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A Lower Bound for the Distance Signless Laplacian Spectral Radius of Graphs in Terms of Chromatic Number

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Abstract Let G be a connected graph on n vertices with chromatic number k, and let $\rho(G)$ be the distance signless Laplacian spectral radius of G. We show that $\rho(G) \geq 2n + 2\lfloor \frac{n}{k} \rfloor - 4$, with equality if and only if G is a regular Turán graph.

Keywords distance matrix; distance signless Laplacian; spectral radius; chromatic number.

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1. Introduction

Let G be a connected simple graph with vertex set $V(G)$ and edge set $E(G)$. The distance between two vertices u, v of G, denoted by d_{uv}, is defined as the length of the shortest path between u and v in G. The distance matrix of G, denoted by $D(G)$, is defined by $D(G) = (d_{uv})_{u,v\in V(G)}$. The transmission $Tr(v)$ of a vertex v is defined to be the sum of the distances from v to all other vertices in G, i.e., $Tr(v) = \sum_{u\in V(G)} d_{uv}$. The distance matrix is very useful in different fields, including the design of communication networks [1], graph embedding theory [2–4] as well as molecular stability [5, 6]. Balaban et al. [7] proposed the use of the distance spectral radius as a molecular descriptor. Gutman et al. [8] used the distance spectral radius to infer the extent of branching and model boiling points of an alkane. Therefore, maximizing or minimizing the distance spectral radius over a given class of graphs is of great interest and significance. Recently, the maximal (minimal) distance spectral radius of a given class of graphs has been studied extensively [9–18].

Similarly to the Laplacian or signless Laplacian of graphs, Aouchiche and Hanse [19] defined the distance Laplacian of a connected graph G as the matrix $D^L(G) = \mathrm{Diag}(Tr) - D(G)$, where

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Diag(Tr) denotes the diagonal matrix of the vertex transmissions in G. Along this line, they [20] defined the distance signless Laplacian of a connected graph G to be $D^Q(G) = \mathrm{Diag}(Tr) + D(G)$. Since $D^Q(G)$ is symmetric, its eigenvalues are all real. In addition, as $D^Q(G)$ is positive, by Perron-Frobenius Theorem, the spectral radius $\rho(G)$ of $D^Q(G)$, called the distance signless Laplacian spectral radius of G, is exactly the largest eigenvalue of $D^Q(G)$ with multiplicity one; and there exists a unique (up to a multiple) positive eigenvector corresponding to this eigenvalue, called the Perron vector of $D^Q(G)$.

Recall that the chromatic number of a connected graph G is the smallest number of colors needed to color the vertices of G such that any two adjacent vertices have different colors. A subset of vertices assigned to the same color is called a color class; every such class forms an independent set. The Turán graph $T_{n,k}$ is a complete k-partite graph on n vertices for which the numbers of vertices of vertex classes are as equal as possible.

In this paper we prove that $T_{n,k}$ is the unique graph with minimum distance signless Laplacian spectral radius in the class of simple connected graphs with n vertices and chromatic number k, and give an lower bound of distance signless Laplacian spectral radius of graphs in terms of chromatic number.

2. Main results

Given a graph G on n vertices, a vector $x \in \mathbb{R}^n$ is considered as a function defined on G, if there is a 1-1 map φ from $V(G)$ to the entries of x; simply written $x_u = \varphi(u)$ for each $u \in V(G)$. If x is an eigenvector of $D^Q(G)$, then it is naturally defined on $V(G)$, i.e., x_u is the entry of x corresponding to the vertex u. One can find that

$$x^T D^Q(G)x = \sum_{\{u,v\}\subseteq V(G)} d_{uv}(x_u + x_v)^2, \tag{2.1}$$

and λ is an eigenvalue of $D^Q(G)$ corresponding to the eigenvector x if and only if $x \neq 0$ and

$$[\lambda - Tr(v)]x_v = \sum_{u\in V(G)} d_{uv}x_u, \quad \text{for each vertex } \ v \in V(G). \tag{2.2}$$

In addition, for an arbitrary unit vector $x \in \mathbb{R}^n$,

$$x^T D^Q(G)x \le \rho(G), \tag{2.3}$$

with the equality holding if and only if x is a Perron vector of $D^Q(G)$.

