1a_1^{n-1}\\ 0 & 0 & a_1^n\end{pmatrix},$$

where $*=a_1^{n-2}b_1d_1+a_1^{n-3}b_1a_1d_1+a_1^{n-3}b_1d_1a_1+\cdots+b_1d_1a_1^{n-2}$.

By $AB=\begin{pmatrix} a_1a_2 & a_1b_2+b_1a_2 & a_1c_2+b_1d_2\\ 0 & a_1a_2 & a_1d_2+d_1a_2\\ 0 & 0 & a_1a_2\end{pmatrix}=0$, we have $a_1a_2=0\cdots(5)$, $a_1b_2+b_1a_2=0\cdots(6)$, $a_1c_2+b_1d_2=0\cdots(7)$, $a_1d_2+d_1a_2=0\cdots(8)$. By the hypothesis and (5), there exists a positive integer n_1 such that $a_1^{n_1}Ra_2 = 0 \cdots$ (9). By (6) and (9), it is easy to get $a_1^{n_1+1}b_2 + a_1^{n_1}b_1a_2 = 0$ and $a_1^{n_1+1}b_2 = 0$, then there exists a positive integer n_2 such that $a_1^{n_2}Rb_2 = 0$. By (8) and (9), then there exists a positive integer n_3 such that $a_1^{n_3}Rd_2 = 0 \cdots$ (10). By (7) and (10), then there exists a positive integer n_4 such that $a_1^{n_4}Rc_2 = 0$. Write $N = \max\{n_1, n_2, n_3, n_4\}$. So $a_1^N Ra_2 = 0$, $a_1^N Rb_2 = 0$, $a_1^N Rc_2 = 0$, $a_1^N Rd_2 = 0$.

Let $n = 3N$. It is easy to get

$$(a_1^{n-1}b_1 + a_1^{n-2}b_1a_1 + \cdots + b_1a_1^{n-1})aa_2 = 0,$$
$$(a_1^{n-1}b_1 + a_1^{n-2}b_1a_1 + \cdots + b_1a_1^{n-1})(ad_2 + da_2) = 0,$$
$$(a_1^{n-1}d_1 + a_1^{n-2}d_1a_1 + \cdots + d_1a_1^{n-1})aa_2 = 0,$$
$$(a_1^{n-2}b_1d_1 + a_1^{n-3}b_1a_1d_1 + a_1^{n-3}b_1d_1a_1 + \cdots + b_1d_1a_1^{n-2})aa_2 = 0.$$

Hence we obtain $A^nS_3(R)B = 0$. There exists a positive integer n such that $A^nS_3(R)B = 0$.

Case 4 Set $a_1 \neq 0$, $c_1 = 0$, $b_1 = 0$ and $d_1 \neq 0$. It is similar to Case 3.

Case 5 Set $a_1 \neq 0$, $c_1 = 0$, $d_1 = 0$ and $b_1 \neq 0$. It is similar to Case 3.

Case 6 Set $a_1 \neq 0$, $c_1 \neq 0$, $d_1 = 0$ and $b_1 = 0$. Then

$$A = \begin{pmatrix} a_1 & 0 & c_1 \\ 0 & a_1 & 0 \\ 0 & 0 & a_1 \end{pmatrix}, \quad A^2 = \begin{pmatrix} a_1^2 & 0 & a_1c_1 + c_1a_1 \\ 0 & a_1^2 & 0 \\ 0 & 0 & a_1^2 \end{pmatrix} \cdots,$$

$$A^n = \begin{pmatrix} a_1^n & 0 & a_1^{n-1}c_1 + a_1^{n-2}c_1a_1 + \cdots + c_1a_1^{n-1} \\ 0 & a_1^n & 0 \\ 0 & 0 & a_1^n \end{pmatrix}.$$

Since $AB = \begin{pmatrix} a_1a_2 & a_1b_2 & a_1c_2 + c_1a_2 \\ 0 & a_1a_2 & a_1d_2 \\ 0 & 0 & a_1a_2 \end{pmatrix}$, that is, $a_1a_2 = 0$, $a_1b_2 = 0$, $a_1c_2 + c_1a_2 = 0$, $a_1d_2 = 0$. It is similar to Case 3, there exists a positive integer N such that $a_1^N Ra_2 = 0$, $a_1^N Rb_2 = 0$, $a_1^N Rc_2 = 0$, $a_1^N Rd_2 = 0$. There exists a positive integer $n = 3N$ such that $A^nS_3(R)B = 0$.

