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Generalizations of Nil-Injective Rings

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Abstract The definition of AP-injectivity motivates us to generalize the injectivity to almost wnil-injectivity and almost nil n-injectivity. The aim of this paper is to investigate characterizations and properties of almost wnil-injective rings and almost nil n-injective rings. Various results are developed, and many conclusions extend known results.

Keywords almost wnil-injective rings; almost nil-injective rings; almost nil n-injective rings; nil-injective rings; nil n-injective rings.

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1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary. Denote by $M_R({}_RM)$ a right(left) R-module, and by $R^{m\times n}$ the set of all $m\times n$ matrices over R. We write $N\leq M$ when N is a submodule of M. For $A\in R^{m\times n}, A^T$ denotes the transpose of A. We write $R^n=R^{1\times n}, R_n=R^{n\times 1}$. If $X\in R^n$, $Y\in R_n$, $C\in R^{n\times n}$, define $r_{R_n}(X)=\{s\in R_n : Xs=0\}$, $l_{R^n}(Y)=\{s\in R^n : sY=0\}$, $l_{R^n}(C)=\{s\in R^n : sC=0\}$. For singletons $\{x\}$ and $\{a\}$, we abbreviate to $r_R(x)$ and $l_R(a)$. As usual, $J(R), N(R), Z(R_R), \mathrm{Soc}(R_R)$ denote the Jacobson radical, the set of nilpotent elements, the right singular ideal, the right socle of R, respectively. $N|M$ denotes that N is a direct summand of M.

It is known to all that generalizations of injectivity have been discussed in many papers. R is called a P-injective ring [1], if every right R-homomorphism from aR to R can be extended to an endomorphism of R, where $a\in R$. In [2], P-injective rings were extended to AP-injective rings and AGP-injective rings. A ring R is called right AP-injective, if, for any $a\in R$, there exists a left ideal X_a of R such that $lr(a)=Ra\oplus X_a$. A ring R is called right AGP-injective, if, for any $0\neq a\in R$, there exists a positive integer $n=n(a)$ and a left ideal X_a of R such that $a^n\neq 0$ and $lr(a^n)=Ra^n\oplus X_a$. Clearly, AP-injectivity and AGP-injectivity are the generalizations of P-injectivity, and they have many meaningful properties [2]. In [3], a ring R is called PS-injective, if every R-homomorphism $f: I\to R$, for every principally small right ideal I, can be extended to R. In [4], PS-injectivity was extended to APS-injectivity. A ring

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R is called right APS-injective if, for any $a \in J(R)$, there exists a left ideal X_a of R such that $lr(a) = Ra \oplus X_a$. In [5], a ring R is called right nil-injective, if, for any $a \in N(R)$, $lr(a) = Ra$. A ring R is called right wnil-injective, if for any $a \in N(R)$, there exists a positive integer n such that $a^n \neq 0$ and $lr(a^n) = Ra^n$. In [6], nil n-injectivity has been studied. A ring R is called right nil n-injective, if for any $\alpha \in R^n$ such that $l_{R^n} r_{R_n}(\alpha) = R\alpha$, where every component of α is nilpotent. So, in this paper, we study the almost wnil-injectivity and almost nil n-injectivity which are the generalizations of nil-injectivity.

In the second section, we prove that: (1) Let R be a ring. For any $0 \neq a \in N(R)$, there exists a positive integer n and a left ideal X_{a^n} of R such that $a^n \neq 0$ and $lr(a^n) = Ra^n \oplus X_{a^n}$ $\Leftrightarrow$ for any $0 \neq a \in N(R), b \in R$, there exists a positive integer n such that $(ab)^n \neq 0$ and $l(bR \cap r((ab)^{n-1}a)) = (X_{(ab)^n} : b)_l + R(ab)^{n-1}a$ with $ab \in N(R)$, and $(X_{(ab)^n} : b)_l \cap R(ab)^{n-1}a \subseteq l(b)$, where $(X_{(ab)^n} : b)_l = \{x \in R : xb \in X_{(ab)^n}\}$; (2) If R is a commutative ring whose every simple singular right R-module is almost wnil-injective, then $J(R) \cap N(R) = 0$; (3) If $aR, a \in N(R)$ is a commutative right almost wnil-injective ring, then R is right noetherian $\Leftrightarrow$ R is right artinian.

In the third section, we prove that: (1) If $M_n(R)$ is right almost wnil-injective, and R has no non-zero zero divisor except nilpotent elements, then R is right almost nil n-injective; (2) If $R \propto R$ is right almost wnil-injective, then R is right AP-injective.

