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The Signless Laplacian Spectral Radius of Tricyclic Graphs with a Given Girth

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Abstract A tricyclic graph $G = (V(G), E(G))$ is a connected and simple graph such that $|E(G)| = |V(G)|+2$. Let $\mathscr{T}_n^g$ be the set of all tricyclic graphs on n vertices with girth g. In this paper, we will show that there exists the unique graph which has the largest signless Laplacian spectral radius among all tricyclic graphs with girth g containing exactly three (resp., four) cycles. And at the same time, we also give an upper bound of the signless Laplacian spectral radius and the extremal graph having the largest signless Laplacian spectral radius in $\mathscr{T}_n^g$, where g is even.

Keywords tricyclic graph; signless Laplacian spectral radius; girth.

MR(2010) Subject Classification 05C50; 15A18

1. Introduction

All graphs considered here are connected and simple. Let $G = (V, E)$ be a graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G) = \{e_1, e_2, \ldots, e_m\}$. The order of a graph is the cardinality of its vertex set. Especially, if $m = n + 2$, then G is called a tricyclic graph. Let $N_G(v)$ or $N(v)$ denote the adjacent vertex set of v in G and d_v or $d(v)$ the degree of v. Let $\Delta = \Delta(G)$ be the maximum degree of G. The girth $g = g(G)$ of G is the length of the shortest cycle contained in G. The adjacency matrix of G is $A(G) = (a_{ij})$, where $a_{ij} = 1$ if and only if v_i and v_j are adjacent in G and $a_{ij} = 0$ otherwise. The characteristic polynomial $P(G, x) = |xI_n - A(G)|$ of the adjacency matrix $A(G)$ of G is called the characteristic polynomial of G. The spectrum of $A(G)$ is also called the spectrum of G. Let $D = D(G) = \mathrm{diag}(d_{v_1}, d_{v_2}, \ldots, d_{v_n})$ be the vertex degree diagonal matrix of G. The spectral radius of G, denoted by $\rho_1(G)$, is the largest eigenvalue of its adjacency matrix $A(G)$. The Laplacian spectral radius of G, denoted by $\mu_1(G)$, is the largest eigenvalue of its Laplacian matrix $L(G) = D(G) - A(G)$, and the signless Laplacian spectral radius of G, denoted by $q_1(G)$, is the largest eigenvalue of its signless Laplacian matrix $Q(G) = D(G) + A(G)$. Moreover, if G is connected, by the Perron-Frobenius Theorem, we have that Q-spectral radius is simple and has a unique unit positive eigenvector. We refer to such an eigenvector as Perron vector of G.

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The adjacency matrix $A(G)$ and Laplacian matrix $L(G)$ are studied extensively in the literature (see e.g., books [2–4] and survey papers [1, 19]), respectively. Recently, the problem about determining the extremal graphs with the maximal signless Laplacian spectral radius for a class of graphs attracts people's attention. Some properties of signless Laplacian spectra of graphs and some possibilities for developing the spectral theory of graphs based on $Q(G)$ are discussed in [6–8]. Fan and Yang studied the signless Laplacian spectral radius of graphs with a given number of pendent vertices in [9]. Feng and Yu studied the signless Laplacian spectral radius of unicyclic graphs with a given number of pendent vertices or independence number in [10]. Li, Wang and Zhao studied the signless Laplacian spectral radius of tricyclic graphs and trees with k pendant vertices in [14], and the signless Laplacian spectral radius of unicyclic and bicyclic graphs with a given girth in [13]. Liu, Tan and Liu studied the (signless) Laplacian spectral radius of unicyclic and bicyclic graphs with n vertices and k pendent vertices in [17]. Zhai, Yu and Shu determined the extremal graph with the maximal Laplacian spectral radius among all bicyclic graphs with a given girth in [20]. Li and Yan discussed the Laplacian spectral radius of tricyclic graphs with a given girth [15]. In this paper, we will show that there exists the unique graph which has the largest signless Laplacian spectral radius among all tricyclic graphs with girth g and containing exactly three (resp., four) cycles. Meanwhile, we also give an upper bound of the signless Laplacian spectral radius and the extremal graph having the largest signless Laplacian spectral radius in $\mathscr{T}_n^g$, where g is even.

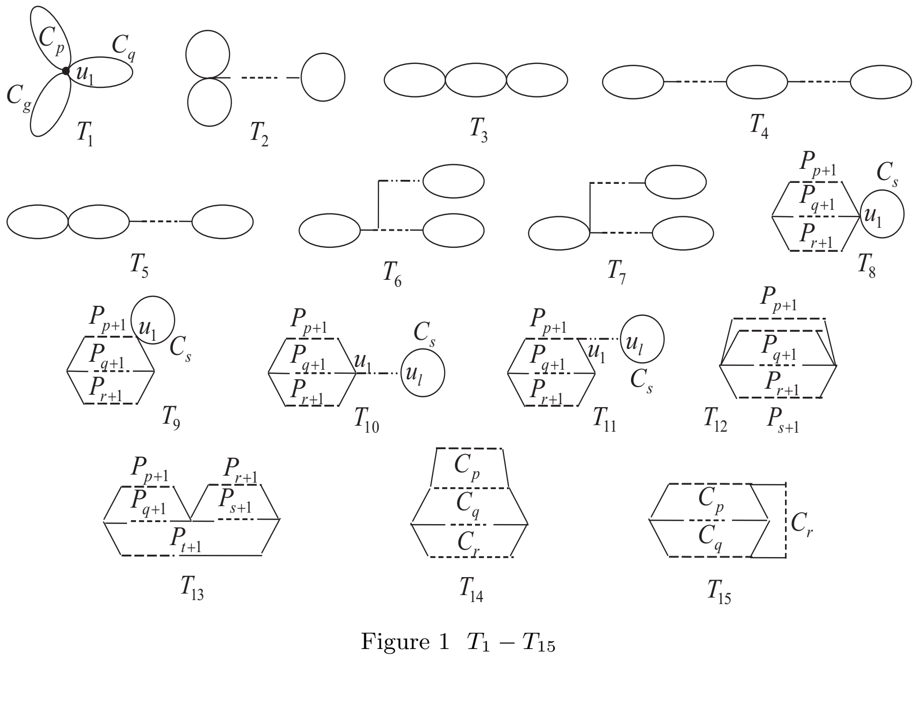

Figure 1 $T_1 - T_{15}$

2. Preliminaries

Denote by C_n and P_n the cycle and the path, respectively, each on n vertices. A pendent edge is an edge incident with a pendent vertex. A path $P = vv_1v_2\cdots v_k$ of G is said to be a pendent path from a vertex v if $d(v_1) = d(v_2) = \cdots = d(v_{k-1}) = 2$, and $d(v_k) = 1$. For convenience, we denote by $\mathscr{T}_n^g$ the set of all the n-vertex tricycle graphs of girth g. We know, by Geng and Li [11], that a tricyclic graph G contains at least 3 cycles and at most 7 cycles, furthermore, there does not exist 5 cycles in G. Let $\mathscr{T}_n^{g,i} \subset \mathscr{T}_n^g$ be the set of all graphs with exact i cycles for $i = 3, 4, 6, 7$. Then $\mathscr{T}_n^g = \mathscr{T}_n^{g,3} \cup \mathscr{T}_n^{g,4} \cup \mathscr{T}_n^{g,6} \cup \mathscr{T}_n^{g,7}$.