Case 7 Set $a_1 \neq 0$, $c_1 \neq 0$, $d_1 \neq 0$ and $b_1 \neq 0$. By $AB = 0$,

$$\begin{pmatrix} a_1a_2 & a_1b_2 + b_1a_2 & a_1c_2 + b_1d_2 + c_1a_2 \\ 0 & a_1a_2 & a_1d_2 + d_1a_2 \\ 0 & 0 & a_1a_2 \end{pmatrix} = 0.$$

Similarly to Case 3, there exists a positive integer N such that $a_1^N Ra_2 = 0$, $a_1^N Rb_2 = 0$, $a_1^N Rc_2 = 0$, $a_1^N Rd_2 = 0$.

$$A^n = \begin{pmatrix} a_1^n & a_1^{n-1}b_1 + a_1^{n-2}b_1a_1 + \cdots + b_1a_1^{n-1} & * \\ 0 & a_1^n & a_1^{n-1}d_1 + a_1^{n-2}d_1a_1 + \cdots + d_1a_1^{n-1} \\ 0 & 0 & a_1^n \end{pmatrix},$$

where $* = (a_1^{n-2}b_1d_1 + a_1^{n-3}b_1a_1d_1 + a_1^{n-3}b_1d_1a_1 + \cdots + b_1d_1a_1^{n-2}) + (a_1^{n-1}c_1 + a_1^{n-2}c_1a_1 + \cdots + c_1a_1^{n-1})$. Similarly to Case 3, there exists a positive integer n such that $A^nS_3(R)B = 0$.

Case 8 Set $a_1 \neq 0$, $c_1 \neq 0$, $d_1 \neq 0$ and $b_1 = 0$. It is similar to Case 7.

Case 9 Set $a_1 \neq 0$, $c_1 \neq 0$, $d_1 = 0$ and $b_1 \neq 0$. It is similar to Case 7.
Hence we obtain $S_3(R)$ is left GWZI.

Conversely, if $S_3(R)$ is left GWZI, set $b = c = d = 0$. It is clear that R is left GWZI. $\square$

Remark 2.18 We do not know whether $S_n(R)$ is still a GWZI ring when R is left GWZI. But we have the following remark.

Remark 2.19 If R is a left GWZI ring, then $T_n(R)$ is not a left GWZI ring.

Set $A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$. It is easy to get $AB = 0$, $A^2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$ and $A^3 = A^2$. But

$A^2CB = \begin{pmatrix} 0 & 0 & 0 & c_{23} & 0 & \dots & 0 \\ 0 & 0 & 0 & c_{23} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \neq 0$, $A^mCB = A^2CB = \begin{pmatrix} 0 & 0 & 0 & c_{23} & 0 & \dots & 0 \\ 0 & 0 & 0 & c_{23} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$

$\neq 0$, for any $C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ 0 & c_{22} & c_{23} & \cdots & c_{2n} \\ 0 & 0 & c_{33} & \cdots & c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{nn} \end{pmatrix} \in T_n(R)$.

This implies that $T_n(R)$ is not left GWZI.

Next example will show that R is left GWZI, but $R[x]$ is not left GWZI.

Remark 2.20 ([9]) Let $k = F_2\langle a_0, a_1, a_2, a_3, b_0, b_1\rangle$ be the free associative algebra (with 1) over F_2 generated by six indeterminates. Let I be the ideal generated by the following relations:

$$\langle a_0b_0, \quad a_0b_1 + a_1b_0, \quad a_1b_1 + a_2b_0, \quad a_2b_1 + a_3b_0, \quad a_3b_1,$$

$$a_0a_j\,(0 \le j \le 3), \quad a_3a_j\,(0 \le j \le 3), \quad a_1a_j + a_2a_j\,(0 \le j \le 3),$$

$$b_ib_j\,(0 \le i, j \le 1), \quad b_ia_j\,(0 \le i \le 1, 0 \le j \le 3)\rangle.$$

We let $R = k/I$. Think of $\{a_0, a_1, a_2, a_3, b_0, b_1\}$ as elements (sometimes called letters) of R satisfying the relations in I , suppressing the bar notation.

Put $F(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and $G(x) = b_0 + b_1x$. The first row of relations in I guarantees that $F(x)G(x) = 0$ in $R[x]$.

By computing, we have $F(x)^n = (a_1 + a_2x)^nx^n$ and $F(x)^nH(x)G(x) = x^n[a_1^nc_0b_0 + (*)x + (**)x^2 + \cdots] \neq 0$ whenever $H(x) = c_0 + c_1x$.

This shows that $R[x]$ is not left GWZI. But it was proved in [9, Claim 6] that R is semi-commutative, then R is also left GWZI.

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