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Unique Weighted Representation Basis of Integers

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Abstract Let k_1, k_2 be nonzero integers with $(k_1, k_2) = 1$ and $k_1k_2 \neq -1$. In this paper, we prove that there is a set $A \subseteq \mathbb{Z}$ such that every integer can be represented uniquely in the form $n = k_1a_1 + k_2a_2$, $a_1, a_2 \in A$.

Keywords additive basis; representation function.

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1. Introduction

For sets A and B of integers and integers k_1, k_2, let

$$k_1A + k_2B = \{k_1a + k_2b : a \in A, b \in B\}.$$

Let

$$r_{k_1,k_2}(A, n) = \text{card}\{(a_1, a_2) : n = k_1a_1 + k_2a_2, a_1, a_2 \in A\}.$$

The counting function for the set A is

$$A(y, x) = \text{card}\{a \in A : y \le a \le x\}.$$

We call A a weighted representation basis if $r_{k_1,k_2}(A, n) \ge 1$ for all $n \in \mathbb{Z}$. In 2003, Nathanson [3] constructed a family of arbitrarily sparse bases $A \subseteq \mathbb{Z}$ satisfying $r_{1,1}(n) = 1$ for all $n \in \mathbb{Z}$. In 2011, Tang et al. [5] proved that there exists a family of bases of $A \subseteq \mathbb{Z}$ satisfying $r_{1,-1}(n) = 1$ for all $n \neq 0$. For related problems, see [1, 2, 4, 6].

In this paper, we obtain the following result.

Theorem 1.1 *Let $f(x)$ be a function such that $\lim_{x\to\infty} f(x) = +\infty$ and k_1, k_2 be nonzero integers with $(k_1, k_2) = 1$, $k_1k_2 \neq -1$. Then there exists a set A of integers such that*

$$r_{k_1,k_2}(A, n) = 1 \quad \textit{for all} \quad n \in \mathbb{Z},$$

and $A(-x, x) \le f(x)$ for all sufficiently large x.

2. Proof of Theorem 1.1

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By the result of Nathanson [3], we may assume that $|k_1| > |k_2| \geq 1$. We shall construct an ascending sequence of finite sets $A_1 \subseteq A_2 \subseteq \cdots$ such that

(i) $\mathrm{card}(A_l) = 2l$ for all $l \geq 1$,

(ii) $r_{k_1,k_2}(A_l, n) \leq 1$ for all $n \in \mathbb{Z}$,

(iii) If l is even, then $r_{k_1,k_2}(A_l, n) = 1$ for $-\frac{l}{2}+1 \leq n \leq \frac{l}{2}$. If l is odd, then $r_{k_1,k_2}(A_l, n) = 1$ for $|n| \leq \frac{l-1}{2}$.

We shall show that the infinite set

$$A = \bigcup_{l=1}^{\infty} A_l$$

is a unique (k_1, k_2)-weighted basis of $\mathbb{Z}$.

We construct A_l by induction. Let $A_1 = \{0, k_1\}$. Assume that for some l, we have constructed

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_l$$

satisfying (i), (ii), (iii). Now we construct A_{l+1}.

We define the integer

$$d_l = \max\{|a| : a \in A_l\}.$$

Then

$$A_l \subseteq [-d_l, d_l]$$

and

$$k_1A_l + k_2A_l \subseteq [-(|k_1| + |k_2|)d_l, (|k_1| + |k_2|)d_l].$$

Define

$$b_l = \min\{b > 0 : b \notin k_1A_l + k_2A_l\}$$

and

$$b'_l = \max\{b < 0 : b \notin k_1A_l + k_2A_l\}.$$

Then

$$\frac{l-1}{2} \leq b_l, -b'_l \leq (|k_1| + |k_2|)d_l + 1.$$

To construct the set A_{l+1}, we choose an integer c_l such that $c_l \geq 3k_1^2 d_l$.

Case 1 l is odd. Since $(k_1, k_2) = 1$, we know that there exist two different integers x, y satisfying $k_1x + k_2y = b_l$ with $k_2x \geq |k_2|c_l, |y| \geq c_l$. Put $x_{l+1} = x$, $y_{l+1} = y$, $A_{l+1} = A_l \cup \{x_{l+1}, y_{l+1}\}$. We have

$$k_1A_{l+1} + k_2A_{l+1} = \bigcup_{l=1}^{6} S_l,$$

where

$$S_1 = \{b_l, (k_1 + k_2)x_{l+1}, (k_1 + k_2)y_{l+1}, k_1y_{l+1} + k_2x_{l+1}\},$$

$$S_2 = k_1A_l + k_2A_l,$$

$$S_3 = k_1x_{l+1} + k_2A_l, \quad S_4 = k_1y_{l+1} + k_2A_l,$$

$$S_5 = k_1 A_l + k_2 x_{l+1}, \quad S_6 = k_1 A_l + k_2 y_{l+1}.$$

We shall show that $k_1 A_{l+1} + k_2 A_{l+1}$ is the disjoint union of the above six sets.