For any $G \in \mathscr{T}_n^g$, G can be obtained from some T_i in Figure 1 by attaching trees to some vertices. It is easy to see that each of T_i in Figure 1 is a minimal tricycle graph, i.e., it contains no pendent vertices. For convenience, denote by T_G the minimal tricyclic graph contained in G, if G is an n-vertex tricyclic graph. Furthermore, we say that T_G is of type T_i, $i \in 1, 2, \ldots, 15$, if the arrangement of cycles contained in T_G is the same as that of T_i. Let $T^k_{p,q,r}, T^k_{p,q,r;s}, T^k_{p,q,r,l}$ be the graphs as shown in Figure 2.

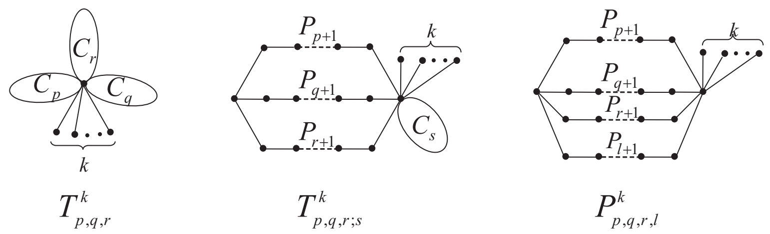

Figure 2 The tricyclic graphs $T^k_{p,q,r}, T^k_{p,q,r;s}, T^k_{p,q,r,l}$

In order to complete the proof of our main results, we need the following lemmas.

Lemma 2.1 ([18]) *Let G be a graph on n vertices. Then*

$$q_1(G) \leq \max\{d_u + m_u : u \in V(G)\},$$

where $m_u = (\sum_{uv\in E(G)} d_v)/d_u$ is the average of the degrees of the vertices of G adjacent to u, the equality holds if and only if G is regular or semi-regular bipartite.

Lemma 2.2 ([18]) *Let G be a simple and connected graph, its degree sequence is $d_{v_1}, d_{v_2}, \ldots, d_{v_n}$. Then we have*

(1) $q_1(G) \leq \max\{\frac{d_u(d_u+m_u)+d_v(d_v+m_v)}{d_u+d_v} : uv \in E\}$.

(2) $q_1(G) \leq \max\{d_u + d_v : uv \in E\}$.

Lemma 2.3 ([12]) *Let G be a connected graph and u, v be the two vertices of G. Suppose $v_1, v_2, \ldots, v_s \in N(v)\backslash\{N(u) \cup \{u\}\}$ $(1 \leq s \leq d_v)$ and G^* is the graph obtained from G by deleting the edges vv_i and adding uv_i $(1 \leq i \leq s)$. Let $X = (x_1, x_2, \ldots, x_n)^t$ be the principal eigenvector of $Q(G)$, where x_i corresponds to v_i $(1 \leq i \leq n)$. If $x_u \geq x_v$, then $q_1(G) < q_1(G^*)$.*

Lemma 2.4 ([6]) *Suppose G is a nontrivial simple and connected graph. Let v be some vertex of G. For nonnegative integers k, l, let $G(k, l)$ denote the graph obtained from G by adding pendant paths of length k, l at v. If $k \geq l \geq 1$, then $q_1(G(k, l)) > q_1(G(k + 1, l - 1))$.*

Lemma 2.5 ([5]) *Let G be a graph on n vertices with at least an edge and the maximum degree* *of G be Δ. Then $q_1(G) \geq \Delta + 1$, the equality holds if only if G is a star $S_n = K_{1,n-1}$.*

Lemma 2.6 ([16]) *If a graph G is a bipartite graph, then $\mu_1(G) = q_1(G)$.*

Lemma 2.7 ([16]) *If G is a graph with at least one edge, then $q_1(G) \geq \mu_1(G) \geq \Delta(G) + 1$. If G is connected, the first equality holds if and only if G is bipartite, the second equality holds if and only if $\Delta = n - 1$.*

Lemma 2.8 *Let G^* have the maximal signless Laplacian spectral radius among all graphs in $\mathscr{T}_n^{g,3}$ (resp., $\mathscr{T}_n^{g,4}$ $\mathscr{T}_n^{g,6}$ $\mathscr{T}_n^{g,7}$). Then G^* is obtained from T_{G^*} by attaching all the pendant edges (if exists) to a unique vertex of T_{G^*}.*

Proof Let $X = (x_1, x_2, \ldots, x_n)^T$ be the principal eigenvector of G^*. Above all, we claim that all pendant vertices of G^* have a unique neighbour. Otherwise, let u_1', u_2' be two pendant vertices with different neighbours u_1, u_2, respectively.

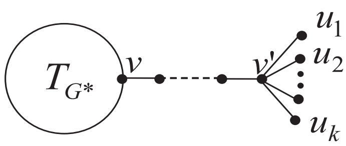

Figure 3 G^* is obtained by jioning T_{G^*} and a star $K_{1,k}$ by a path

Without loss of generality, suppose $x_{u_1} \geq x_{u_2}$. By Lemma 2.3, $q_1(G^*) < q_1(G^* - u_2u_2' + u_1u_2')$, but G^* is an extremal graph, a contradiction. This claim implies that G^* is the graph obtained from T_{G^*} and a star S by joining a path between a vertex v of T_{G^*} and the center v' of the star $K_{1,k}$ (see Figure 3). Now it suffices to show that $v = v'$. Assume to the contrary that $v \neq v'$. If $x_v \geq x_{v'}$, then by Lemma 2.3, $q_1(G^*) < q_1(G')$, a contradiction, where

$$G' = G^* - \bigcup_{i=1}^{k} v'u_i + \bigcup_{i=1}^{k} vu_i.$$

Similarly if $x_v < x_{v'}$, by Lemma 2.3, $q_1(G^*) < q_1(G'')$, a contradiction, where

$$G'' = G^* - \bigcup_{v_i \in N_{T_{G^*}}(v)} vv_i + \bigcup_{v_i \in N_{T_{G^*}}(v)} v'v_i.$$

Thus the proof is completed. □

3. The graph with the largest signless Laplacian spectral radius in $\mathscr{T}_n^{g,3}$

In this section, we will determine the graph with the largest signless Laplacian spectral radius in $\mathscr{T}_n^{g,3}$.

