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On Non-Bi-Lipschitz Homogeneity of Some Hyperspaces

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Abstract A metric space (X, d) is called bi-Lipschitz homogeneous if for any points $x, y \in X$, there exists a self-homeomorphism h of X such that both h and h^{-1} are Lipschitz and $h(x) = y$. Let $2^{(X,d)}$ denote the family of all non-empty compact subsets of metric space (X, d) with the Hausdorff metric. In 1985, Hohti proved that $2^{([0,1],d)}$ is not bi-Lipschitz homogeneous, where d is the standard metric on $[0, 1]$. We extend this result in two aspects. One is that $2^{([0,1],\varrho)}$ is not bi-Lipschitz homogeneous for an admissible metric ϱ satisfying some conditions. Another is that $2^{(X,d)}$ is not bi-Lipschitz homogeneous if (X, d) has a nonempty open subspace which is isometric to an open subspace of m-dimensional Euclidean space $\mathbb{R}^m$.

Keywords non-bi-Lipschitz homogeneity; hyperspace; Hilbert cube.

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1. Introduction and main results

Let (X_1, d_1) and (X_2, d_2) be metric spaces. We call a homeomorphism $\varphi : X_1 \to X_2$ a bi-Lipschitz homeomorphism if there exists a real number $L > 0$ such that

$$L^{-1}d_1(x, y) \le d_2(\varphi(x), \varphi(y)) \le Ld_1(x, y)$$

for all points $x, y \in X_1$. This concept plays an important role in fractal geometry. Recently, there are a lot of interests in related topics [2, 5]. Trivially, $L \ge 1$ and there exists the minimum positive number L with the property above. Let bilipφ denote the least such constant L. We say that φ is a K-bi-Lipschitz homeomorphism if bilip$\varphi \le K$. Let us recall that a topological space X is called homogeneous if for all points $x, y \in X$ there exists a self-homeomorphism h of X such that $h(x) = y$. A metric space X is called bi-Lipschitz homogeneous, if for all points $x, y \in X$ there exists a bi-Lipschitz self-homeomorphism $h : X \to X$ such that $h(x) = y$. 2^X is used to denote the metric space of all non-empty compact subsets of metric space (X, d) with Hausdorff metric d_H defined by

$$d_H(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\},$$

for each pair of points $A, B \in 2^{(X,d)}$. $2^{(X,d)}$ will be called the hyperspace of (X, d) and it may be denoted as 2^X for short if there is no confusion.

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Let $\mathbb{I}$ denote $[0,1]$ with the standard metric unless otherwise stated. Recall that the admissible metric on a metrizable topological space X means the metric which induces the original topology of X.

Earlier in 1931, Keller in [4] proved that Hilbert cube Q, countable infinite product of $\mathbb{I}$ with product topology, is homogeneous. This fact is very different from a well-known theorem that no finite dimensional cube $\mathbb{I}^n$ is homogeneous for any $n \in \mathbb{N}$. Later, Curtis and Schori proved in [1] that the hyperspace 2^X is homeomorphic to Hilbert cube Q for every non-degenerate connected local-connected compact metrizable topological space X. This remarkable work built a bridge between hyperspace theory and infinite dimensional topology. In particular, $2^{\mathbb{I}}$ is homeomorphic to Hilbert cube Q. From these two results, we know that $2^{\mathbb{I}}$ is homogeneous. However, Hohti in [3, Proposition 7.2] showed that $2^{(\mathbb{I},d)}$ is not bi-Lipschitz homogeneous, where d is the standard metric on $\mathbb{I}$. In [3, Theorem 3.1] the author also proved that Hilbert cube Q can be bi-Lipschitz homogeneous if it is endowed with some special admissible metric. Therefore, we know that bi-Lipschitz homogeneity is not a topological invariant.

We extend [3, Proposition 7.2] in two aspects. One is the following theorem which is a generalization in the sense of metric.

