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Multiplicative Perturbation Bounds for the SR Factorization

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Abstract In this paper, we present the first order perturbation bounds for the SR factorization with respect to left multiplicative perturbation, and the first order and rigorous perturbation bounds for this factorization with respect to right multiplicative perturbation. Moreover, taking the properties of SR factors into consideration, we also provide some refined perturbation bounds.

Keywords multiplicative perturbation; SR factorization; rigorous perturbation bound; first order perturbation bound.

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1. Introduction

Let $\mathbb{R}^{m\times n}$ be the set of $m\times n$ real matrices, A^T be the transpose of the matrix A, and I_r be the identity matrix of order r. Let $A\in\mathbb{R}^{2n\times 2n}$, and $P=[e_1,e_3,\ldots,e_{2n-1},e_2,e_4,\ldots,e_{2n}]$ with e_k representing the k-th unit vector. If all even leading principal submatrices of PA^TJAP^T are nonsingular, Bunse-Gerstner [1] showed that A has the following SR factorization

$$A=SR=\begin{bmatrix} S_{11} & S_{12}\\ S_{21} & S_{22}\end{bmatrix}\begin{bmatrix} R_{11} & R_{12}\\ R_{21} & R_{22}\end{bmatrix}, \tag{1.1}$$

where S is a symplectic matrix, i.e., it satisfies

$$S^TJS=J,\quad J=\begin{bmatrix} 0 & I_n\\ I_n & 0\end{bmatrix}, \tag{1.2}$$

R_{ij} $(i,j=1,2)$ are upper triangular, and $\mathrm{diag}(R_{21})=0$. Further, if

$$\mathrm{diag}(R_{11})=|\mathrm{diag}(R_{22})| \text{ and } \mathrm{diag}(R_{12})=0 \tag{1.3}$$

hold, then the SR factorization (1.1) is unique [2]. Here, for any matrix $X=(x_{ij})\in\mathbb{R}^{m\times n}$, $|X|$ is defined by $(|x_{ij}|)$. In the following, when it comes to the unique SR decomposition, it means that its factor R satisfies (1.3).

Since the SR factorization is widely used in the computation of some optimal control problems (e.g., [3–5]) and is a key step for some important structure-preserving eigenproblems (e.g.,

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[1, 6–8]), some authors studied its algorithms, error analysis, and perturbation analysis (e.g., [1, 2, 5, 9–13]). The first order additive perturbation bounds of this factorization were derived by Bhatia [9] and Chang [2]. In this paper, using the classical and refined matrix equation approaches from [14, 15], we consider the perturbation bounds for the SR factorization with respect to multiplicative perturbations. That is, the original A is perturbed to

$$\widetilde{A}_L = D_L A, \tag{1.4}$$

or

$$\widetilde{A}_R = A D_R, \tag{1.5}$$

where $D_L \in \mathbb{R}^{2n\times 2n}$ and $D_R \in \mathbb{R}^{2n\times 2n}$ approach to the identity matrix I_{2n}, multiples of the identity matrix, or symplectic matrices, and they are called the left and right multiplicative perturbations, respectively. Of course, the multiplicative perturbations can easily be turned into additive perturbation. However, in this case, the perturbations will lose their nature and the obtained additive perturbation bounds will not reveal the special structures of multiplicative perturbations. Therefore, it is worth deriving the multiplicative perturbation bounds for the SR factorization.

The rest of this paper is organized as follows. Section 2 gives some notation and preliminaries. In Section 3, we present some lemmas which will be necessary for the following two sections. Sections 4 and 5 are devoted to deriving the multiplicative perturbation bounds for the SR factorization.