Lemma 3.1 *Let G^* be the graph with the largest signless Laplacian spectral radius among all graphs in $\mathcal{T}_n^{g,p,q} = \{G \in \mathscr{T}_n^{g,3}$: G contains three cycles $C_p, C_q, C_g\}$. Then $G^* \cong T_{g,p,q}^{n-(g+p+q-2)}$ for some $p, q \geq g$.*

Proof Since T_{G^*} is the minimal tricyclic graph contained in G^*, T_{G^*} is a tricyclic graph without pendent vertices. Thus we consider two following cases.

Case 1 G^* has no pendant vertices.

In this case, if G^* is of type T_i (see Figure 1), $i = 2, 4, 6, 7$. By Lemma 2.1, $q_1(G^*) < \max\{d_u + m_u : u \in V(G^*)\} < 5 + 3 = 8$ since G^* cannot be regular or semiregular in such case. However, $q_1(T^1_{g,p,q}) \geq \Delta + 1 = 8$. For $|V(G^*)| \geq |V(T^1_{g,p,q,})|$, we can find a graph $G \in \mathcal{T}_n^{g,p,q}$ that contains $T^1_{g,p,q}$ as a subgraph. This means $q_1(G) \geq q_1(T^1_{g,p,q}) > q_1(G^*)$, a contradiction.

If G^* is of type T_3, by Lemma 2.1 $q_1(G^*) < \max\{d_u + m_u : u \in V(G^*)\} = 6.5$ for G^* cannot be regular or semiregular in this case. Since $q_1(T^0_{g,p,q}) \geq \Delta + 1 = 7$ and $|V(G^*)| \geq |V(T^0_{g,p,q})|$, we can find a graph $G \in \mathcal{T}_n^{g,p,q}$ that contains $T^0_{g,p,q}$ as a subgraph. Then $q_1(G) \geq q_1(T^0_{g,p,q}) > q_1(G^*)$, a contradiction.

Thus we obtain that $G^* \cong T_{g,p,q}^{n-(g+p+q-2)}$ for some $p, q \geq g$ with $n = g + p + q - 2$.

Case 2 G^* has pendant vertices.

In this case, by Lemma 2.8, we know that G^* is a tricyclic graph obtained from T_{G^*} and a star by identifying the center of the star with a vertex, say v, of T_{G^*}. Denote $V^* = \{u : d_{T_{G^*}}(u) \geq 3\}$. It is clear that $|V^*| \leq 4$ in such case. Next we will show that $v \in V^*$ and $|V^*| = 1$.

First, assume that $v \notin V^*$. It is apparent that $|V^*| \geq 1$, thus we choose $u \in V^*$ on some cycle, say C_p and v is not on C_p. Denote $N_{C_p}(u) = \{w_1, w_2\}$. Let $X = (x_1, x_2, \ldots, x_n)$ be the principal eigenvector of G^*. Hence $x_i \geq 0, i = 1, 2, \ldots, n$. Let

$$G' = G^* - \{uw_1, uw_2\} + \{vw_1, vw_2\}.$$

Then $G' \in \mathcal{T}_n^{g,p,q}$. If $x_v \geq x_u$, by Lemma 2.3 we get $q_1(G^*) < q_1(G')$, a contradiction. Denote by $\{u_1, u_2, \ldots, u_k\}$ the set of pendant vertices adjacent to v. Let

$$G'' = G^* - \{vu_1, vu_2, \ldots, vu_k\} + \{uu_1, uu_2, \ldots, uu_k\}.$$

Then $G'' \in \mathcal{T}_n^{g,p,q}$. If $x_v < x_u$, then by Lemma 2.3 we have $q_1(G^*) < q_1(G'')$, a contradiction. Thus we get $v \in V^*$.

Next, we show that $|V^*| = 1$. To the contrary, we assume that $|V^*| \geq 2$. Then there exists another vertex $w \in V^*$ different from v. By a similar discussion as above, we can get that in $\mathcal{T}_n^{g,p,q}$ there is a graph G such that $q_1(G^*) < q_1(G)$, a contradiction.

Finally, we obtain that $G^* \cong T_{g,p,q}^{n-(g+p+q-2)}$ for some $p, q \geq g$. Thus the proof is completed. □

Theorem 3.2 *Let G^* be the graph with the largest signless Laplacian spectral radius among all graphs in $\mathscr{T}_n^{g,3}$, where $n \geq 3g - 2$. Then $G^* \cong T_{g,g,g}^{n-3g+2}$ and $q_1(G^*) < n - 3g + 9 + \frac{6}{n-3g+8}$.*

Proof From Lemma 3.1, we know that $G^* \cong T_{g,p,q}^{n-(g+p+q-2)}$ for some $p, q \geq g$. Now we show $p = q = g$. Suppose that $p \geq g + 1$, then $n \geq 3g - 1$. Let $k = n - (g + p + q - 2)$ be the number of pendant vertices of G^*. Then $k \leq n - 3g + 1$. Since G^* cannot be regular or semiregular, by

Lemma 2.1, we get

$$q_1(G^*) < \max\{d_u + m_u : u \in V(G^*)\} = k + 6 + \frac{k+12}{k+6} = k + 7 + \frac{6}{k+6}.$$

Clearly, $k + 7 + \frac{6}{k+6}$ is increasing with nonegative number k. Hence

$$q_1(G^*) < n - 3g + 8 + \frac{6}{n-3g+9} \le n - 3g + 9$$

for $n \ge 3g - 1$. However, by Lemma 2.7, $q_1(T^{n-3g+2}_{g,g,g}) \ge \Delta + 1 = n - 3g + 9 > q_1(G^*)$, a contradiction. Thus $p = g$. Similarly, we can obtain that $q = g$, which implies $G^* \cong T^{n-3g+2}_{g,g,g}$. Since $T^{n-3g+2}_{g,g,g}$ is neither regular nor semiregular,

$$q_1(G^*) < \max\{d_u + m_u : u \in V(T^{n-3g+2}_{g,g,g})\} = n - 3g + 9 + \frac{6}{n-3g+8}.$$

Thus the proof is completed. $\Box$

4. The graph with the largest signless spectral radius in $\mathscr{T}_n^{g,4}$

Denote by $P(p,q,r)$ the graph consisting of three pairwise internal disjoint paths P_{p+1}, P_{q+1}, P_{r+1} with common endpoints. For all $G \in \mathscr{T}_n^{g,4}$, we have that T_G is of type T_i, $i = 8, 9, 10, 11$ (see Figure 1). Let C_s be a cycle with $s \ge g$ and $P_l = u_1u_2\cdots u_l$ the path connecting $P(p,q,r)$ and C_s, where $u_1 \in V(P(p,q,r))$ and $u_l \in V(C_s)$. When $l = 1$, T_G is of type T_8 or T_9. When $l \ge 2$ we have that T_G is of type T_{10} or T_{11}. Let G and H be two disjoint graphs with $u \in V(G)$ and $v \in V(H)$. we denote by $GuvH$ the graph obtained from G and H by identifying u with v.