Theorem 1.1 *Let $f : \mathbb{I} \to [0,+\infty)$ be a continuous strictly monotone increasing function which satisfies:*

(1) $f(0) = 0$ and $f(x+y) \le f(x) + f(y)$;

(2) For $L \in (0,+\infty)$ and $1/n \in [0, f(1)/L)$, we have $nf^{-1}(\frac{1}{Ln}) = O(1)$ and $\alpha_n f^{-1}(\frac{1}{Ln}) \to 0$ as $n \to +\infty$, where $\alpha_n = \frac{f^{-1}(Lf(f^{-1}(1/n)/n))}{f^{-1}(\frac{1}{Ln})/n}$.

Define a function $\varrho^f : \mathbb{I} \times \mathbb{I} \to [0,+\infty)$ by $(x,y) \mapsto f(|x-y|)$. Then ϱ^f is an admissible metric on $\mathbb{I}$. Moreover, $2^{(\mathbb{I},\varrho^f)}$ is not bi-Lipschitz homogeneous. We call ϱ^f a metric on $\mathbb{I}$ induced by the function f.

Example 1.1 Let $g(x)$ be a linear combination, with positive coefficient, of functions x^μ, where $\mu \in (0,1]$. Clearly, $g(x)$ satisfies the condition of Theorem 1.1. Hence the hyperspace of $\mathbb{I}$ with the metric induced by g is not bi-Lipschitz homogeneous.

Denote by $\mathbb{R}^m$ the m-fold products of real line $\mathbb{R}$ with the Euclidean metric. Another generalization of [3, Proposition 7.2] is the following theorem:

Theorem 1.2 *For a metric space X, 2^X is not bi-Lipschitz homogeneous if there exists a nonempty open subspace of X which is bi-Lipschitz homeomorphic to an open subspace of $\mathbb{R}^m$.*

Remark 1.1 By Theorem 1.2, we may study the non-bi-Lipschitz homogeneity of a hyperspace 2^X by studying the local metrical property of X. In particular, 2^X is not bi-Lipschitz homogeneous if X has a nonempty open subspace which is isometric to an open subspace of $\mathbb{R}^m$.

Example 1.2 According to Theorem 2, we can deduce that no hyperspace of surface in $\mathbb{R}^m$ like sphere or torus is bi-Lipschitz homogeneous. Moreover, no hyperspace of subspace in $\mathbb{R}^m$ which has non-empty interior is bi-Lipschitz homogeneous.

In this paper, $\mathbb{N}$ denotes the set of all positive integers, $P(A)$ denotes the power set of a set A and $P^*(A) = P(A)\backslash\{\emptyset\}$. For a finite set B, $|B|$ denotes the number of all elements in B. For $m \in \mathbb{R}$, we use $[m]$ to denote the largest integer n such that $n \le m$. We say that a subset S of a metric space (X, d) is ε-discrete if $d(x, y) \ge \varepsilon$ for every pair of points $x, y \in S$. For a compact metric space X and $\varepsilon > 0$, we use $N(X, \varepsilon)$ to denote the maximum cardinality of all ε-discrete subsets of X. For a metric space (X, d) and a compact set A in (X, d), $\overline{B}_{d_H}(A, \varepsilon)=\{B \in 2^X | d_H(B, A) \le \varepsilon\}$, where $\varepsilon > 0$.

In [3], Hohti gave the following result which will be useful in our proofs:

Comparison Principle *Let X and Y be two compact metric spaces. If there is an L-bi-Lipschitz homeomorphism $\varphi : X \to Y$, then for every $\varepsilon > 0$ we have $N(Y, L\varepsilon) \le N(X, \varepsilon)$. If for every $L \ge 1$ there exists an $n \in \mathbb{N}$ with $N(Y, L/n) > N(X, 1/n)$, then Y is not bi-Lipschitz homeomorphic to X.*

Using the Comparison Principle and some estimations of the cardinality of ε-discrete set, the author proved in [3, Proposition 7.2] that $2^{\mathbb{I}}$ is not bi-Lipschitz homogeneous. We prove Theorems 1.1 and 1.2 by similar but more complicated estimations.