2. Almost wnil-injective rings

We begin with the following definition.

Definition 2.1 *A module M_R is called right almost wnil-injective, if for any $0 \neq a \in N(R)$, there exists a positive integer n and an S-submodule X_a such that $a^n \neq 0$ and $l_M r_R(a^n) = Ma^n \oplus X_{a^n}$. If R_R is an almost wnil-injective module, then we call R a right almost wnil-injective ring. Similarly, we can define a left almost wnil-injective ring.*

In [7], almost nil-injective rings are defined and studied. A ring R is called almost nil-injective [7], if for any $k \in N(R)$, there exists an S-submodule X_k of M such that $l_M r_R(k) = Mk \oplus X_k$ as left S-modules. Obviously, almost nil-injective rings are almost wnil-injective. Of course, AP-injective rings and AGP-injective rings are also almost wnil-injective.

Example 2.2 The three examples of [2, Example 1.5(1)] are almost nil-injective, but they are not nil-injective (or wnil-injective).

Theorem 2.3 *Let R be a commutative right almost wnil-injective ring. Then the following statements hold.*

(1) $J(R) \cap N(R) \subseteq Z(R_R) \cap N(R)$.

(2) $\mathrm{Soc}(R_R) \subseteq r(J)$.

Proof Let $a \in N(R) \cap J(R)$. If $a \notin Z(R_R)$, then $r(a) \cap I = 0$ for some $0 \neq I \subseteq R_R$. Take an $s \in I$ such that $as \neq 0$. Since R is commutative, $as \in N(R)$. Then there exists a positive integer n and a left ideal $X_{(as)^n}$ of R such that $(as)^n \neq 0$ and $lr((as)^n) = R(as)^n \oplus X_{(as)^n}$. For any $x \in R$, if $(as)^n x = 0$, then $s(as)^{n-1}x \in r(a) \cap I = 0$, implying $r_R((as)^n) = r_R(s(as)^{n-1})$. Note that $s(as)^{n-1} \in lr(s(as)^{n-1}) = lr((as)^n) = R(as)^n \oplus X_{(as)^n}$. Write $s(as)^{n-1} = d(as)^n + y, y \in X_{(as)^n}$, then $(1-da)s(as)^{n-1} = y$, and so $s(as)^{n-1} = (1-da)^{-1}y$. Then $(as)^n = a(1-da)^{-1}y \in X_{(as)^n}$, a contradiction. Hence $J(R) \cap N(R) \subseteq Z(R_R) \cap J(R)$.

(2) Let $tR \subseteq R$ be simple. Suppose $jt \neq 0$ for some $j \in N(R)$. Then $r(jt) = r(t)$. Since R is right almost wnil-injective and $(jt)^2 = 0$, there is a left ideal X_{jt} such that $lr(jt) = R(jt) \oplus X_{jt}$. Note that $t \in lr(jt)$. Write $t = rjt + x$, where $x \in X_{jt}$. Then $(1-rj)t = x$, so $t = (1-rj)^{-1}x \in X_{jt}$. This means that $jt \in X_{jt}$, a contradiction. □

Corollary 2.4 *Let R be commutative semiperfect and right almost wnil-injective. Then $J(R) \cap N(R) = Z(R_R) \cap N(R)$.*

Proof Since R is semiperfect, $Z(_RR) + Z(R_R) \subseteq J(R)$ by [8, Lemma 2]. Then $Z(_RR) \cap N(R) + Z(R_R) \cap N(R) \subseteq J(R) \cap N(R)$, so $Z(R_R) \cap N(R) \subseteq J(R) \cap N(R)$. By Theorem 2.3(1), $J(R) \cap N(R) = Z(R_R) \cap N(R)$. □

Theorem 2.5 *If R is right almost wnil-injective, so is eRe for all $e^2 = e \in R$ satisfying $ReR = R$.*

Proof Write $S = eRe$, and let $k \in N(S)$, so $k \in N(R)$. By the assumption, there exists a positive integer n and a left ideal X_{k^n} of R such that $k^n \neq 0$ and $lr(k^n) = Rk^n \oplus X_{k^n}$. It is easy to see that $elr(k^n) = l_S r_S(k^n), eRk^n = eRek^n$ and eX is a left ideal of eRe, then $l_S r_S(k^n) = (eRe)k^n \oplus eX_{k^n}$, so eRe is right almost wnil-injective. □