Lemma 4.1 *Let G^* be the graph with the largest sigless Laplacian spectral radius among graphs in $\mathscr{T}_n^{g,4}$. Then $G^* \cong T_{G^*}u_1vS$, where T_{G^*} is of type T_8 or T_9, u_1 is the vertex of maximum degree in T_{G^*} and v is the center of the star S.*

Proof By Lemma 2.8 we know that G^* is a graph obtained from T_{G^*} and a star S by identifying a vertex u of T_{G^*} with the center v of S. That is $G^* \cong T_{G^*}uvS$. Let k be the number of pendant vertices of G^*. Then $k \ge 0$ and $|V(S)| = k + 1$.

First, we show that T_{G^*} is neither of type T_{10} nor T_{11}. Assume that T_{G^*} is of type T_{10} or T_{11}. By Lemma 2.1, we have

$$q_1(G^*) < \max\{d_u + m_u : u \in V(G^*)\} = \max\{k + 4 + \frac{k+10}{k+4}, k + 3 + \frac{k+9}{k+3}\} < k + 7$$

for $k \ge 0$. But $q_1(T^{k+l-1}_{p,q.r;s}) \ge \Delta+1 = k+l+5 \ge k+7$ for $l \ge 2$. Thus $q_1(G^*) < k+7 \le q_1(T^{k+l-1}_{p,q.r;s})$. This contradicts the hypothesis.

Next we will show that $u = u_1$. By contradiction, suppose that $u \ne u_1$. Let $X = (x_1, x_2, \ldots, x_n)^T$ be the principal eigenvector of G^*. Then $x_i \ge 0$, $i = 1, 2, \ldots, n$. If $x_u \le x_{u_1}$, by Lemma 2.3, $q_1(G^*) < q_1(T_{G^*}u_1vS)$, a contradiction. If $x_u > x_{u_1}$, then we have two subcases. If $u \in C_s$, denote by v_1, v_2, v_3 the neighbours of u_1 not in C_s, then $q_1(G^*) < q_1(G')$, where $G' = G^* - \{u_1v_1, u_1v_2, u_1v_3\} + \{uv_1, uv_2, uv_3\}$, a contradiction. If $u \notin C_s$, then $q_1(G^*) < q_1(T_{G^*}uvS)$, a contradiction. That means $u = u_1$ as desired. The proof is completed. $\Box$

Theorem 4.2 *Let G^* be the graph with the maximal signless Laplacian spectral radius among all graphs in $\mathscr{T}_n^{g,4}$, where $n \geq \lceil \frac{5}{2}g \rceil - 2$. Then $G^* \cong T^{n-\lceil \frac{5}{2}g \rceil+2}_{\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lceil \frac{g}{2} \rceil; g}$ and $q_1(G^*) < n - \lceil \frac{5}{2}g \rceil + 8 + \frac{6}{n-\lceil \frac{5}{2}g \rceil+7}$.*

Proof By Lemma 4.1, we have that $G^* \cong T_{G^*}u_1vS$, where $T_{G^*}u_1vS$ is defined in Lemma 4.1. Clearly T_{G^*} is obtained by attaching a cycle C_s to a vertex of $P(p,q,r)$. Without loss of generality, let $s \geq g, p \leq q \leq r$ and $p+q=g$.

Case 1 T_{G^*} is of type T_8.

In this case, let $g = 2a$ if g is even and $g = 2a+1$ otherwise. Then we only need to show that $p = q = r = a$ and $s = 2a$ if $g = 2a$, or $p = a, q = r = a+1$ and $s = 2a+1$ if $g = 2a+1$. Here we prove the latter case only. The former case can be proved similarly, and we omit the procedure here. It is clear that $p+q = 2a+1, p+r \geq 2a+1, q+r \geq 2a+1$ and $s \geq 2a+1$ when $g = 2a+1$, then we have $p+q+r+s \geq 5a+\frac{5}{2} > 5a+2$. Thus $n \geq (p+q+r+s)-2 \geq 5a+1$ and $0 \leq k \leq n-5a-1$.

If $n = \lceil \frac{5}{2}g \rceil - 2 = 5a+1$, then G^* contains no pendent edges and $p+q+r+s = 5a+3$. For $p+q = 2a+1$, we have $r+s = 3a+2$. Note that $p+q+r \geq 3a+2$, thus $r \geq a+1$ and then $s \leq 2a+1$. For $s \geq g = 2a+1$, we have $s = 2a+1$, whence $r = a+1$. Therefore, $p \geq q \geq r = a+1$. If $p = a-1$, then $q = a+2$, a contradiction to $q \leq r = a+1$. Hence, $p \geq a$. If $p = a+1$, then $q = a$, a contradiction to $p \leq q$. Therefore, $p = a$. Hence, $q = a+1$. That is $p = a, q = r = a+1, s = 2a+1$.

If $n = \lceil \frac{5}{2}g \rceil - 1 = 5a+2$, then $p+q+r+s \leq n+2 = 5a+4$. Since $p+q = 2a+1, s \geq 2a+1$, we have $a+1 \leq r \leq a+2$, which implies that $q \leq a+2$ and $p \geq a-1$. Thus $(p,q,r,s) \in \{(a,a+1,a+2,2a+1),(a-1,a+2,a+2,2a+1),(a,a+1,a+1,2a+1),(a,a+1,a+1,2a+2)\}$. If $(p,q,r,s) \in \{(a,a+1,a+2,2a+1),(a-1,a+2,a+2,2a+1),(a,a+1,a+1,2a+2)\}$, then G^* contains no pendant edges. When $a = 2$, then $G^* \cong G_1, G_2$, or G_3 (see Figure 4). By direct calculations, we have $q_1(G_1) = 6.2716, q_1(G_2) = 6.3494, q_1(G_3) = 6.2752$, whereas $q_1(T^1_{2,3,3;5}) = 7.1809$, thus $q_1(G_i) < q_1(T^1_{2,3,3;5}), i = 1,2,3$, a contradiction. When $a \geq 3$, then the 3-vertex and 5-vertex are not adjacent in G^*. Hence by Lemma 2.2(2), $q_1(G^*) \leq \max\{d_u + d_v : uv \in E(G^*)\} = 7$. But, $q_1(T^1_{a,a+1,a+1;2a+1}) > \Delta + 1 = 7$ by Lemma 2.7. Thus $q_1(G^*) < q_1(T^1_{a,a+1,a+1;2a+1})$, a contradiction. That is $p = a, p = r = a+1, s = 2a+1$.