2. Proof of Theorem 1.1

Lemma 2.1 *Let $f : \mathbb{I} \to \mathbb{R}$ be a continuous strictly monotone increasing function with $f(0) = 0$ and $f(x + y) \le f(x) + f(y)$. Define a function $\varrho^f : \mathbb{I} \times \mathbb{I} \to [0, +\infty)$ by $\varrho^f(x, y) = f(|x - y|)$. Then ϱ^f is an admissible metric on $\mathbb{I}$ and we have, in $2^{(\mathbb{I}, \varrho^f)}$,*

$$N(\overline{B}_{\varrho^f_H}(\{0\}, \frac{1}{n}), f(\frac{f^{-1}(\frac{1}{n})}{n})) = 2^{n+1} - 1$$

for large enough n.

Proof Clearly, $\varrho^f(x, y) = \varrho^f(y, x)$ and $\varrho^f(x, y) \ge 0$. $\varrho^f(x, y) = 0$ if and only if $x = y$ since $f(0) = 0$ and the function f is strictly monotone increasing. It follows from $f(x+y) \le f(x)+f(y)$ that the function ϱ^f satisfies the triangle inequality. Therefore, ϱ^f is a metric on $\mathbb{I}$. Moreover, ϱ^f is an admissible metric on $\mathbb{I}$ since f is continuous strictly monotone increasing.

The rest of proof is based on large enough n such that $f^{-1}(\frac{1}{n}) \in \mathbb{I}$. Since f is strictly monotone increasing on $\mathbb{I}$,

$$\overline{B}_{\varrho^f_H}(\{0\}, \frac{1}{n}) = 2^{[0, f^{-1}(\frac{1}{n})]}$$

for large enough n. Let $A = \{i\frac{f^{-1}(\frac{1}{n})}{n} \mid 0 \le i \le n\}$. For any two distinct numbers $i_1, i_2 \in \{0, \ldots, n\}$, let $x = i_1\frac{f^{-1}(\frac{1}{n})}{n}$, $y = i_2\frac{f^{-1}(\frac{1}{n})}{n}$. Clearly,

$$\varrho^f(x, y) = f(|i_1 - i_2|\frac{f^{-1}(\frac{1}{n})}{n}) > f(\frac{f^{-1}(\frac{1}{n})}{n}).$$

Hence $P^*(A)$ is an $f(\frac{f^{-1}(\frac{1}{n})}{n})$-discrete subset of $2^{([0, f^{-1}(\frac{1}{n})], \varrho^f)}$. Moreover, $|P^*(A)| = 2^{n+1} - 1$. Suppose Ω is an $f(\frac{f^{-1}(\frac{1}{n})}{n})$-discrete subset of $2^{([0, f^{-1}(\frac{1}{n})], \varrho^f)}$. We define a map $F : \Omega \to P^*(A)$

by

$$D \mapsto \{i\frac{f^{-1}(\frac{1}{n})}{n} \in A \mid ((i-1)\frac{f^{-1}(\frac{1}{n})}{n}, i\frac{f^{-1}(\frac{1}{n})}{n}] \cap D \neq \emptyset, i \in \{1, \dots, n\}\}.$$

Trivially, F is injective. Hence $|\Omega| \leq |P^*(A)| = 2^{n+1} - 1$. We are done. $\Box$