2. Notation and preliminaries

Throughout this paper, for any matrix $A \in \mathbb{R}^{m\times n}$, the symbols $\|A\|_2$ and $\|A\|_F$ represent its spectral norm and Frobenius norm, respectively. For these two norms, the following relations hold [16, pp. 80]

$$\|XYZ\|_F \leqslant \|X\|_2\|Y\|_F\|Z\|_2, \quad \|XYZ\|_2 \leqslant \|X\|_2\|Y\|_2\|Z\|_2, \tag{2.1}$$

whenever the matrix product XYZ is defined. If the matrix $A \in \mathbb{R}^{n\times n}$ is nonsingular, define its condition number as $k_2(A) = \|A\|_2\|A^{-1}\|_2$. For any matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, define

$$\mathrm{up}(A) = \begin{bmatrix} \frac{1}{2}a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \frac{1}{2}a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2}a_{nn} \end{bmatrix}, \quad \mathrm{low}(A) = \begin{bmatrix} \frac{1}{2}a_{11} & 0 & \cdots & 0 \\ a_{21} & \frac{1}{2}a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \frac{1}{2}a_{nn} \end{bmatrix},$$

$$\mathrm{sut}(A) = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \mathrm{lt}(A) = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

where the symbols “up”, “low,” and “sut” can be found in [14, 15, 17].

For any matrix

$$B=\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \in \mathbb{R}^{2n\times 2n},$$

where $B_{ij}\in\mathbb{R}^{n\times n}$ $(i,j=1,2)$, define

$$\begin{aligned} \operatorname{bup}(B) &= \begin{bmatrix} \operatorname{sut}(B_{11}) & \operatorname{up}(B_{12}) \\ \operatorname{up}(B_{21}) & \operatorname{sut}(B_{22}) \end{bmatrix}, \operatorname{blow}(B) = \begin{bmatrix} \operatorname{lt}(B_{11}) & \operatorname{low}(B_{12}) \\ \operatorname{low}(B_{21}) & \operatorname{lt}(B_{22}) \end{bmatrix}, \\ B_{(2k)} &= B(1:2k, 1:2k),\ k=1,2,\ldots,n, \end{aligned}$$

where "b" in "bup" and "blow" means "block" and $B(1:2k,1:2k)$ is the MATLAB notation. Clearly,

$$B-\operatorname{bup}(B)=\operatorname{blow}(B), \tag{2.2}$$

$$\|\operatorname{bup}(B)\|_F \leqslant \|B\|_F. \tag{2.3}$$

Let $\mathbb{D}_{2n}$ denote the set of all $2n\times 2n$ real positive definite diagonal matrices. Then for any matrix

$$D\equiv \begin{bmatrix} D^{(1)} & \\ & D^{(2)} \end{bmatrix} = \operatorname{diag}\left(\delta_1^{(1)},\ldots,\delta_n^{(1)},\delta_1^{(2)},\ldots,\delta_n^{(2)}\right)\in\mathbb{D}_{2n}, \tag{2.4}$$

it is easy to verify that the following properties hold

$$\operatorname{bup}(BD)=\operatorname{bup}(B)\,D, \quad \operatorname{bup}(DB)=D\operatorname{bup}(B). \tag{2.5}$$

Through out this paper, unless otherwise specified, $D\in\mathbb{D}_{2n}$ is defined as in (2.4).

3. Some lemmas

The following lemmas will be used later in this paper.

Lemma 3.1 ([2]) *For any matrix $B\in\mathbb{R}^{2n\times 2n}$ and $D\in\mathbb{D}_{2n}$,*

$$\|\operatorname{bup}(B)-D^{-1}\operatorname{bup}(B^T)D\|_F \leqslant \sqrt{1+\zeta_D^2}\,\|B\|_F, \tag{3.1}$$

where

$$\zeta_D=\max\left\{\max_{i<j}\left\{\frac{\delta_j^{(1)}}{\delta_i^{(1)}},\frac{\delta_j^{(2)}}{\delta_i^{(2)}}\right\},\ \max_{i\leqslant j}\left\{\frac{\delta_j^{(1)}}{\delta_i^{(2)}},\frac{\delta_j^{(2)}}{\delta_i^{(1)}}\right\}\right\}.$$

In particular, if $B^T=-B$ and $D=I_{2n}$, then (3.1) reduces to

$$\|\operatorname{bup}(B)\|_F \leqslant (1/\sqrt{2})\|B\|_F. \tag{3.2}$$

Lemma 3.2 *For any matrix $B\in\mathbb{R}^{2n\times 2n}$,*

$$\|\operatorname{blow}(B)+\operatorname{bup}(B^T)\|_F \leqslant \sqrt{2}\|B\|_F. \tag{3.3}$$

This lemma follows from [2, Lemma 3] with $D=I_{2n}$. It is worth pointing out that $\|B-\operatorname{bup}(B)-D^{-1}\operatorname{bup}(B^T)D\|_F$ in Lemma 3 in [2] should be $\|B-\operatorname{bup}(B)+D^{-1}\operatorname{bup}(B^T)D\|_F$.