If $n \geq \lceil \frac{5}{2}g \rceil = 5a+3$, note that $p \leq q \leq r$ and $p+q = g = 2a+1$, then $r \geq a+1$. If $r \geq a+2$, then $p+q+r+s \geq 5a+4$ and $k \leq n-5a-2$. By Lemma 2.1, we have

$$q_1(G^*) < \max\{d_u + m_u : u \in V(G^*)\} = k+5+\frac{k+11}{k+5} = k+6+\frac{6}{k+5}$$

as G^* cannot be regular or semiregular. Note that $k+6+\frac{6}{k+5}$ is increasing with nonnegative number k. Hence $q_1(G^*) < n-5a+4+\frac{6}{n-5a+3} \leq n-5a+5$ for $n \geq 5a+3$. However, by Lemma 2.5, $q_1(T^{n-5a-1}_{a,a+1,a+1;2a+1}) \geq \Delta+1 = n-5a+5 > q_1(G^*)$, a contradiction. Then $r < a+2$. That is $r = a+1$. Then it is clear that $p = a, q = a+1$. Now it suffices to show that $s = 2a+1$.

Assume that $s \geq 2a+2$, then $k = n-(p+q+r+s)+2 \leq n-5a-2$. By Lemma 2.1, we have

$$q_1(G^*) < \max\{d_u + m_u : u \in V(G^*)\} = k+5+\frac{k+11}{k+6} = k+6+\frac{6}{k+5}$$

for G^* cannot be regular or semiregular. Note that $k+6+\frac{6}{k+5}$ is increasing with nonnegative number k. Thus $q_1(G^*) < n-5a+4+\frac{6}{n-5a+3} \leq n-5a+5$ for $n \geq 5a+3$. However by Lemma 2.5, $q_1(T^{n-5a-1}_{a,a+1,a+1;2a+1}) \geq \Delta+1 = n-5a+5 > q_1(G^*)$, a contradiction. That is $s = 2a+1$.

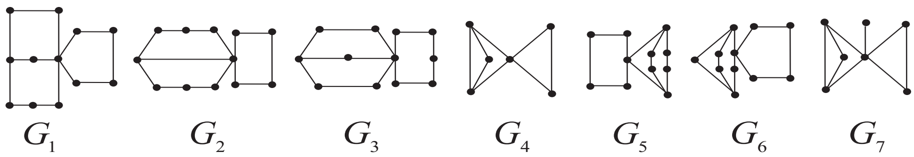

Figure 4 $G_1, G_2, \ldots, G_7$

Therefore, $p = a, q = r = a+1, s = 2a+1$ and then $G^* \cong T^{n-\lceil \frac{5}{2}g\rceil+2}_{\lfloor \frac{g}{2}\rfloor,\lceil \frac{g}{2}\rceil,\lceil \frac{g}{2}\rceil;g}$.

Case 2 T_{G^*} is of type T_8.

Similarly to Case 1, we can obtain that, in this case, $p = q = r = a$ and $s = 2a$ in T_{G^*} if the girth g is even, and $p = a, q = r = a+1$ and $s = 2a+1$ in T_{G^*} if g is odd. We only discuss the latter case here. The former case can be proved by the similar way, and we omit the procedure here. In the latter case, let k be the number of pendant vertices of G^*. Then $k = n-(p+q+r+s-2) = n-5a+1$. Note there exists a 4-vertex, denote by u_1, and two 3-vertices, say u_2, u_3 in T_{G^*}. We now show that T_{G^*} is not of type T_9.

If $k = 0$, then $n = 5a+1$. First, if $a = 1$, then $G^* \cong G_4$ (see Figure 4). By direct calculations, we have $q_1(G_4) = 6.0000, q_1(T^0_{1,2,2;3}) = 6.6262$. Hence $q_1(G^*) < q_1(T^0_{1,2,2;3})$, a contradiction. Second, if $a = 2$, then $G^* \cong G_5$ or G_6 (see Figure 4). By direct calculations, we have $q_1(G_5) = 5.6585, q_1(G_6) = 5.5560$, but $q_1(T^0_{2,3,3;5}) = 6.2791$. Hence $q_1(G^*) < q_1(T^0_{2,3,3;5})$. Third, if $a \geq 3$, then $g = 2a+1 \geq 7$ and $G^*[u_1,u_2,u_3]$ and C_3 are not isomorphic. Thus by Lemma 2.2(1), we get

$$q_1(G^*) < \max\Big\{\frac{4(4+\frac{9}{4})+3(3+\frac{8}{3})}{4+3}, \frac{4(4+\frac{9}{4})+2(2+\frac{6}{2})}{4+2}, \frac{3(3+\frac{8}{3})+2(2+\frac{5}{2})}{3+2}, \frac{2(2+\frac{5}{2})+2(2+\frac{6}{2})}{2+2}\Big\} = 6,$$

for G^* is neither regular nor semiregular. And by Lemma 2.5, $q_1(T^0_{a,a+1,a+1;2a+1}) \geq \Delta+1 = 6$, hence $q_1(T^0_{a,a+1,a+1;2a+1}) \geq 6 > q_1(G^*)$, a contradiction.

If $k = 1$, thus there exists exactly one pendant vertex attached to u_1 in G^*. We firstly consider $a = 1$. That is $G^* \cong G_7$ (see Figure 4). We can obtain $q_1(G_7) = 6.6352, q_1(T^1_{1,2,2;3}) = 7.4346$ by direct calculations. That is $q_1(G_7) < q_1(T^1_{1,2,2;3})$. Secondly, we discuss $a \geq 2$. By Lemma 2.2(1), we get

$$q_1(G^*) < \max\Big\{\frac{5(5+\frac{11}{5})+3(3+\frac{9}{3})}{5+3}, \frac{5(5+\frac{11}{5})+2(2+\frac{7}{2})}{5+2}, \frac{5(5+\frac{11}{5})+6}{5+1}, \frac{3(3+\frac{9}{3})+2(2+\frac{5}{2})}{3+2}, \frac{2(2+\frac{5}{2})+2(2+\frac{7}{2})}{2+2}\Big\} = 7,$$

for G^* is neither regular nor semiregular. And by Lemma 2.5, $q_1(T^1_{a,a+1,a+1;2a+1}) \geq \Delta + 1 = 7$, hence $q_1(T^1_{a,a+1,a+1;2a+1}) \geq 7 > q_1(G^*)$, a contradiction.

If $k \geq 2$, we can get, by Lemma 2.1,

$$q_1(G^*) < \max\{d_u + m_u : u \in V(V^*)\} = k + 4 + \frac{k+10}{k+4} \leq k + 6,$$

for $k \geq 2$. However by Lemma 2.5, we have $q_1(T^k_{a,a+1,a+1;2a+1}) \geq \Delta + 1 = k + 6$. Thus $q_1(T^k_{a,a+1,a+1;2a+1}) \geq k + 6 > q_1(G^*)$, a contradiction.