(i) $S_1 \cap S_2 = \varnothing$. In fact, by the definition of b_l, we know that $b_l \notin S_2$. Moreover,

$$|(k_1+k_2)x_{l+1}|, \quad |(k_1+k_2)y_{l+1}| \geq c_l > (|k_1|+|k_2|)d_l,$$

$$\begin{aligned}|k_1 y_{l+1} + k_2 x_{l+1}| &= \frac{|k_1(b_l - k_1 x_{l+1}) + k_2^2 x_{l+1}|}{|k_2|} \geq \frac{(k_1^2 - k_2^2)|x_{l+1}|}{|k_2|} - \frac{|k_1|}{|k_2|} b_l \\ &\geq \frac{3(k_1^2 - k_2^2)k_1^2 d_l}{|k_2|} - \frac{|k_1|}{|k_2|}((|k_1|+|k_2|)d_l + 1) \\ &> \frac{6k_1^2}{|k_2|} d_l > (|k_1|+|k_2|)d_l.\end{aligned}$$

(ii) $S_1 \cap S_3 = \varnothing$. Firstly, we have b_l, $(k_1+k_2)x_{l+1} \notin S_3$. Secondly, we can show that $(k_1+k_2)y_{l+1}$, $k_1 y_{l+1} + k_2 x_{l+1} \notin S_3$. In fact, if $(k_1+k_2)y_{l+1}$ or $k_1 y_{l+1} + k_2 x_{l+1} \in S_3$, then there exists $a \in A_l$ such that

$$(k_1+k_2)y_{l+1} = k_1 x_{l+1} + k_2 a \tag{1}$$

or

$$k_1 y_{l+1} + k_2 x_{l+1} = k_1 x_{l+1} + k_2 a. \tag{2}$$

That is,

$$(k_1 + 2k_2)y_{l+1} = b_l + k_2 a \tag{3}$$

or

$$(k_2^2 - k_1^2 - k_1 k_2)x_{l+1} = k_2^2 a - k_1 b_l. \tag{4}$$

It follows from (4) that

$$c_l \leq |x_{l+1}| \leq \frac{k_2^2 d_l}{|k_2^2 - k_1^2 - k_1 k_2|} + \frac{|k_1 b_l|}{|k_2^2 - k_1^2 - k_1 k_2|} < 3k_1^2 d_l \leq c_l,$$

a contradiction. If $k_1 + 2k_2 = 0$, by (3), we have $b_l = -k_2 a = k_1 a + k_2 a \in k_1 A_l + k_2 A_l$, a contradiction. If $k_1 + 2k_2 \neq 0$, then

$$c_l \leq |y_{l+1}| \leq \frac{|k_2| d_l}{|k_1 + 2k_2|} + \frac{|b_l|}{|k_1 + 2k_2|} \leq (|k_1| + 2|k_2|)d_l + |k_1| < c_l,$$

a contradiction. Hence, $S_1 \cap S_3 = \varnothing$.

Similarly, we can show $S_1 \cap S_i = \varnothing$ for $4 \leq i \leq 6$.

(iii) $S_i \cap S_j = \varnothing$, $2 \leq i < j \leq 6$. We can assume that $k_1 x_{l+1} > 0$ (The condition $k_1 x_{l+1} < 0$ is similar). For $u_4 = k_1 y_{l+1} + k_2 a_4 \in S_4$, $u_6 = k_1 a_6 + k_2 y_{l+1} \in S_6$, we have

$$u_6 = k_1 a_6 + b_l - k_1 x_{l+1} \leq -3|k_1|^3 d_l + (2|k_1| + |k_2|)d_l + 1 < -(|k_1| + |k_2|)d_l,$$

and

$$\begin{aligned}u_4 &= \frac{k_1(b_l - k_1 x_{l+1})}{k_2} + k_2 a_4 \\ &\leq -\frac{k_1^2}{k_2} x_{l+1} + \frac{k_1}{k_2} b_l + |k_2| d_l \leq \frac{-3k_1^4 + 3k_1^2}{|k_2|} d_l\end{aligned}$$