Lemma 3.3 ([14]) *Let $a,b>0$. Let $c(\cdot)$ be a continuous function of a parameter $t\in[0,1]$ such that $b^2-4ac(t)>0$ holds for all t. Suppose that a continuous function $x(t)$ satisfies the quadratic* *inequality* $ax(t)^2 - bx(t) + c(t) \geqslant 0$. *If* $c(0) = x(0) = 0$, *then* $x(1) \leqslant (1/2a)\left(b - \sqrt{b^2 - 4ac(1)}\right)$.

Lemma 3.4 *Suppose that* $A \in \mathbb{R}^{2n\times 2n}$ *has the SR factorization (1.1) and all even leading principal submatrices of* $P(A+\Delta A)^T J(A+\Delta A)P^T$ *are nonsingular. Then* $A+\Delta A$ *has the unique SR factorization*

$$A + \Delta A = (S+\Delta S)(R+\Delta R),$$

and

$$\|\Delta S\|_F \lesssim \sqrt{2}k_2(S)\|R^{-1}\|_2\|\Delta A\|_F, \tag{3.4}$$

$$\|\Delta R\|_F \lesssim \sqrt{2}k_2(R)\|S^{-1}\|_2\|\Delta A\|_F. \tag{3.5}$$

The above two inequalities were presented in [9, Corollary 4.4].

4. Left multiplicative perturbation

In this section, we present the first order perturbation bounds when A suffers from left multiplicative perturbation. Moreover, based on the properties of SR factors, some refined bounds are provided.

Theorem 4.1 *Assume that* $A \in \mathbb{R}^{2n\times 2n}$ *has the unique SR factorization (1.1). Let* $D_L = I_{2n} + E \in \mathbb{R}^{2n\times 2n}$ *and* $\widetilde{A}_L$ *be defined as in (1.4). If*

$$k_2(S)\|E\|_2 < \sqrt{2} - 1, \tag{4.1}$$

then $\widetilde{A}_L$ *has the unique SR factorization*

$$\widetilde{A}_L = D_L A = \widetilde{S}\widetilde{R} = (S+\Delta S)(R+\Delta R), \tag{4.2}$$

and

$$\frac{\|\Delta S\|_F}{\|S\|_2} \lesssim \sqrt{2}k_2(S)\|D_L - I_{2n}\|_F, \tag{4.3}$$

$$\frac{\|\Delta R\|_F}{\|R\|_2} \lesssim \sqrt{2}k_2(S)\|D_L - I_{2n}\|_F. \tag{4.4}$$

Proof Noting (1.4), (1.1), and the fact that S is nonsingular, we have

$$\widetilde{A}_L = D_L A = (I_{2n} + E)SR = S(I_{2n} + S^{-1}ES)R. \tag{4.5}$$

Considering (1.2), we get

$$\begin{aligned} P\left[(I_{2n}+E)S\right]^T J(I_{2n}+E)SP^T &= P(I_{2n}+S^{-1}ES)^T J(I_{2n}+S^{-1}ES)P^T \\ &= P(J+M)P^T = \widehat{J} + N, \end{aligned} \tag{4.6}$$

where $M = JS^{-1}ES + S^TE^TS^{-T}J + S^TE^TS^{-T}JS^{-1}ES, N = PMP^T$, and

$$\widehat{J} = PJP^T = \mathrm{diag}(J_0, J_0, \ldots, J_0), \quad J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{4.7}$$

Noting (2.1) and (4.1), we obtain

$$\|N\|_2 = \|PMP^T\|_2 = \|M\|_2 \leqslant 2\|S^{-1}ES\|_2 + \|S^{-1}ES\|_2^2 \leqslant 2k_2(S)\|E\|_2 + k_2^2(S)\|E\|_2^2 < 1.$$