Let $e = uv$ be an edge of G such that $G - e$ is also connected. The removal of e does not decrease distance, while it does increase the distance by at least one distance, as the distance between u and v is 1 in G and at least 2 in $G - e$. Similarly, adding a new edge to G does not increase distances, while it does decrease the distance by at least one.

By Perron-Frobenius Theorem, we have the following lemma immediately.

Lemma 2.1 *Let G be a connected graph with $u, v \in V(G)$. If $uv \notin E(G)$, then $\rho(G) > \rho(G+uv)$. If $uv \in E(G)$ such that $G - uv$ is also connected, then $\rho(G) < \rho(G - uv)$.*

According to Lemma 2.1, for a connected graph G on n vertices, we have $\rho(G) \geq 2n-2$, with equality if and only if $G = K_n$; and $\rho(G) \leq \rho(T_G)$, with equality if and only if G is a tree, where T_G is a spanning tree of G.

Let G be a graph and v be a vertex of G. Denote by $N(v)$ the set of neighbors of v in the graph G.

Lemma 2.2 *Let G be a connected graph containing two vertices u, v.*

(1) If $N(u)\backslash\{v\} = N(v)\backslash\{u\}$, then $Tr(u) = Tr(v)$.

(2) If $N(u)\backslash\{v\} \subsetneq N(v)\backslash\{u\}$, then $Tr(u) > Tr(v)$.

Proof For any $w \in V(G)$ and $w \neq u, v$. We prove the result (1) firstly. If $w \in N(u)\backslash\{v\} = N(v)\backslash\{u\}$, then $d_{uw} = d_{vw} = 1$. If $w \notin N(u)\backslash\{v\} = N(v)\backslash\{u\}$, then there exists a shortest path P connecting w and u. Let w_1 be the vertex on the path P that is adjacent to u and let P' be the remaining part of P connecting w_1 and w. Then the union of P' and w_1v connects w and v, which implies that $d_{vw} \leq d_{uw}$. By the symmetry of u, v, we can show that $d_{uw} \leq d_{vw}$, and hence $d_{vw} = d_{uw}$.

(2) If $w \in N(u)\backslash\{v\}$, then $d_{uw} = d_{vw} = 1$. If $w \in N(v)\backslash\{u\} - N(u)\backslash\{v\}$, then $d_{uw} > 1$ and $d_{vw} = 1$. If $w \notin N(v)\backslash\{u\}$, by what we have proved in (1), $d_{vw} \leq d_{uw}$. The result follows by the above discussion. $\square$

By Lemma 2.2, we can get the following result.

Lemma 2.3 *Let G be a connected graph containing two vertices u, v, and let x be a Perron vector of $D^Q(G)$.*

(1) If $N(u)\backslash\{v\} = N(v)\backslash\{u\}$, then $x_u = x_v$.

(2) If $N(u)\backslash\{v\} \subsetneq N(v)\backslash\{u\}$, then $x_u > x_v$.

Proof Let $\rho := \rho(G)$. By (2.2) we have

$$[\rho - Tr(u)]x_u = \sum_{w\in V(G)} d_{uw}x_w = d_{uv}x_v + \sum_{w\in V(G)\backslash\{u,v\}} d_{uw}x_w, \tag{2.4}$$

$$[\rho - Tr(v)]x_v = \sum_{w\in V(G)} d_{vw}x_w = d_{uv}x_u + \sum_{w\in V(G)\backslash\{u,v\}} d_{vw}x_w. \tag{2.5}$$

For the assertion (1), as $N(u)\backslash\{v\} = N(v)\backslash\{u\}$, by Lemma 2.2 we have $Tr(u) = Tr(v)$, and $d_{uw} = d_{vw}$ for each $w \in V(G)\backslash\{u, v\}$ by what we proved in Lemma 2.2(1). Thus

$$[\rho - Tr(u) + d_{uv}]x_u = [\rho - Tr(v) + d_{uv}]x_v.$$

By (2.4) and (2.5) we have $\rho > \max\{Tr(u), Tr(v)\}$ as both right sides are positive. Therefore $x_u = x_v$.