Proof of Theorem 1.1 Since we have proved that ϱ^f is an admissible metric on $\mathbb{I}$ in Lemma 2.1, it suffices to prove that $2^{(\mathbb{I},\varrho^f)}$ is not bi-Lipschitz homogeneous. If $2^{(\mathbb{I},\varrho^f)}$ is bi-Lipschitz homogeneous, then there exists an L-bi-Lipschitz homeomorphism $\psi : 2^{(\mathbb{I},\varrho^f)} \to 2^{(\mathbb{I},\varrho^f)}$ for some positive number L, such that

$$\psi(\{0\}) = \mathbb{I}.$$

The rest of our proof is based on large enough n such that all terms with n are well defined. According to the Comparison Principle and Lemma 2.1,

$$N\big(\psi(\overline{B}_{\varrho^f_H}(\{0\}, \frac{1}{n})), Lf(\frac{f^{-1}(\frac{1}{n})}{n})\big) \leq N\big(\overline{B}_{\varrho^f_H}(\{0\}, \frac{1}{n}), f(\frac{f^{-1}(\frac{1}{n})}{n})\big) \leq 2^{n+1}.$$

Clearly,

$$\overline{B}_{\varrho^f_H}(\mathbb{I}, \frac{1}{Ln}) \subseteq \psi\big(\overline{B}_{\varrho^f_H}(\{0\}, \frac{1}{n})\big).$$

Hence we have

$$N\big(\overline{B}_{\varrho^f_H}(\mathbb{I}, \frac{1}{Ln}), Lf(\frac{f^{-1}(\frac{1}{n})}{n})\big) \leq N\big(\overline{B}_{\varrho^f_H}(\{0\}, \frac{1}{n}), f(\frac{f^{-1}(\frac{1}{n})}{n})\big) \leq 2^{n+1}. \tag{2.1}$$

So we derive an upper bound of $N(\overline{B}_{\varrho^f_H}(\mathbb{I}, \frac{1}{Ln}), Lf(\frac{f^{-1}(\frac{1}{n})}{n}))$.

Now we estimate a lower bound of $N(\overline{B}_{\varrho^f_H}(\mathbb{I}, \frac{1}{Ln}), Lf(\frac{f^{-1}(\frac{1}{n})}{n}))$. Let $r = [\frac{1}{f^{-1}(\frac{1}{Ln})}]$ and $D = \{if^{-1}(\frac{1}{Ln}) | 0 \leq i \leq r\}$. Clearly,

$$\varrho^f_H(D, \mathbb{I}) = f(f^{-1}(\frac{1}{Ln})) = \frac{1}{Ln}.$$

Let D_i be a maximal $Lf(\frac{f^{-1}(\frac{1}{n})}{n})$-discrete subset of open interval $((if^{-1}(\frac{1}{Ln}), (i+1)f^{-1}(\frac{1}{Ln})), \varrho^f)$, $i = 0, \dots, r-1$. Clearly, there exists a number $\alpha_n \in [1, +\infty)$ such that

$$\alpha_n f^{-1}(\frac{1}{Ln})/n = f^{-1}\big(Lf(\frac{f^{-1}(\frac{1}{n})}{n})\big).$$

Hence every $Lf(\frac{f^{-1}(\frac{1}{n})}{n})$-discrete subset of $2^{(\mathbb{I},\varrho^f)}$ is an $\big(\alpha_n f^{-1}(\frac{1}{Ln})/n\big)$-discrete subset of $2^{(\mathbb{I},d)}$, where d is the standard metric on $\mathbb{I}$. Moreover,

$$|D_i| \geq [\frac{f^{-1}(\frac{1}{Ln})}{\alpha_n f^{-1}(\frac{1}{Ln})/n}] = [\frac{n}{\alpha_n}] > 2$$

for large enough n, since $\alpha_n f^{-1}(\frac{1}{Ln}) \to 0$ and $nf^{-1}(\frac{1}{Ln}) = O(1)$ as $n \to +\infty$. Let $E_i = D_i \backslash \{\max D_i, \min D_i\}$. Trivially,

$$|E_i| \geq [\frac{n}{\alpha_n}] - 2 > 0$$

for large enough n. Denote $C = D \cup \bigcup_{i=0}^{r-1} E_i$. Then C is an $Lf(\frac{f^{-1}(\frac{1}{n})}{n})$-discrete subset of $(\mathbb{I}, \varrho^f)$.