As a result [15],

$$\left\|N_{(2k)}\right\|_2 < 1, \quad k = 1, 2, \ldots, n.$$

Furthermore, obviously, $\|\widehat{J}_{(2k)}\|_2 = 1$. Then $\|\widehat{J}_{(2k)} N_{(2k)}\|_2 < 1$, which implies $I_{2k} - \widehat{J}_{(2k)} N_{(2k)}$ is nonsingular. Therefore,

$$(\widehat{J} + N)_{(2k)} = \widehat{J}_{(2k)} + N_{(2k)} = \widehat{J}_{(2k)}[I_{2k} - \widehat{J}_{(2k)} N_{(2k)}]$$

is nonsingular since $\widehat{J}_{(2k)}$ is nonsingular. Noting (4.6), we obtain that all even leading principal submatrices of $P\left[(I_{2n} + E)S\right]^T J (I_{2n} + E) S P^T$ are nonsingular. Thus, $(I_{2n} + E)S$ has the unique SR factorization

$$(I_{2n} + E)S = \widetilde{S}\overline{R}, \tag{4.8}$$

which, combined with (4.5), leads to

$$\widetilde{A}_L = (I_{2n} + E)SR = \widetilde{S}\overline{R}R = \widetilde{S}\widetilde{R}, \tag{4.9}$$

where $\widetilde{R} = \overline{R}R$. It is easy to check that the structure of $\widetilde{R}$ is the same as that of R defined in (1.1) and (1.3). Thus, (4.9) is the unique SR factorization of $\widetilde{A}_L$. So, the proof of (4.2) is completed. □

In the following, we consider (4.3) and (4.4). Noting (4.8) and (4.9), by Lemma 3.4 and using (2.1), we have

$$\|\Delta S\|_F = \|\widetilde{S} - S\|_F \lesssim \sqrt{2} k_2(S) \|ES\|_F \lesssim \sqrt{2} k_2(S) \|S\|_2 \|E\|_F, \tag{4.10}$$

$$\begin{aligned} \|\Delta R\|_F &= \|\widetilde{R} - R\|_F = \|\overline{R}R - R\|_F \leqslant \|\overline{R} - I_{2n}\|_F \|R\|_2 \\ &\lesssim \sqrt{2} \|S^{-1}\|_2 \|R\|_2 \|ES\|_F \lesssim \sqrt{2} k_2(S) \|R\|_2 \|E\|_F. \end{aligned} \tag{4.11}$$

Note $E = D_L - I_{2n}$. It is easy to get (4.3) and (4.4) from (4.10) and (4.11).

Note that if D_L is a scalar multiple of the identity matrix, the real perturbation bound of the factor S is zero and if D_L is a symplectic matrix, the real perturbation bound of the factor R is zero. However, in these cases, the bounds of S and R in (4.3) and (4.4) are far from zero. Therefore, the bounds (4.3) and (4.4) are not good enough. Next, following the idea of [15] and employing the following properties of SR factors:

- the factor S is invariant when A suffers from a left multiplicative perturbation by any positive constant,
- the factor R is invariant when A suffers from a left multiplicative perturbation by any symplectic matrix,

we refine the bounds (4.3) and (4.4), respectively.

We first consider the factor S. For any constant $\alpha > 0$, if the condition (4.1) holds, then

$$\alpha D_L A = [I_{2n} + (\alpha D_L - I_{2n})]A = \widetilde{S}(\alpha \widetilde{R}). \tag{4.12}$$

Further, from Theorem 4.1, if

$$k_2(S) \|\alpha D_L - I_{2n}\|_2 < \sqrt{2} - 1,$$

then (4.12) also holds and

$$\frac{\|\Delta S\|_F}{\|S\|_2} \lesssim \sqrt{2}k_2(S)\|\alpha D_L - I_{2n}\|_F. \tag{4.13}$$

Thus, we can find out $\widehat{\alpha}$ such that $\|\alpha D_L - I_{2n}\|_F$ is minimal, and then substitute $\widehat{\alpha}$ into (4.13) to better the bound. Since

$$\|\alpha D_L - I_{2n}\|_F^2 = \mathrm{trace}[(\alpha D_L - I_{2n})^T(\alpha D_L - I_{2n})] = \|D_L\|_F^2\alpha^2 - 2\mathrm{trace}(D_L)\alpha + n,$$

we get

$$\widehat{\alpha} = \frac{\mathrm{trace}(D_L)}{\|D_L\|_F^2}. \tag{4.14}$$

In this case, $k_2(S)\|\widehat{\alpha}D_L - I_{2n}\|_2 \leqslant k_2(S)\|\widehat{\alpha}D_L - I_{2n}\|_F = 0 < \sqrt{2} - 1$. In addition, we can also verify that $\widehat{\alpha} > 0$ if (4.1) holds. As a result, we have the following theorem which improves the bound of factor S in Theorem 4.1.