Remark 2.6 The condition that $ReR = R$ in Theorem 2.5 is needed. Let F be a field,

$$R = \begin{pmatrix} F & F & 0 & 0 & 0 & 0 \\ 0 & F & 0 & 0 & 0 & 0 \\ 0 & 0 & F & F & 0 & 0 \\ 0 & 0 & 0 & F & 0 & 0 \\ 0 & 0 & 0 & 0 & F & F \\ 0 & 0 & 0 & 0 & 0 & F \end{pmatrix}.$$

By [9, Example 9], R is a QF-ring, then R is right almost wnil-injective. Let $e = e_{11} + e_{22} + e_{44} + e_{55}$ be a sum of canonical matrix units. Then e is an idempotent of R such that $ReR \neq R$ and $eRe \cong S = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then S is not right almost wnil-injective. In fact, for any $\bar{d} = \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} \in N(S)$, by [4, Remark 2.17], there does not exist a left ideal X of S such that $l_S r_S(\bar{d}) = S\bar{d} \oplus X$.

Corollary 2.7 *If the matrix ring $M_n(R)$ over a ring R is right almost wnil-injective ($n \geq 1$),* *then so is R.*

Proof Suppose $S = M_n(R)$ is right almost wnil-injective. Since $Se_{11}S = S$ and $R \cong e_{11}Se_{11}$, R is right almost wnil-injective by Theorem 2.5, where e_{ij} is the matrix unit. $\square$

Theorem 2.8 *The following conditions are equivalent for a ring R.*

(1) For any $0 \neq a \in N(R)$, there exists a positive integer n and a left ideal X_{a^n} of R such that $a^n \neq 0$ and $lr(a^n) = Ra^n \oplus X_{a^n}$.

(2) For any $0 \neq a \in N(R), b \in R$, there exists a positive integer n such that $(ab)^n \neq 0$ and $l(bR \cap r((ab)^{n-1}a)) = (X_{(ab)^n} : b)_l + R(ab)^{n-1}a$ with $ab \in N(R)$, and $(X_{(ab)^n} : b)_l \cap R(ab)^{n-1}a \subseteq l(b)$, where $(X_{(ab)^n} : b)_l = \{x \in R : xb \in X_{(ab)^n}\}$.

Proof (1)$\Rightarrow$(2). For any $a \in N(R), b \in R$ with $ab \in N(R)$, by (1), there exists a positive integer n and a left ideal X_{a^n} of R such that $lr((ab)^n) = R(ab)^n \oplus X_{(ab)^n}$. Let $x \in l(bR \cap r((ab)^{n-1}a))$. Then $r((ab)^n) \subseteq r(xb)$ and so $xb \in lr(xb) \subseteq lr((ab)^n) = R(ab)^n \oplus X_{(ab)^n}$. Write $xb = t(ab)^n + y$, where $t \in R, y \in X_{(ab)^n}$. Then $(x - t(ab)^{n-1}a)b = y \in X_{(ab)^n}$. Hence $x - t(ab)^{n-1}a \in (X_{(ab)^n} : b)_l$. It follows that $x \in (X_{(ab)^n} : b)_l + R(ab)^{n-1}a$. Obviously, $R(ab)^{n-1}a \subseteq l(bR \cap r((ab)^{n-1}a))$. If $y \in (X_{(ab)^n} : b)_l$, then $yb \in X_{(ab)^n} \subseteq lr((ab)^n)$. Let $bs \in bR \cap r((ab)^{n-1}a)$. Then $(ab)^n s = 0$. Hence $ybs = 0$ since $yb \in lr((ab)^n)$. This follows that $y \in l(bR \cap r((ab)^{n-1}a))$. We have proved that $l(bR \cap r((ab)^{n-1}a)) = (X_{(ab)^n} : b)_l + R(ab)^{n-1}a$. If $s(ab)^{n-1}a \in (X_{(ab)^n} : b)_l \cap R(ab)^{n-1}a$, then $s(ab)^n \in X_{(ab)^n} \cap R(ab)^n$, showing that $s(ab)^n = 0$. Hence $s(ab)^{n-1}a \in l(b)$.

(2)$\Rightarrow$(1). Let $b = 1$. $\square$

In [10], a ring R is π-N-regular, if for any $0 \neq a \in N(R)$, there exists a positive integer n and $b \in R$ such that $a^n \neq 0$, and $a^n = a^n b a^n$.