By the above discussion, we obtain $G^* \cong T^{n-\lceil \frac{5}{2}g\rceil+2}_{\lfloor \frac{g}{2}\rfloor,\lceil \frac{g}{2}\rceil,\lceil \frac{g}{2}\rceil;g}$. Finally we can get, by Lemma 2.1,

$$q_1(G^*) < \max\{d_u + m_u : u \in V(G^*)\} = k + 6 + \frac{6}{k+5} = n - \lceil \frac{5}{2}g\rceil + 8 + \frac{6}{n - \lceil \frac{5}{2}g\rceil + 7}.$$

Thus the proof is completed. □

5. The graph with the largest signless Laplacian spectral radius in $\mathscr{T}_n^{g,6}$ $\cup \mathscr{T}_n^{g,7}$

For all $G \in \mathscr{T}_n^{g,6} \cup \mathscr{T}_n^{g,7}$, we have that T_G is of type T_i, $i = 12, 13, 14, 15$. Let $P(p, q, r, l)$ be the graph consisting of four pairwise internal disjoint paths P_{p+1}, P_{q+1}, P_{r+1}, P_{l+1} with common endpoints. Then we get $T_{12} = P(p, q, r, l)$.

Lemma 5.1 *Let G^* be the graph with the largest signless Laplacian spectral radius among all graphs in $\mathscr{T}_n^{g,6} \cup \mathscr{T}_n^{g,7}$, where $n \geq 2g-2$. If T_{G^*} is of type T_{12} and g is even, then $G^* \cong P^{n-2g+2}_{\frac{g}{2},\frac{g}{2},\frac{g}{2},\frac{g}{2}}$ and $q_1(G^*) < n - 2g + 7 + \frac{6}{n-2g+6}$.*

Proof By Lemma 2.8, we have that G^* can be obtained from $P(p, q, r, l)$ by attaching $n - (p + q + r + l) + 2$ pendent vertices to a unique vertex of T_{G^*}. Then we suppose that $T_{G^*} = P(p, q, r, l)$ with $p \leq q \leq r \leq l$. Let $g = 2a$. Then $p + q = g = 2a$.

First, we show that $p = q = r = l = \frac{g}{2}$. As $g = 2a$, we obtain $p + q + r + l \geq 4a$ and $n \geq p + q + r + l \geq 4a - 2$. Then we have four cases as follows:

(1) If $n = 4a + 2$, then $G^* \cong T_{G^*}$ and $p + q + r + l = 4a$. Thus $r + l = 2a$ for $p + q = 2a$. Since $p \leq q \leq r \leq l$, we have $l \geq r \geq a$. Therefore we have $p = q = r = l = a$.

(2) If $n = 4a - 1$, then $p + q + r + l \leq 4a - 1$, and $r + l \leq 2a + 1$. And obviously $r + l \geq g = 2a$, hence $r + l = 2a$ or $r + l = 2a + 1$. Since $p + q = 2a$ and $p \leq q \leq r \leq l$, we have that the 4-tuple $(p, q, r, l) \in \{(a, a, a, a), (a, a, a, a + 1)\}$. If $(p, q, r, l) = (a, a, a, a + 1)$, $G^* \cong P(a, a, a, a + 1)$. Since $g > 3$, we get that $a \geq 2$ and the two 4-vertices of G^* are not adjacent. Then we have $q_1(G^*) \leq \max\{d_u + d_v : uv \in E(G^*)\} = 6$ by Lemma 2.2(2). And by Lemma 2.5, $q_1(P^1_{a,a,a,a}) > \Delta + 1 = 6$ since G^* is not a star. Thus we have $q_1(G^*) < q_1(P^1_{a,a,a,a})$, a contradiction. Therefore we have $p = q = r = l = a$.

(3) If $n = 4a$, then $p + q + r + l \leq 4a + 2$. As $p + q = 2g$, we have $r + l \leq 2a + 2$. And apparently $r + l \geq g = 2a$, hence $r + l = 2a, r + l = 2a + 1$, or $r + l = 2a + 2$. Since $p + q = 2a$ and $p \leq q \leq r \leq l$, we have that the 4-tuple $(p, q, r, l) \in \{(a, a, a, a), (a, a, a, a + 1), (a, a, a, a +$

$2), (a-1, a+1, a+1, a+1), (a, a, a+1, a+1)\}$. It suffices to show that $(p, q, r, l) \notin \{(a, a, a, a+1), (a, a, a, a+2), (a-1, a+1, a+1, a+1), (a, a, a+1, a+1)\}$. If $(p, q, r, l) = (a, a, a, a+1)$, then $T_{G^*} \cong P(a, a, a, a+1)$. By Lemma 2.2(2), we obtain that $q_1(G^*) \leq \max\{d_u + d_v : uv \in E(G^*)\} = \max\{3+4, 4+2, 3+2, 2+2, 5+2, 5+1, 4+1\} = 7$. However, $q_1(P^2_{a,a,a,a,}) > \Delta + 1 = 7$ since $P^2_{a,a,a,a}$ is not a star. Then $q_1(G^*) < q_1(P^2_{a,a,a,a})$, a contradiction. If $(p, q, r, l) \in \{(a, a, a, a+2), (a-1, a+1, a+1, a+1), (a, a, a+1, a+1)\}$, then we have $G^* \cong P(a, a, a, a+2), P(a-1, a+1, a+1, a+1)$ or $P(a, a, a+1, a+1)$. If $a = 2$, by direct calculations, we obtain $\max\{q_1(P(2,2,2,4)), q_1(P(1,3,3,3)), q_1(P(2,2,3,3))\} < q_1(P^2_{2,2,2,2})$, a contradiction. If $a \geq 3$, then the two vertices of degree 4 of G^* are not adjacent. Thus, by Lemma 2.2(2), we get $q_1(G^*) \leq \max\{d_u + d_v : uv \in E(G^*)\} = 6$. However $q_1(P^2_{a,a,a,a}) > \Delta + 1 = 7$ since $P^2_{a,a,a,a}$ is not a star. Hence $q_1(G^*) < q_1(P^2_{a,a,a,a})$, a contradiction. Therefore, we have $p = q = r = l = a$.

(4) If $n \geq 4a+1$, then it is clear that $p = q = r = a$ when $l = a$. Suppose that $l \geq a+1$. Let k be the number of pendant vertices of G^*. Then $k = n - (p+q+r+l) + 2$. Thus we have $k \leq n - 4a + 1$. By Lemma 2.1, we obtain that

$$q_1(G^*) \leq \max\{d_u + m_u : u \in V(G^*)\} = k + 4 + \frac{k+10}{k+4} = k + 5 + \frac{6}{k+5}.$$

Note that $k + 5 + \frac{6}{k+5}$ is increasing with nonnegative number k. Hence $q_1(G^*) \leq n - 4a + 6 + \frac{6}{n-4a+5} \leq n - 4a + 7$ for $n \geq 4a+1$. By Lemma 2.5 we have $q_1(P^{n-4a+2}_{a,a,a,a}) > \Delta + 1 = n - 4a + 7 \geq q_1(G^*)$, a contradiction. Hence we have $p = q = r = l = a$.