Clearly, $C \in \overline{B}_{\varrho_H^f}(\mathbb{I}, \frac{1}{Ln})$. Thus we have

$$N\big(\overline{B}_{\varrho_H^f}(\mathbb{I}, \frac{1}{Ln}), Lf(\frac{f^{-1}(\frac{1}{n})}{n})\big) \geq \prod_{i=0}^{r-1} |P(E_i)| \geq \prod_{k=0}^{r-1} 2^{\left[\frac{n}{\alpha_n}\right]-2} = 2^{\left(\left[\frac{n}{\alpha_n}\right]-2\right)r}$$

for large enough n. Since

$$\big([\frac{n}{\alpha_n}] - 2\big)\big[\frac{1}{f^{-1}(\frac{1}{Ln})}\big] \geq \frac{n}{4\alpha_n f^{-1}(\frac{1}{Ln})}$$

for large enough n and $\alpha_n f^{-1}(\frac{1}{Ln}) \to 0$ as $n \to +\infty$, we have

$$\frac{n}{4\alpha_n f^{-1}(\frac{1}{Ln})} > n+1$$

for large enough n. It follows that

$$N\big(\overline{B}_{\varrho_H^f}(\mathbb{I}, \frac{1}{Ln}), Lf(\frac{f^{-1}(\frac{1}{n})}{n})\big) > 2^{n+1}$$

for large enough n. However, it contradicts the inequality (2.1). $\square$

3. Proof of Theorem 1.2

In this section, we use d^X to denote the metric on X, where X is a metric space. $[a,b]^m$ is endowed with the maximal metric unless otherwise stated for every non-degenerate closed interval $[a,b]$.

To prove Theorem 1.2, we have to prove the following lemmas.

Lemma 3.1 *Suppose $k \in \mathbb{N}$ and $\varepsilon > 0$. For any non-degenerate closed interval $[a,b]$ with $l = \frac{a-b}{\varepsilon} \in \mathbb{N}$, we have*

$$N(2^{[a,b]^k}, \varepsilon) = 2^{(l+1)^k} - 1.$$

Proof Let $A = \{(a + i_1\varepsilon, \dots, a + i_k\varepsilon) \mid 0 \leq i_1, i_2, \dots, i_k \leq l\}$. Then $P^*(A)$ is an ε-discrete subset of $[a,b]^k$ and $|P^*(A)| = 2^{(l+1)^k} - 1$. Suppose Ω is an ε-discrete subset of $2^{[a,b]^k}$. Define a map $F : \Omega \to P^*(A)$ by

$$F(D) = \big\{(a + i_1\varepsilon, \dots, a + i_k\varepsilon) \mid (a + (i_1 - 1)\varepsilon, a + i_1\varepsilon] \times \cdots \times (a + (i_k - 1)\varepsilon, a + i_k\varepsilon] \cap D \neq \emptyset\big\}.$$

Clearly, F is injective. Hence $|\Omega| \leq |P^*(A)| = 2^{(l+1)^k} - 1$. We are done. $\square$

Lemma 3.2 *Let X, Y be two compact metric spaces and $\varphi : X \to Y$ be an L-bi-Lipschitz homeomorphism. Then for $W \in 2^X$ and $\varepsilon > \delta > 0$, we have*

$$N\big(\overline{B}_{d_H^X}(W, \varepsilon), \delta\big) \leq N\big(\overline{B}_{d_H^Y}(\varphi(W), L\varepsilon), \frac{\delta}{L}\big).$$