Theorem 2.9 *Let R be a right nonsingular right almost wnil-injective ring, and $l(I \cap K) = l(I) + l(K)$, where I and K are any right ideals of R. Then R is π-N-regular.*

Proof For any $0 \neq a \in N(R)$, since R is right almost wnil-injective, there exists a positive integer n and a left ideal of R such that $a^n \neq 0$, and $lr(a^n) = Ra^n \oplus X_{a^n}$. Because R is right nonsingular, $r(a^n)$ is not essential in R. So there exists a nonzero right ideal L such that $r(a^n) \oplus L$ is essential in R. By the assumption, $l(r(a^n)) + l(L) = l(r(a^n) \cap L) = R$, and $lr(a^n) \cap l(L) \subseteq l(r(a^n) + L)$. For any $x \in l(r(a^n) + L)$, then $x(r(a^n) + L) = 0$, i.e., $r(a^n) + L \subseteq r(x) \subseteq R$, thus $r(x)$ is essential in R, so $x = 0$ since R is right nonsingular. Therefore $lr(a^n) \cap l(L) \subseteq l(r(a^n) + L) = 0$, thus $R = l(r(a^n)) \oplus l(L) = a^n R \oplus X_{a^n} \oplus l(L)$, so $a^n R$ is a direct summand of R. Let $a^n R = eR, e^2 = e \in R$. Then $e = a^n r, r \in R$, and $a^n = ea^n = a^n r a^n$, so R is π-N-regular. $\square$

Recall that R is a right quasi-duo ring [11], if every maximal right ideal of R is a two-sided ideal.

Theorem 2.10 *Let R be a ring such that every simple right R-module is almost wnil-injective. If R is right quasi duo, then R is reduced.*

Proof Let $a^2=0$. Then $a\in N(R)$. Suppose $a\neq 0$. Then there exists a maximal right ideal M of R containing $r(a)$. Thus R/M is right almost wnil-injective, so $l_{(R/M)}r_R(a)=(R/M)a\oplus X$. Let $f: aR\to R/M$ be defined by $f(ar)=r+M$. Note that f is a well-defined R-homomorphism. Thus $f(a)=ca+M+x, c\in R, x\in X$, and $f(a)=1+M$, so $(1-ca)+M=x\in (R/M)\cap X=0$. Since R is right quasi-duo, $ca\in M$. Hence $1\in M$, which is a contradiction. Therefore, $a=0$, R is reduced. □

Theorem 2.11 *Let R be a ring. If every simple right R-module is almost wnil-injective, then R is semiprime.*

Proof Assume that $a\in R$ such that $aRa=0$. Then $RaR\subseteq r(a)$. If $a\neq 0$, then there exists a maximal right ideal M containing $r(a)$. By hypothesis, R/M is almost wnil-injective. Hence $l_{R/M}r_R(a)=(R/M)a\oplus X_a$. Define $f: aR\to R/M$ by $f(ar)=r+M$. Note that f is well-defined. So there exists a $c\in R$ such that $1+M=f(a)=ca+M+x$, where $c\in R, x\in X_a$. Hence $1-ca+M=x\in (R/M)\cap X_a=0$, which implies $1-ca\in M$. Since $ca\in RaR\subseteq r(a)\subseteq M$, $1\in M$, which is a contradiction. So $a=0$ and then R is a semiprime ring. □

In [5], a ring R is callled n-regular if $a\in aRa$ for all $a\in N(R)$.

Theorem 2.12 *If R is right quasi duo, then the following statements are equivalent.*

(1) *Every right R-module is almost wnil-injective.*

(2) *Every cyclic right R-module is almost wnil-injective.*

(3) *Every simple right R-module is almost wnil-injective.*

(4) *Every nilpotent element of R is strongly regular.*

(5) *R is n-regular.*

Proof Obviously, (1)⇒(2)⇒(3) and (4)⇒(5). (5)⇒(1) is easy by [5, Theorem 2.18]. Thus it remains to prove that (3) implies (4). By Theorem 2.10, R is reduced. For any $0\neq a\in N(R)$, we will show that $aR+r(a)=R$. Suppose not. Then there exists a maximal right ideal K of R containing $aR+r(a)$. Since R/K is right almost wnil-injective, $l_{(R/K)}r_R(a)=(R/K)a\oplus X_a$. Let $f: aR\to R/K$ be defined by $f(ar)=r+K$. Note that f is a well-defined R-homomorphism. Thus $f(a)=ca+K+x, c\in R, x\in X_a$, and $f(a)=1+K$, and so $1-ca+K=x\in (R/K)\cap X_a=0$, $1-ca\in K$. Since R is right quasi-duo, $ca\in K$, thus $1\in K$, which is a contradiction. Hence $aR+r(a)=R$. So R is strongly regular. □

Corollary 2.13 *If R is right quasi-duo, then the following statements are equivalent.*

(1) *R is n-regular.*

(2) *Every right R-module is almost nil-injective.*

(3) *Every cyclic right R-module is almost nil-injective.*

(4) *Every simple right R-module is almost nil-injective.*

(5) *Every right R-module is almost wnil-injective.*

(6) *Every cyclic right R-module is almost wnil-injective.*

(7) *Every simple right R-module is almost wnil-injective.*

(8) Every right R-module is nil-injective.