Theorem 4.2 *Assume that the conditions of Theorem 4.1 hold and $\widehat{\alpha}$ is defined as in (4.14). Then*

$$\frac{\|\Delta S\|_F}{\|S\|_2} \lesssim \sqrt{2}k_2(S)\|\widehat{\alpha}D_L - I_{2n}\|_F. \tag{4.15}$$

Remark 4.3 We can see that if $D_L = kI_{2n}$ for any $k \in \mathbb{R}$, then $\widehat{\alpha}D_L - I_{2n} = 0$. As a result, the bound in (4.15) is zero, which shows that the bound (4.15) is also valid for some special perturbations.

Now we consider the factor R. For any symplectic matrix $G \in \mathbb{R}^{2n\times 2n}$, if the condition (4.1) holds, then

$$GD_LA = [I_{2n} + (GD_L - I_{2n})]A = (G\widetilde{S})\widetilde{R}. \tag{4.16}$$

Further, from Theorem 4.1, if

$$k_2(S)\|GD_L - I_{2n}\|_2 < \sqrt{2} - 1,$$

then (4.16) also holds and

$$\frac{\|\Delta R\|_F}{\|R\|_2} \lesssim \sqrt{2}k_2(S)\|GD_L - I_{2n}\|_F. \tag{4.17}$$

Therefore, we can refine the bound (4.3) by a symplectic matrix G. What we only need to do is to find a symplectic matrix $\widehat{G}$ such that $\|GD_L - I_{2n}\|_F$ is minimal and then substitute it into (4.17). Unfortunately we have not found the suitable symplectic matrix so far.

5. Right multiplication perturbation

In this section, we first present the rigorous perturbation bounds for the SR factorization when A suffers from right multiplicative perturbation. Then, the first order perturbation bounds for this factorization are presented as the special case.

Theorem 5.1 *Assume that $A \in \mathbb{R}^{2n\times 2n}$ has the unique SR factorization (1.1). Let $D_R =$*

$I_{2n}+F \in \mathbb{R}^{2n\times 2n}$ and $\widetilde{A}_R$ be defined as in (1.5). If

$$k_2(R)\|F\|_F < \sqrt{3/2}-1 < \sqrt{2}-1, \tag{5.1}$$

then $\widetilde{A}_R$ has the unique SR factorization

$$\widetilde{A}_R = A(I_{2n}+F) = (S+\Delta S)(R+\Delta R), \tag{5.2}$$

and

$$\frac{\|\Delta S\|_F}{\|S\|_2} \leqslant (2+2\sqrt{2}+2\sqrt{3}+\sqrt{6})k_2(R)\|D_R - I_{2n}\|_F, \tag{5.3}$$

$$\frac{\|\Delta R\|_F}{\|R\|_2} \leqslant (\sqrt{3}+\sqrt{6}) \inf_{D\in\mathbb{D}_{2n}} \left[\sqrt{1+\zeta_D^2}k_2(D^{-1}R)\right]\|D_R - I_{2n}\|_F. \tag{5.4}$$