Proof Suppose $A \in \overline{B}_{d_H^X}(W, \varepsilon)$. Since $d_H^Y\big(\varphi(A), \varphi(W)\big)/L \leq \varepsilon$, $\varphi(A) \in \overline{B}_{d_H^Y}\big(\varphi(W), L\varepsilon\big)$. Moreover, if $E, F \in \overline{B}_{d_H^X}(W, \varepsilon)$ and $d_H^X(E, F) \geq \delta$, then $d_H^Y\big(\varphi(E), \varphi(F)\big) \geq \frac{\delta}{L}$. Hence every δ-discrete subset of $\overline{B}_{d_H^X}(W, \varepsilon)$ is mapped to a $\frac{\delta}{L}$-discrete subset of $\overline{B}_{d_H^Y}\big(\varphi(W), L\varepsilon\big)$. It follows that Lemma 3.2 holds. $\square$

Lemma 3.3 *Let s be a positive constant. Then for large enough n we have*

$$N\big(\overline{B}_{d_H^{[0,2]^m}}([0,2]^m, \frac{1}{ns}), \frac{s}{n^2}\big) \geq 2^{\left([\frac{n}{s^2}]-1\right)^m [2ns]^m}.$$

Proof Let n be a large enough number such that $\frac{1}{ns} < 2$ and $\frac{n}{s^2} > 2$. Denote

$$D = \big\{\big(\frac{i_1}{ns}, \dots, \frac{i_m}{ns}\big) \mid 0 \leq i_j \leq [2ns],\ 0 \leq j \leq m\big\},$$

$$D_1 = \big\{\big(\frac{i_1}{ns}, \dots, \frac{i_m}{ns}\big) \in D \mid 0 \leq i_j < [2ns],\ 0 \leq j \leq m\big\}.$$

Clearly,

$$D \in \overline{B}_{d_H^{[0,2]^m}}([0,2]^m, \frac{1}{ns})$$

and

$$|D_1| = [2ns]^m.$$

For every point $\big(\frac{p_1}{ns}, \dots, \frac{p_m}{ns}\big) \in D_1$, define

$$D_{p_1\cdots p_m} = \big\{\big(\frac{p_1}{ns} + \frac{j_1}{n^2}, \dots, \frac{p_m}{ns} + \frac{j_m}{n^2}\big) \mid 1 \leq j_k \leq [\frac{n}{s^2}] - 1,\ k = 1, \dots, m\big\}.$$

Clearly, $D_{p_1\cdots p_m}$ is $\frac{s}{n^2}$-discrete in $[0,2]^m$ and

$$|D_{p_1\cdots p_m}| = \big([\frac{n}{s^2}] - 1\big)^m.$$

Moreover, let

$$B = \bigcup_{\left(\frac{p_1}{ns}, \dots, \frac{p_m}{ns}\right) \in D_1} D_{p_1\cdots p_m} \cup D.$$

Then B is an $\frac{s}{n^2}$-discrete subset of $[0,2]^m$. It follows that $\Gamma = \big\{A \in P^*(B) \mid A \supseteq D\big\}$ is an $\frac{s}{n^2}$-discrete subset of $\overline{B}_{d_H^{[0,2]^m}}\big([0,2]^m, \frac{1}{ns}\big)$. Moreover,

$$|\Gamma| = |P(B \backslash D)| = 2^{\left([\frac{n}{s^2}]-1\right)^m |D_1|} = 2^{\left([\frac{n}{s^2}]-1\right)^m [2ns]^m}.$$

Therefore,

$$N\big(\overline{B}_{d_H^{[0,2]^m}}([0,2]^m, \frac{1}{ns}), \frac{s}{n^2}\big) \geq |\Gamma| = 2^{\left([\frac{n}{s^2}]-1\right)^m [2ns]^m}. \quad \square$$