(9) Every cyclic right R-module is nil-injective.

(10) Every simple right R-module is nil-injective.

Theorem 2.14 *If R is a ring whose every simple singular right R-module is almost wnil-injective, then $J(R) \cap Z(R_R)$ contains no nonzero nilpotent elements.*

Proof Take any $b \in J(R) \cap Z(R_R)$ with $b^2 = 0$. If $b \neq 0$, then $RbR + r(b)$ is an essential right ideal of R. We will show that $RbR + r(b) = R$. If not, there exists a maximal right ideal M of R containing $RbR + r(b)$. By assumption, R/M is right almost wnil-injective, and $l_{(R/M)}r_R(b) = (R/M)b \oplus X_b$. Let $f : bR \to R/M$ be defined by $f(br) = r + M$. Then $1 + M = f(b) = cb + M + x, 1 - cb + M = x \in (R/M) \cap X_b = 0$, thus $1 - cb \in M$, since $cb \in M$, $1 \in M$, which is a contradiction. Hence $RbR + r(b) = R$, and thus $b = bd, d \in RbR \subseteq J(R)$. This implies $b = 0$, which is a required contradiction. □

Corollary 2.15 *If R is a commutative ring whose every simple singular right R-module is almost wnil-injective, then $J(R) \cap N(R) = 0$.*

Proof By Theorem 2.3, $J(R) \cap N(R) \subseteq J(R) \cap Z(R_R)$, so $J(R) \cap N(R) = J(R) \cap Z(R_R) \cap N(R)$. By Theorem 2.14, $J(R) \cap Z(R_R) \cap N(R) = 0$, hence $J(R) \cap N(R) = 0$. □

Theorem 2.16 *Let R be a commutative right almost wnil-injective ring. For any $a \in N(R), b \in R$, $aR \subseteq bR|R$, then aR is a direct summand of R.*

Proof Let $a \in N(R)$ such that $a \in bR = eR$, where $e = e^2$, and that $\sigma : aR \to bR$ is an R-isomorphism. We want to show that aR is a direct summand of R. There exists $c \in R$ such that $\sigma(ac) = e$ and hence $acR = aR$. Since R is commutative, $ac \in N(R)$. Hence there exists a positive integer n such that $(ac)^n \neq 0$ and $lr((ac)^n) = R(ac)^n \oplus X_{(ac)^n}$ for some left ideal $X_{(ac)^n}$ of R. We write $(ac)^0 = e$ and we have $\sigma((ac)^{i+1}R) = \sigma(ac)(ac)^iR = e(ac)^iR = (ac)^iR$ for $i = 0, 1, \ldots, n-1$. This implies that $(ac)^iR|(ac)^{i-1}R \Leftrightarrow \sigma((ac)^{i+1}R)|\sigma((ac)^iR) \Leftrightarrow (ac)^{i+1}R|(ac)^iR$ for $i = 1, 2, \ldots, n-1$. Therefore, we have $acR|eR \Leftrightarrow (ac)^2R|acR \Leftrightarrow \cdots \Leftrightarrow (ac)^nR|(ac)^{n-1}R$. So, to show $(ac)R|R$, it suffices to show that $(ac)^nR|R$. We note that $(ac)^nR \cong (ac)^{n-1}R \cong \cdots \cong aR \cong eR$, that is to say $\sigma^n : (ac)^nR \to eR$ is an isomorphism, and $\sigma^n((ac)^n) = e$. Then we can conclude that $(ac)^nR$ is a direct summand of R by [9, Theorem 2.8]. □

Recall that a ring R is right GC_2 (see [12]), if every right ideal that is isomorphic to R is itself a direct summand of R.