Therefore, we have that $T_{G^*} \cong P(\frac{g}{2}, \frac{g}{2}, \frac{g}{2}, \frac{g}{2})$.

Secondly, we show that $G^* \cong P^{n-2g+2}_{\frac{g}{2},\frac{g}{2},\frac{g}{2},\frac{g}{2}}$. If not, we have that G^* is obtained from $P(a, a, a, a)$ by attaching $n - 2g + 2$ pendent vertices to a unique vertex u of $P(a, a, a, a)$, which is of degree 2. Let $k = n - 2g + 2$. Then $k \geq 1$, otherwise $G^* \cong P^0_{a,a,a,a}$. By Lemma 2.1, we obtain

$$q_1(G^*) < \max\{d_u + m_u : u \in V(G^*)\} = k + 2 + \frac{k+8}{k+2} \leq k + 5,$$

since G^* is neither regular nor semiregular and $k \geq 1$. However we get $q_1(P^k_{a,a,a,a}) \geq \Delta + 1 = k + 5 > q_1(G^*)$, a contradiction. Thus we have $G^* \cong P^{n-2g+2}_{\frac{g}{2},\frac{g}{2},\frac{g}{2},\frac{g}{2}}$.

Now we show that $q_1(G^*) < n - 2g + 7 + \frac{6}{n-2g+6}$. Note that $P^0_{2,2,2,2} \cong K_{2,4}$, then $g = 4, n = 2 \times 4 - 2 = 6$. Thus we have $q_1(P^0_{2,2,2,2}) = 6 < \frac{13}{2} = n - 2g + 7 + \frac{6}{n-2g+6}$. Therefore, if $(n, g) \neq (6, 4)$, $P^{n-2g+2}_{\frac{g}{2},\frac{g}{2},\frac{g}{2},\frac{g}{2}}$ is neither regular nor semiregular. By Lemma 2.1, we have that

$$q_1(G^*) < \max\{d_u + m_u : u \in V(G^*)\} \leq n - 2g + 7 + \frac{6}{n-2g+6}.$$

Therefore, the proof is completed. □

Lemma 5.2 *Let G^* be the graph with the largest signless Laplacian spectral radius among all graphs in $\{G : G \in \mathscr{T}^{g,6}_n \cup \mathscr{T}^{g,7}_n$ and G contains at least three pendant vertices$\}$ with $T_{G^*} \cong T_{13}, T_{14}$ or T_{15} (see Figure 1). Then all the pendent edges of G^* are attached to the vertex of maximum degree of T_{G^*} and the length of each of the three independent cycles is the girth g.*

Proof Denote by C_p, C_q, C_r the three independent cycles in T_{13}, T_{14}, or T_{15}, where p, q, r are the length of the cycles. Thus we have $p, q, r \geq g$. Let k be the number of pendant vertices of G^*. Thus $3 \leq k = n - |V(T_{G^*})|$. We will show that the Lemma is true when T_{G^*} is of type T_{13}. The other two cases can be proved similarly, and we omit the procedure here.

First, we show that G^* can be obtained from T_{G^*} by attaching all the pendent edges to one of the two vertices of degree four in T_{G^*}. If not, by Lemma 2.8 we have that the neighbour of all pendant vertices is of degree 3 or 2. By Lemma 2.1, we obtain

$$\begin{aligned} q_1(G^*) &< \max\{d_u + m_u : u \in V(G^*)\} \\ &= \max\{k+3+\frac{k+9}{k+3}, k+2+\frac{k+7}{k+2}, 4+\frac{k+10}{4}, 3+\frac{k+9}{3}, 2+\frac{k+7}{2}\} \\ &\leq k+5, \end{aligned}$$

since G^* is neither regular nor semiregular and $k \geq 3$. However, by Lemma 2.5 we have $q_1(G') \geq \Delta + 1 = k + 5$, where G' is obtained from T_{G^*} by attaching k pendant edges to one of the two vertices of degree four in T_{G^*}. Then $q_1(G^*) < q_1(G')$, a contradiction.

Secondly, we show that $p = q = r = g$. If not, then there exists at least one number, say p, such that $p \geq g + 1$. Then by Lemma 2.1, we have

$$q_1(G^*) < \max\{d_u + m_u : u \in V(G^*)\} = k + 4 + \frac{k+10}{k+4} \leq k + 6.$$

Let G'' be a graph obtained from T_{13} by attaching k' pendant vertices to its 4-vertex such that the three independent cycles are C_g, C_q, C_r and $|V(G'')| = n$. Hence $k' \geq k + 1$, and then $q_1(G'') \geq \Delta + 1 = k' + 5 \geq k + 6 > q_1(G^*)$, a contradiction. Thus we obtain $p = q = r = g$.

With the similar method, we can show that the Lemma is true when T_{G^*} is of type T_{14} or T_{15}. Thus the proof is completed. $\Box$

Theorem 5.3 *Let G^* be the graph with the largest signless Laplacian spectral radius among graphs in $\{G : G \in \mathscr{T}_n^{g,6} \cup \mathscr{T}_n^{g,7}$ and G contains at least three pendant vertices$\}$. If g is even, then $G^* \cong P^{n-2g+2}_{\frac{g}{2},\frac{g}{2},\frac{g}{2},\frac{g}{2}}$ and $n \geq 2g + 3$.*

Proof By Lemmas 5.1 and 5.2, it is enough for us to show that T_{G^*} is not of type T_{13}, T_{14} or T_{15}. Let k be the number of pendant vertices of G^*. Thus $k = n - |V(G^*)| \geq 3$.

If T_{G^*} is of type T_{13}, then by Lemma 5.2 we have that the length of the three independent cycles in G^* is g. Thus $|V(G^*)| \geq 2g - 1$. Therefore, $3 \leq k \leq n - |V(G^*)| \leq n - 2g + 1$, and $n \geq 2g + 2$. By Lemma 2.1 we have that

$$q_1(G^*) < \max\{d_u + m_u : u \in V(G^*)\} = k + 4 + \frac{k+10}{k+4} = k + 5 + \frac{6}{k+4}.$$

Since $k + 5 + \frac{6}{k+4}$ is increasing with nonnegative number k, we obtain $q_1(G^*) < n - 2g + 6 + \frac{6}{n-2g+5} < n - 2g + 7$ for $n \geq 2g + 2$. However $q_1(P^{n-2g+2}_{\frac{g}{2},\frac{g}{2},\frac{g}{2},\frac{g}{2}}) \geq \Delta + 1 = n - 2g + 7$. That is $q_1(G^*) < q_1(P^{n-2g+2}_{\frac{g}{2},\frac{g}{2},\frac{g}{2},\frac{g}{2}})$, a contradiction.