For the assertion (2), as $N(u)\backslash\{v\} \subsetneq N(v)\backslash\{u\}$, by Lemma 2.2 we have $Tr(u) > Tr(v)$, $d_{uw} \geq d_{vw}$ for each $w \in V(G)\backslash\{u, v\}$, and there exists at least one vertex $w_0 \in V(G)\backslash\{u, v\}$ such that $d_{uw_0} > d_{vw_0}$. So

$$[\rho - Tr(u) + d_{uv}]x_u > [\rho - Tr(v) + d_{uv}]x_v,$$

which implies $x_u > x_v$. $\Box$

For a connected graph G with n vertices and chromatic number k, if $k = 1$, then G is an isolated vertex, and if $k = n$, then G is a complete graph. In the following, we consider the graphs with $2 \le k \le n-1$.

Lemma 2.4 *Let G be a connected graph with minimal distance signless Laplacian spectral radius among all connected graphs with n vertices and chromatic number k, where $2 \le k \le n-1$. Then G is the unique graph $T_{n,k}$.*

Proof Observe that $V(G)$ can be partitioned into k color classes $V_1, V_2, \ldots, V_k$, where $|V_i| = n_i$ $(i = 1, 2, \ldots, k)$ and $\sum_{i=1}^{k} n_i = n$. By Lemma 2.1, $G = K_{n_1,n_2,\ldots,n_k}$, a complete k-partite graph whose parts have size $n_1, n_2, \ldots, n_k$, respectively. Without loss of generality, assume $n_1 \ge n_2 \ge \cdots \ge n_k$.

Suppose that G is not the Turán graph. Then we have $n_1 - n_k \ge 2$. Consider the graph $G' = K_{n_1-1,n_2,\ldots,n_{k-1},n_k+1}$ whose color classes are $V_1', V_2', \ldots, V_k'$, where $|V_1'| = n_1 - 1$, $|V_k'| = n_k + 1$, and $|V_i'| = n_i$ for $i = 2, \ldots, k-1$. Let x be the unit Perron vector of $D^Q(G')$. By Lemma 2.3, x may be written as

$$x = (\underbrace{x_1, \ldots, x_1}_{n_1-1}, \underbrace{x_2, \ldots, x_2}_{n_2}, \ldots, \underbrace{x_{k-1}, \ldots, x_{k-1}}_{n_{k-1}}, \underbrace{x_k, \ldots, x_k}_{n_k+1}).$$

We will show $x_1 \ge x_k$. Let $u \in V_1'$ and $v \in V_k'$. By (2.2) we have

$$[\rho(G') - Tr(u)]x_1 = 2(n_1 - 2)x_1 + n_2x_2 + \cdots + n_{k-1}x_{k-1} + (n_k + 1)x_k,$$
$$[\rho(G') - Tr(v)]x_k = (n_1 - 1)x_1 + n_2x_2 + \cdots + n_{k-1}x_{k-1} + 2n_kx_k.$$

So

$$[\rho(G') - Tr(u) - n_1 + 3]x_1 = [\rho(G') - Tr(v) - n_k + 1]x_k. \tag{2.6}$$

Note that $Tr(u) = n + n_1 - 3$, $Tr(v) = n + n_k - 1$ and $Tr(w_i) = n + n_i - 2$ for $w_i \in V_i'$, $i = 2, \ldots, k-1$. So (2.6) becomes

$$[\rho(G') - n - 2n_1 + 6]x_1 = [\rho(G') - n - 2n_k + 2]x_k. \tag{2.7}$$

By the theory of nonnegative matrices [21], $\rho(G')$ is at least the minimum row sum of $D^Q(G')$, so

$$\rho(G') \ge 2\min\{n + n_1 - 3, n + n_k - 1, n + n_i - 2, i = 2, \ldots, k-1\} \ge 2n - 2.$$

Hence, $\rho(G') - n - 2n_k + 2 \ge 2n - 2 - n - 2n_k + 2 \ge (n_1 + n_k) - 2n_k = n_1 - n_k > 0$, and $\rho(G') - n - 2n_1 + 6 \le \rho(G') - n - 2n_k + 2$, which implies $x_1 \ge x_k$.