Corollary 2.17 *If $R = aR, a \in N(R)$, and R is a commutative almost wnil-injective ring, then $J(R) = Z(R_R)$.*

Proof By Theorem 2.16, R is right GC_2, so $Z(R_R) \subseteq J(R)$ by [12, Proposition 2.6]. By the proof of Theorem 2.3, $J(R) \subseteq Z(R_R)$. Hence $J(R) = Z(R_R)$. □

Corollary 2.18 *If $R = aR, a \in N(R)$, and R is a commutative almost wnil-injective ring:*

(1) If R_R has finite Goldie dimension, then R is semilocal;

(2) R is right noetherian if and only if R is right artinian.

Proof (1) By Theorem 2.16, R_R satisfies GC_2. Since R_R has finite Goldie dimension, it is semilocal by [13, Lemma 1.1].

(2) Let R be right noetherian. Then R is semilocal by (1). Further, since R has ACC on right annihilators, $Z(R_R)$ is nilpotent by [14, Lemma 3.29]. By Corollary 2.17, $J(R) = Z(R_R)$. Hence R is right noetherian and semiprimary, and therefore R is right artinian. The converse is clear. □

Lemma 2.19 *If $R = aR, a \in N(R)$, and R is a commutative almost wnil-injective ring. If $b \notin J(R)$, then the inclusion $r(b) \subset r(b - bdb)$ is strict for some $d \in R$.*

Proof By Corollary 2.17, $r(b)$ is not essential in R_R, and so there exists a nonzero right ideal I of R such that $r(b) \oplus I$ is essential in R_R. Take $0 \neq c \in I$, then $bc \neq 0$. It is easy to see that $bc \in N(R)$, and there exists a positive integer n and a left ideal $X_{(bc)^n}$ of R such that $(bc)^n \neq 0$ and $lr((bc)^n) = R(bc)^n \oplus X_{(bc)^n}$. We claim that $r((bc)^n) = r(c(bc)^{n-1})$. In fact, if $(bc)^n x = 0$, then $c(bc)^{n-1}x \in r(b) \cap I = 0$, so $r((bc)^n) \subseteq r(c(bc)^{n-1})$. Thus, $c(bc)^{n-1} \in lr(c(bc)^{n-1}) = lr((bc)^n) = R(bc)^n \oplus X_{(bc)^n}$. Hence there exists $d \in R, y \in X_{(bc)^n}$ such that $c(bc)^{n-1} = d(bc)^n + y$, $bc(bc)^{n-1} = bd(bc)^n + by$, $(1 - bd)(bc)^n = by \in R(bc)^n \cap X_{(bc)^n} = 0$, then $c(bc)^{n-1} \in r(b - bdb)$, and $(bc)^n \neq 0$, $c(bc)^{n-1} \notin r(b)$. So $r(b) \subset r(b - bdb)$ is strict. □

Theorem 2.20 *Let $R = aR(a \in N(R))$ be a commutative almost wnil-injective ring. Then the following statements are equivalent.*

(1) R is right perfect;

(2) The ascending chain $r(a_1) \subseteq r(a_2a_1) \subseteq r(a_3a_2a_1) \subseteq \cdots$ terminates for every sequence $\{a_1, a_2, \ldots\} \subseteq R$.

Proof By Lemma 2.19 and [15, Theorem 3.11], it is easy to be proved. □

3. Almost nil n-injective rings

In [6], a ring R is called right nil n-injective, if $l_{R^n} r_{R_n}(\alpha) = R\alpha$ for any $\alpha = (a_1, a_2, \ldots, a_n)$, where $a_1, a_2, \ldots, a_n \in N(R)$. Hence we give the following definition.

Definition 3.1 *A ring R is called right almost nil n-injective, if for any $\alpha = (a_1, a_2, \ldots, a_n)$, $l_{R^n} r_{R_n}(\alpha) = R\alpha \oplus X_\alpha$, where $a_1, a_2, \ldots, a_n \in N(R)$, X_α is a left ideal of R.*

Lemma 3.2 *Let M_R be a module, $S = \mathrm{End}_R(R)$ and $\alpha \in R$ and every component of α be nilpotent.*

(1) If $l_{R^n} r_{R_n}(\alpha) = R\alpha \oplus X_\alpha$ for some $X_\alpha \subseteq R^n$ as left S-modules, then $\mathrm{Hom}_R(\alpha R^n, R) = \mathrm{Hom}_R(R, R) \oplus \Gamma$ as left S-modules, where $\Gamma = \{f \in \mathrm{Hom}_R(\alpha R_n, R) : f(\alpha) \in X_\alpha\}$.