Proof of Theorem 1.2 By the assumption of Theorem 2, there exists an open subset U of X such that U and $(a,b)^m$ are bi-Lipschitz homeomorphic. It follows that there exists an open subset V of U such that $\overline{V} \subseteq U$ and $\overline{V}$ is bi-Lipschitz homeomorphic to $[c,d]^m$ for some $[c,d] \subseteq (a,b)$. Obviously, $[c,d]^m$ with the Euclidean metric is bi-Lipschitz homeomorphic to $[0,2]^m$ with the maximal metric. So we endow $[0,2]^m$ with the maximum metric for convenience. We assume that

$$h : \overline{V} \to [0,2]^m$$

is a K-bi-Lipschitz homeomorphism. Let

$$x = h^{-1}\big((1, \dots, 1)\big).$$

Suppose that the hyperspace 2^X is bi-Lipschitz homogeneous. Let $H : 2^X \to 2^X$ be an L-bi-Lipschitz homeomorphism for some positive number L with

$$H(\{x\}) = \overline{V}.$$

According to the Comparison Principle,

$$N\big(H(\overline{B}_{d_H^X}(\{x\}, \frac{1}{n})), \frac{L}{n^2}\big) \leq N\big(\overline{B}_{d_H^X}(\{x\}, \frac{1}{n}), \frac{1}{n^2}\big).$$

The rest of our proof is based on large enough n such that

$$\overline{B}_d(x, \frac{1}{n}) \subseteq \overline{V}.$$

Then

$$\overline{B}_{d_H^X}(\{x\}, \frac{1}{n}) = \overline{B}_{d_H^{\overline{V}}}(\{x\}, \frac{1}{n}),$$

hence

$$H\big(\overline{B}_{d_H^X}(\{x\}, \frac{1}{n})\big) = H\big(\overline{B}_{d_H^{\overline{V}}}(\{x\}, \frac{1}{n})\big).$$

It follows that

$$N\big(H\big(\overline{B}_{d_H^X}(\{x\}, \frac{1}{n})\big), \frac{L}{n^2}\big) = N\big(H\big(\overline{B}_{d_H^{\overline{V}}}(\{x\}, \frac{1}{n})\big), \frac{L}{n^2}\big).$$

Using the Comparison Principle, we have

$$N\big(H\big(\overline{B}_{d_H^{\overline{V}}}(\{x\}, \frac{1}{n})\big), \frac{L}{n^2}\big) \leq N\big(\overline{B}_{d_H^{\overline{V}}}(\{x\}, \frac{1}{n}), \frac{1}{n^2}\big). \tag{3.1}$$

Since H is an L-bi-Lipschitz homeomorphism satisfying $H(\{x\}) = \overline{V}$, we have

$$\overline{B}_{d_H^{\overline{V}}}\big(\overline{V}, \frac{1}{nL}\big) \subseteq H\big(\overline{B}_{d_H^{\overline{V}}}(\{x\}, \frac{1}{n})\big).$$

Hence

$$N\big(\overline{B}_{d_H^{\overline{V}}}(\overline{V}, \frac{1}{nL}), \frac{L}{n^2}\big) \leq N\big(H\big(\overline{B}_{d_H^{\overline{V}}}(\{x\}, \frac{1}{n})\big), \frac{L}{n^2}\big).$$

Therefore,

$$N\big(\overline{B}_{d_H^{\overline{V}}}(\overline{V}, \frac{1}{nL}), \frac{L}{n^2}\big) \leq N\big(\overline{B}_{d_H^{\overline{V}}}(\{x\}, \frac{1}{n}), \frac{1}{n^2}\big).$$

On the other hand, according to Lemma 3.3, for the K-bi-Lipschitz homeomorphism h we have

$$N\big(\overline{B}_{d_H^{\overline{V}}}(\{x\}, \frac{1}{n}), \frac{1}{n^2}\big) \leq N\big(\overline{B}_{d_H^{[0,2]^m)}}(\{(1, \ldots, 1)\}, \frac{K}{n}), \frac{1}{Kn^2}\big). \tag{3.2}$$