By (2.1) we find that

$$\begin{aligned}
&x^T[D^Q(G') - D^Q(G)]x \\
&= (n_1 - 1)(x_1 + x_k)^2 + 2n_k(x_k + x_k)^2 - [2(n_1 - 1)(x_1 + x_k)^2 + n_k(x_k + x_k)^2] \\
&= n_k(x_k + x_k)^2 - (n_1 - 1)(x_1 + x_k)^2 < 0.
\end{aligned}$$

So

$$\rho(G') = x^T D^Q(G')x < x^T D^Q(G)x \leq \rho(G).$$

This completes the proof. $\square$

Lemma 2.5 *Let $T_{n,k}$ be a Turán graph with s $(0 \leq s < k)$ parts of size $d+1$ and $k-s$ parts of size d (i.e., $d = \lfloor \frac{n}{k} \rfloor$). Then*

$$\rho(T_{n,k}) = \frac{3n+4d-6+\sqrt{4-4n+n^2+8sd+8s}}{2} \geq 2n+2d-4$$

where equality holds if and only if $s=0$, that is, $n=kd$ or $T_{n,k}$ is regular.

Proof Let $V_1, V_2, \ldots, V_k$ be the color classes of $T_{n,k}$, and let x be a Perron vector of $D^Q(T_{n,k})$. Firstly, we consider the case of $0 < s < k$. By Lemma 2.3, for $i = 1, 2, \ldots, k$, the vertices in V_i have the same value given by x, denoted by x_i. By (2.2), for each $i = 1, 2, \ldots, k$,

$$[\rho(T_{n,k}) - (n + n_i - 2)]x_i = \sum_{j \neq i} n_j x_j + 2(n_i - 1)x_i,$$

that is,

$$[\rho(T_{n,k}) + 4 - n - 2n_i]x_i = \sum_{j=1}^{k} n_j x_j,$$

which implies $\rho(T_{n,k}) > n + 2n_i - 4$ for all $i = 1, 2, \ldots, k$, i.e., $\rho(T_{n,k}) > n + 2d - 2$. Then

$$\frac{n_i}{\rho(T_{n,k}) + 4 - n - 2n_i} = \frac{n_i x_i}{\sum_{j=1}^{k} n_j x_j},$$

and hence

$$\sum_{i=1}^{k} \frac{n_i}{\rho(T_{n,k}) + 4 - n - 2n_i} = 1.$$

Denote

$$f(\lambda) = \sum_{i=1}^{k} \frac{n_i}{\lambda + 4 - n - 2n_i}.$$

For each $\lambda > n + 2d - 2$, $f(\lambda)$ is positive, strictly decreasing with respect to λ, and goes to 0 as $\lambda \to +\infty$. Therefore the equation $f(\lambda) = 1$ has exactly one root greater than $n + 2d - 2$, namely $\rho(T_{n,k})$. So $\rho(T_{n,k})$ is the largest root of the equation $f(\lambda) = 1$. Since $T_{n,k}$ has s $(0 < s < k)$ parts of size $d+1$ and $k-s$ parts of size d, $f(\lambda)$ can be written as

$$f(\lambda) = \frac{(k-s)d}{\lambda + 4 - n - 2d} + \frac{s(d+1)}{\lambda + 4 - n - 2(d+1)}.$$

Noting that $n = dk + s$, we have

$$\lambda^2 + (6 - 3n - 4d)\lambda + 2[(2 - n - 2d)(2 - n - d) - s(d+1)] = 0,$$

and hence

$$\rho(T_{n,k}) = \frac{3n+4d-6+\sqrt{4-4n+n^2+8sd+8s}}{2} > 2n+2d-4.$$

Secondly, we consider the case of $s = 0$. In this case, $T_{n,k}$ is regular, and $D^Q(T_{n,k})$ has constant row sum $2n + 2d - 4$. So, $\rho(T_{n,k}) = 2n + 2d - 4$. Combining the above two cases, we get the result. □

By Lemmas 2.4 and 2.5, we now arrive at the main result of this paper.

Theorem 2.6 *Let G be a connected graph of order n and chromatic number k. Then*

$$\rho(G) \geq 2n + 2\left\lfloor \frac{n}{k} \right\rfloor - 4,$$

where equality holds if and only if G is a regular Turán graph.

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