(2) If $\mathrm{Hom}_R(\alpha R_n, R) = \mathrm{Hom}_R(R, R) \oplus Y$ as left S-modules, then $l_{R^n} r_{R_n}(\alpha) = R\alpha \oplus X$ as left S-modules, where $X = \{f(\alpha) : f \in Y\}$.

(3) $R\alpha$ is a direct summand of $l_{R^n}r_{R_n}(\alpha)$ as left S-modules if and only if $\mathrm{Hom}_R(R,R)$ *is a direct summand of* $\mathrm{Hom}_R(\alpha R_n, R)$ *as left S-modules.*

Proof The proof is similar to that of [2, Lemma 1.2]. □

Theorem 3.3 *If $M_n(R)$ is right almost wnil-injective, and R has no non-zero zero divisor except nilpotent elements, then R is right almost nil n-injective.*

Proof Let $I = a_1R + \cdots + a_nR$, where $a_i \in N(R)$, and write $A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$. Then A is nilpotent. Since $M_n(R)$ is right almost wnil-injective, there is a positive integer m such that $A^m \neq 0$, and $lr(A^m) = M_n(R)A^m \oplus X_{A^m}$. Let

$$\Gamma = \{\alpha \in \mathrm{Hom}_R(I,R) : \begin{pmatrix} \alpha(a_1) & \cdots & \alpha(a_n) \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} \in X_{A^m}\}.$$

It can be verified that Γ is a left R-submodule of $\mathrm{Hom}_R(I,R)$. We claim that $\mathrm{Hom}_R(I,R) = \mathrm{Hom}_R(R,R) \oplus \Gamma$. In fact, for any $\alpha \in \mathrm{Hom}_R(I,R)$, write $B = \begin{pmatrix} \alpha(a_1) & \cdots & \alpha(a_n) \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix}$. Suppose $A^mX = 0$, for $X = (a_{ij}) \in M_n(R)$. Since $A^mX =$

$$\begin{pmatrix} a_1^{m-1}(a_1x_{11} + a_2x_{21} + \cdots + a_nx_{n1}) & \cdots & a_1^{m-1}(a_1x_{1n} + a_2x_{2n} + \cdots + a_nx_{nn}) \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix},$$

$a_1x_{11} + a_2x_{21} + \cdots + a_nx_{n1} = 0, \ldots, a_1x_{1n} + a_2x_{2n} + \cdots + a_nx_{nn} = 0$. It follows that $BX = 0$. Hence $B \in lr(A^m) = M_n(R)A^m \oplus X_{A^m}$. Write $B = (c_{ij})A^m + (d_{ij})$, where $(c_{ij}) \in M_n(R), (d_{ij}) \in X_{A^m}$. Let $\beta : R \to R$ by $\beta(r) = c_{11}a_1^{m-1}r$ and $\gamma : I \to R$ by $\gamma(a_1r_1 + \cdots + a_nr_n) = d_{11}r_1 + \cdots + d_{nn}r_n$. Then $\beta \in \mathrm{Hom}_R(R,R)$ and $\alpha = \beta + \gamma$. Note that $\begin{pmatrix} d_{11} & \cdots & d_{1n} \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} =$

$\begin{pmatrix} 1 & \cdots & 0 \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix}(d_{ij}) \in X_{A^m}$. So $\gamma \in \Gamma$. thus $\mathrm{Hom}_R(I,R) = \mathrm{Hom}_R(R,R) + \Gamma$. Suppose $\gamma \in \Gamma \cap \mathrm{Hom}_R(R,R)$. Then there exists $c_1 \in R$ such that $(\gamma(a_1),\dots,\gamma(a_n)) = (c_1,0,\dots,0)A^m$.

Therefore, $\begin{pmatrix} \gamma(a_1) & \cdots & \gamma(a_n) \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} c_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} A^m \in M_n(R)A^m \cap X_{A^m} = 0.$

Therefore $\mathrm{Hom}_R(I,R) = \mathrm{Hom}_R(R,R) \oplus \Gamma$. Then by Lemma 3.2, R is right almost nil n-injective. □

In Theorem 3.3, let $n = 1$. We have the following corollary.