$$< -(3|k_1|^3 + |k_1|)d_l < -k_1x_{l+1} + b_l - |k_1|d_l$$
$$\le u_6.$$

For $u_3 = k_1x_{l+1} + k_2a_3 \in S_3$, $u_5 = k_1a_5 + k_2x_{l+1} \in S_5$,

$$u_5 = k_1a_5 + k_2x_{l+1} \ge 3k_1^2|k_2|d_l - |k_2|d_l > (|k_1| + |k_2|)d_l,$$

$$u_3 = k_1x_{l+1} + k_2a_3 \ge k_1x_{l+1} - |k_2|d_l > k_2x_{l+1} + |k_2|d_l \ge u_5.$$

These inequalities imply that the sets S_2, S_3, S_4, S_5, S_6 are pairwise disjoint, hence $S_i \cap S_j = \varnothing$ for $1 \le i < j \le 6$. It follows that $r_{k_1,k_2}(A_{l+1}, n) \le 1$ for all n.

By the above discussion, we know that A_{l+1} satisfies (i) and (ii). By the hypothesis and the definition of b_l, we know $r_{k_1,k_2}(A_l, n) = 1$ for all $|n| \le \frac{l-1}{2}$ and $\frac{l+1}{2} \in k_1A_{l+1} + k_2A_{l+1}$. It follows that

$$r_{k_1,k_2}(A_{l+1}, n) = 1 \quad \text{for} \quad \frac{-l+1}{2} \le n \le \frac{l+1}{2}.$$

Hence, A_{l+1} satisfies (iii).

Case 2 l is even. We can find integers x, y such that $k_1x + k_2y = b'_l$ with $k_2x > |k_2|c_l, |y| > c_l$. Put $x_{l+1} = x$, $y_{l+1} = y$, $A_{l+1} = A_l \cup \{x_{l+1}, y_{l+1}\}$. Similarly, we can show that A_{l+1} satisfies (i), (ii) and (iii).

Let $A = \bigcup_{l=1}^{\infty} A_l$. If l is odd, then

$$\{-\frac{l-1}{2} \cdots -1, 0, 1 \cdots \frac{l-1}{2}\} \subseteq k_1A_{l+1} + k_2A_{l+1}.$$

If l is even, then

$$\{-\frac{l}{2} + 1 \cdots -1, 0, 1 \cdots \frac{l}{2}\} \subseteq k_1A_{l+1} + k_2A_{l+1}.$$

So A is a (k_1, k_2)-weighted basis. If $r_{k_1,k_2}(A, n) \ge 2$ for some n, then there exists a set A_l such that $r_{k_1,k_2}(A_l, n) \ge 2$, which is impossible. Therefore, A is a unique (k_1, k_2)-weighted basis for the integers.

Given a function $f(x)$ that tends to infinity, we use induction to construct a sequence $\{c_l\}_{l=1}^{\infty}$ such that $A(-x, x) \le f(x)$ for all $x > c_1$. We observe that

$$A(-x, x) = A_{l+1}(-x, x) \le 2(l + 1) \quad \text{for} \quad d_l \le x < d_{l+1}.$$

We begin by choosing an integer $c_1 \ge 3k_1^2d_1$ such that

$$f(x) \ge 4 \quad \text{for all} \quad x \ge c_1.$$

Then

$$A(-x, x) \le 4 \le f(x) \quad \text{for} \quad c_1 \le x \le d_2.$$

Let $l \ge 2$, and suppose we have selected an integer $c_{l-1} \ge 3k_1^2d_{l-1}$ such that

$$f(x) \ge 2l \quad \text{for} \quad x \ge c_{l-1}$$

and

$$A(-x, x) \le f(x) \quad \text{for} \quad c_{l-1} \le x \le d_l.$$

There exists an integer $c_l \geq 3k_1^2 d_l$ such that

$$f(x) \geq 2l + 2 \quad \text{for} \quad \geq c_l.$$

Then

$$A(-x, x) = 2l \leq f(x) \quad \text{for} \quad d_l \leq x < c_l$$

and

$$A(-x, x) \leq 2l + 2 \leq f(x) \quad \text{for} \quad c_l \leq x \leq d_{l+1},$$

hence

$$A(-x, x) \leq f(x) \quad \text{for} \quad c_1 \leq x \leq d_{l+1}.$$

It follows that

$$A(-x, x) \leq f(x) \quad \text{for all} \quad x \geq c_1.$$

This completes the proof of Theorem 1.1. □

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