Note that for large enough n,

$$\overline{B}_{d_H^{[0,2]^m}}\big(\{(1, \ldots, 1)\}, \frac{([K]+1)^2}{Kn}\big) = 2^{\left[1 - \frac{([K]+1)^2}{Kn}, 1 + \frac{([K]+1)^2}{Kn}\right]^m}.$$

Moreover,

$$\overline{B}_{d_H^{[0,2]^m}}\big(\{(1, \ldots, 1)\}, \frac{K}{n}\big) \subseteq \overline{B}_{d_H^{[0,2]^m}}\big(\{(1, \ldots, 1)\}, \frac{([K]+1)^2}{Kn}\big).$$

It follows from Lemma 3.1 that

$$\begin{aligned} N\big(\overline{B}_{d_H^{[0,2]^m}}(\{(1, \ldots, 1)\}, \frac{K}{n}), \frac{1}{Kn^2}\big) &\leq N\big(\overline{B}_{d_H^{[0,2]^m}}(\{(1, \ldots, 1)\}, \frac{([K]+1)^2}{Kn}), \frac{1}{Kn^2}\big) \\ &= N\big(2^{\left[1 - \frac{([K]+1)^2}{Kn}, 1 + \frac{([K]+1)^2}{Kn}\right]^m}, \frac{1}{Kn^2}\big) \end{aligned}$$

$$= 2^{(2n([K]+1)^2+1)^m} - 1$$

for large enough n. Using inqualities (3.1) and (3.2) we have for large enough n

$$N\big(\overline{B}_{d_H^{\overline{V}}}(\overline{V}, \frac{1}{nL}), \frac{L}{n^2}\big) \le 2^{(2n([K]+1)^2+1)^m} - 1. \tag{3.3}$$

So we derive an upper bound of $N\big(\overline{B}_{d_H^{\overline{V}}}(\overline{V}, \frac{1}{nL}), \frac{L}{n^2}\big)$.

Next, we give a lower bound of $N\big(\overline{B}_{d_H^{\overline{V}}}(\overline{V}, \frac{1}{nL}), \frac{L}{n^2}\big)$. Using Lemmas 3.2 and 3.3, we have

$$\begin{aligned} N\big(\overline{B}_{d_H^{\overline{V}}}(\overline{V}, \frac{1}{nL}), \frac{L}{n^2}\big) &\ge N\big(\overline{B}_{d_H^{[0,2]^m}}(h(\overline{V}), \frac{1}{nKL}), \frac{KL}{n^2}\big) \\ &= N\big(\overline{B}_{d_H^{[0,2]^m}}([0,2]^m, \frac{1}{nKL}), \frac{KL}{n^2}\big). \\ &\ge 2^{\big(\big[\frac{n}{K^2L^2}\big]-1\big)^m[2nKL]^m} \end{aligned}$$

for the K-bi-Lipschitz homeomorphism $h^{-1}: [0,2]^m \to \overline{V}$ and large enough n. Elementarily,

$$2^{\big(\big[\frac{n}{K^2L^2}\big]-1\big)^m[2nKL]^m} > 2^{(2n([K]+1)^2+1)^m}$$

for large enough n. That is, we give a lower bound of $N\big(\overline{B}_{d_H^{\overline{V}}}(\overline{V}, \frac{1}{nL}), \frac{L}{n^2}\big)$ which is larger than one of its upper bound in inequality (3.3) when n is large enough. A contradiction. $\square$

4. Open Problems

After proving Theorem 1.1, it is natural to ask the following question:

Question 4.1 Whether there exists an admissible metric σ on $[0,1]$ such that $2^{([0,1],\sigma)}$ is bi-Lipschitz homogeneous?

According to the remark of Theorem 1.2, it is natural to ask the following question.

Question 4.2 Suppose that a metric space X has a "nice" open subspace whose hyperspace is not bi-Lipschitz homogeneous. Whether 2^X is not bi-Lipschitz homogeneous either?

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