Corollary 3.4 *Assume that R has no non-zero zero divisor except nilpotent elements, then R is right almost wnil-injective if and only if R is right almost nil-injective.*

Theorem 3.5 *Let R be a commutative ring and $n \geq 1$. If $M_n(R)$ is right almost nil-injective, then for any right R-module $I = \alpha_1 R + \alpha_2 R + \cdots + \alpha_n R$, where $\alpha_i \in R^n$ and every component of α_i $(i = 1, 2, \dots, n)$ is nilpotent, $\mathrm{Hom}_R(R^n, R)$ is a direct summand of $\mathrm{Hom}_R(I,R)$.*

Proof Let $S = M_n(R)$ and let $I = \alpha_1 R + \cdots + \alpha_n R$, where $\alpha_i \in R^n$ and every component of α_i $(i = 1, 2, \dots, n)$ is nilpotent. Write $((\alpha_1)^T, \dots, (\alpha_n)^T) = A$. Since R is commutative, $A \in N(S)$. By hypothesis, we have $l_S r_S(A) = SA \oplus X_A$ for some left ideal X_A of S. Let

$$\Gamma = \{f \in \mathrm{Hom}_R(I,R) : \begin{pmatrix} f(\alpha_1) & \cdots & f(\alpha_n) \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} \in X_A\}.$$

It is easy to verify that Γ is a left R-submodule of $\mathrm{Hom}_R(I,R)$. We claim that $\mathrm{Hom}_R(I,R) = \mathrm{Hom}_R(R^n,R) \oplus \Gamma$ as left R-modules. In fact, for any $g \in \mathrm{Hom}_R(I,R)$, write $B = \begin{pmatrix} g(\alpha_1) & \cdots & g(\alpha_n) \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix}$.

Then $B \in l_S r_S(A)$, and hence $B = (c_{ij})A + (d_{ij})$, where $(c_{ij}) \in S$ and $(d_{ij}) \in X_A$. Let $h : R^n \to R, \sum_{i=1}^n e_i r_i \mapsto \sum_{i=1}^n c_{1i} r_i$, where e_i is the standard basis of R^n over R, and let

$k : I \to R, \sum_{i=1}^{n} \alpha_i r_i \mapsto \sum_{i=1}^{n} d_{1i} r_i$. Then $g = h + k$. Note that $\begin{pmatrix} d_{11} & \cdots & d_{1n} \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} =$

$\begin{pmatrix} 1 & \cdots & 0 \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} (d_{ij}) \in X_A$. So $k \in \Gamma$. Therefore, we have $\mathrm{Hom}_R(I, R) = \mathrm{Hom}_R(R^n, R) + \Gamma$. Suppose $l \in \mathrm{Hom}_R(R^n, R) \cap \Gamma$. Then there exists $(c_1, \ldots, c_n) \in R^n$ such that $(l(\alpha_1), \ldots, l(\alpha_n)) =$

$(c_1, \ldots, c_n)A$. Thus, $\begin{pmatrix} l(\alpha_1) & \cdots & l(\alpha_n) \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} c_1 & \cdots & c_n \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} A \in SA \cap X_A = 0$. Therefore, $\mathrm{Hom}_R(I, R) = \mathrm{Hom}_R(R^n, R) \oplus \Gamma$. $\square$

Let R be a ring and M a bimodule over R. The trivial extension of R and M is $R \propto M = \{(a, x) : a \in R, x \in M\}$ with addition defined componentwise and multiplication defined by $(a, x)(b, y) = (ab, ay + xb)$. In fact, $R \propto M$ is isomorphism to the subring $\{ \begin{pmatrix} a & x \\ 0 & a \end{pmatrix} : a \in R, x \in M\}$ of the formal 2×2 upper triangular matrix ring $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$.

Theorem 3.6 *Let R be a ring. If $R \propto R$ is right almost wnil-injective, then R is right AP-injective.*

Proof Let $S = R \propto R$. For any $0 \neq a \in R$, $(0, a)(0, a) = (0, 0)$, so $(0, a) \in N(S)$. Since S is right almost wnil-injective, there exists a left ideal $X_{(0,a)}$ of S such that $l_S r_S(0, a) = S(0, a) \oplus X_{(0,a)}$. For any $(b, c) \in l_S r_S(0, a), r_S(0, a) \subseteq r_S(b, c)$. Since $(0, 1) \in r_S(0, a), 0 = (b, c)(0, 1) = (0, b)$, showing $b = 0$. If $(m, n) \in S(0, a)$, then $m = 0$ by [4, Corollary 3.3]. So $X_{(0,a)} = 0 \propto X_a$, where X_a is a left ideal of R. By [4, Proposition 3.1], $l_R r_R(a) = Ra \oplus X_a$, proving that R is right AP-injective. $\square$

Corollary 3.7 *Let R be a ring. If $R \propto R$ is right almost nil-injective, then R is right AP-injective.*

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