If T_{G^*} is of type T_{14}, then $|V(G^*)| \geq 2g$. Thus $3 \leq k \leq n - |V(G^*)| = n - 2g$ and $n \geq 2g + 3$. By Lemma 2.1, we have that

$$q_1(G^*) < \max\{d_u + m_u : u \in V(G^*)\} = k + 3 + \frac{k+8}{k+3} = k + 4\frac{5}{k+3}.$$

Since $k+4+\frac{5}{k+3}$ is increasing with nonnegative number k, we have $q_1(G^*) < n-2g+4+\frac{5}{n-2g+3} < n-2g+5$ for $n \geq 2g+3$. Then we obtain $q_1(G^*) < n-2g+5 < n-2g+7 = q_1(P^{n-2g+2}_{\frac{g}{2},\frac{g}{2},\frac{g}{2},\frac{g}{2}})$, a contradiction.

If T_{G^*} is of type T_{15}, then G^* is obtained from T_{G^*} by attaching k pendant vertices to one of the four vertices of degree three in T_{G^*}. Let $m = |E(G^*)|$. Then $m \geq 2g$ for the length of all the three independent cycles in G^* is g. Since G^* is a tricyclic graph, we have that $|E(G^*)| = |V(G^*)| + 2$ and thus $k + m = n + 2$. Hence $3 \leq k = n + 2 - m \leq n - 2g + 2$ and then $n \geq 2g + 1$. By Lemma 2.1, we have that

$$q_1(G^*) < \max\{d_u + m_u : u \in V(G^*)\} = k + 3 + \frac{k+9}{k+3} = k + 4 + \frac{6}{k+3}.$$

Since $k+4+\frac{6}{k+3}$ is increasing with nonnegative number k, we can obtain that $q_1(G^*) < n - 2g + 6 + \frac{6}{n-2g+5} \leq n - 2g + 7$ for $n \geq 2g+1$. Then we have $q_1(G^*) < n-2g+7 \leq q_1(P^{n-2g+2}_{\frac{g}{2},\frac{g}{2},\frac{g}{2},\frac{g}{2}})$, a contradiction.

Therefore, we obtain that $n \geq 2g+3$ and $G^* \cong P^{n-2g+2}_{\frac{g}{2},\frac{g}{2},\frac{g}{2},\frac{g}{2}}$. Thus the proof is completed. $\square$

6. The results

Theorem 6.1 *For each pair of positive integers n, g.*

(1) If $3 \leq g \leq \frac{n+2}{3}$, then $T^{n-3g+2}_{g,g,g}$ is the unique graph with the largest signless Laplacian spectral radius among all graphs in $\mathscr{T}^{g,3}_n$.

(2) If $3 \leq g \leq \frac{2(n+2)}{5}$, then $T^{n-\lceil\frac{5g}{2}\rceil+2}_{\lfloor\frac{g}{2}\rfloor,\lceil\frac{g}{2}\rceil,\lceil\frac{g}{2}\rceil;g}$ is the unique graph with the largest signless Laplacian spectral radius among graphs in $\mathscr{T}^{g,4}_n$.

(3) If g is even, then $4 \leq g \leq \frac{n-3}{2}$ and $P^{n-2g+2}_{\frac{g}{2},\frac{g}{2},\frac{g}{2},\frac{g}{2}}$ is the unique graph with the largest signless Laplacian spectra radius among all graphs in $\{G : G \in \mathscr{T}^{g}_n$ and G contains at least three pendant vertices$\}$.

Proof (1) and (2) can be obtained directly from Theorems 3.2 and 4.2, respectively. We only show that (3) is true. Since $\mathscr{T}^g_n = \mathscr{T}^{g,3}_n \cup \mathscr{T}^{g,4}_n \cup \mathscr{T}^{g,6}_n \cup \mathscr{T}^{g,7}_n$, by Theorems 3.2, 4.2 and 5.3, it is enough for us to show that

$$q_1(T^{n-3g+2}_{g,g,g}) < q_1(T^{n-\lceil\frac{5g}{2}\rceil+2}_{\lfloor\frac{g}{2}\rfloor,\lceil\frac{g}{2}\rceil,\lceil\frac{g}{2}\rceil;g}) < q_1(P^{n-2g+2}_{\frac{g}{2},\frac{g}{2},\frac{g}{2},\frac{g}{2}}).$$

By Theorem 3.2, we have $q_1(T^{n-3g+2}_{g,g,g}) < n-3g+9+\frac{6}{n-3g+8} \leq n-3g+10$ for $n \leq 3g-2$. However $q_1(T^{n-\lceil\frac{5g}{2}\rceil+2}_{\lfloor\frac{g}{2}\rfloor,\lceil\frac{g}{2}\rceil,\lceil\frac{g}{2}\rceil;g}) \geq \Delta+1 = n-\frac{5g}{2}+8$. Note that $n-\frac{5g}{2}+8-(n-3g+10) = \frac{g}{2}-2 \geq 0$ for $g \leq 4$. Thus $q_1(T^{n-3g+2}_{g,g,g}) < q_1(T^{n-\lceil\frac{5g}{2}\rceil+2}_{\lfloor\frac{g}{2}\rfloor,\lceil\frac{g}{2}\rceil,\lceil\frac{g}{2}\rceil;g})$.

We now prove the second inequality. If $n = \frac{5g}{2}-2$, then $T^{n-\lceil\frac{5g}{2}\rceil+2}_{\lfloor\frac{g}{2}\rfloor,\lceil\frac{g}{2}\rceil,\lceil\frac{g}{2}\rceil;g}$ contains no pendant vertices. Thus $n \geq \frac{5g}{2}-1$, and then by Theorem 4.2 we have $q_1(T^{n-\lceil\frac{5g}{2}\rceil+2}_{\lfloor\frac{g}{2}\rfloor,\lceil\frac{g}{2}\rceil,\lceil\frac{g}{2}\rceil;g}) < n-\frac{5g}{2}+8+\frac{6}{n-\frac{5g}{2}+7} \leq n-\frac{5g}{2}+9$. However, by Theorem 5.3, we have $q_1(P^{n-2g+2}_{\frac{g}{2},\frac{g}{2},\frac{g}{2},\frac{g}{2}}) > \Delta+1 = n-2g+7$. Then we have $n-2g+7-(n-\frac{5g}{2}+9) = \frac{g}{2}-2 \leq 0$ for $g \geq 4$. Therefore $q_1(T^{n-\lceil\frac{5g}{2}\rceil+2}_{\lfloor\frac{g}{2}\rfloor,\lceil\frac{g}{2}\rceil,\lceil\frac{g}{2}\rceil;g}) < q_1(P^{n-2g+2}_{\frac{g}{2},\frac{g}{2},\frac{g}{2},\frac{g}{2}})$. By the above discussion, the proof is completed. $\square$

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