Proof For any $t\in[0,1]$, let

$$\widetilde{A}_R(t) = A(I_{2n}+tF) = SR(I_{2n}+tF).$$

Then, using (1.1) and (1.2), we have

$$\begin{aligned} P\widetilde{A}_R^T(t)J\widetilde{A}_R(t)P^T =& P(I_{2n}+tF)^TR^TJR(I_{2n}+tF)P^T \\ =& PR^T[J+K(t)]RP^T = PR^TP^TP[J+K(t)]P^TPRP^T, \end{aligned} \tag{5.5}$$

where $K(t) = tR^{-T}F^TR^TJ + tJRFR^{-1} + t^2R^{-T}F^TR^TJRFR^{-1}$. Employing the same method as Theorem 4.1, we have that all even leading principal submatrices of $P[J+K(t)]P^T$ are nonsingular. Obviously, $P[J+K(t)]P^T$ is skew-symmetric. Thus, from [18], we get

$$P[J+K(t)]P^T = R^T(t)\widehat{J}R(t), \tag{5.6}$$

where $\widehat{J}$ is defined as in (4.7) and $R(t)$ is upper triangular with 2×2 main diagonal blocks:

$$\begin{bmatrix} r_i(t) & \\ & \pm r_i(t) \end{bmatrix}, \quad r_i(t)>0, \ i=1,2,\ldots,n.$$

Substituting (5.6) into (5.5) gives

$$P\widetilde{A}_R^T(t)J\widetilde{A}_R(t)P^T = (PR^TP^T)R^T(t)\widehat{J}R(t)(PRP^T),$$

where PRP^T is upper triangular with 2×2 main diagonal blocks:

$$\begin{bmatrix} |r_i| & \\ & r_i \end{bmatrix}, \quad i=1,2,\ldots,n.$$

Here, r_i are the diagonal elements of R_{22}, which is defined as in (1.1). Let

$$R(t) = \begin{bmatrix} R_{11}(t) & R_{12}(t) & \cdots & R_{1n}(t) \\ 0 & R_{22}(t) & \cdots & R_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{nn}(t) \end{bmatrix}, \quad PRP^T = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ 0 & R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{nn} \end{bmatrix},$$

where $R_{ij}(t)$ and R_{ij} $(i,j=1,2,\ldots,n)$ are 2×2 matrices, and $R_{ii}(t)$ and R_{ii} $(i=1,2,\ldots,n)$

are defined above. After some computations, we have

$$R(t)(PRP^T)=\begin{bmatrix} R_{11}(t)R_{11} & R_{11}(t)R_{12}+R_{12}(t)R_{22} & \cdots & \Sigma_{k=1}^n R_{1k}(t)R_{kn} \\ 0 & R_{22}(t)R_{22} & \cdots & \Sigma_{k=2}^n R_{2k}(t)R_{kn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{nn}(t)R_{nn} \end{bmatrix}.$$

Further, noting (4.7), we have

$$\begin{aligned} &P\widetilde{A}_R^T(t)J\widetilde{A}_R(t)P^T \\ &=\begin{bmatrix} R_{11}^TR_{11}^T(t)J_0 & 0 & \cdots & 0 \\ R_{12}^T(t)R_{11}^T(t)+R_{22}^TR_{12}^T(t) & R_{22}^TR_{22}^T(t)J_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{k=1}^n R_{kn}^TR_{1k}^T(t) & \Sigma_{k=2}^n R_{kn}^TR_{2k}^T(t) & \cdots & R_{nn}^TR_{nn}^T(t)J_0 \end{bmatrix}\times \\ &\begin{bmatrix} R_{11}(t)R_{11} & R_{11}(t)R_{12}+R_{12}(t)R_{22} & \cdots & \Sigma_{k=1}^n R_{1k}(t)R_{kn} \\ 0 & R_{22}(t)R_{22} & \cdots & \Sigma_{k=2}^n R_{2k}(t)R_{kn} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{nn}(t)R_{nn} \end{bmatrix}. \end{aligned}$$

Hence, the even leading principle submatrices of $P\widetilde{A}_R^T(t)J\widetilde{A}_R(t)P^T$ are:

$$\begin{aligned} &\begin{bmatrix} R_{11}^TR_{11}^T(t)J_0 & 0 & \cdots & 0 \\ R_{12}^T(t)R_{11}^T(t)+R_{22}^TR_{12}^T(t) & R_{22}^TR_{22}^T(t)J_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{k=1}^m R_{km}^TR_{1k}^T(t) & \Sigma_{k=2}^m R_{km}^TR_{2k}^T(t) & \cdots & R_{mm}^TR_{mm}^T(t)J_0 \end{bmatrix}\times \\ &\begin{bmatrix} R_{11}(t)R_{11} & R_{11}(t)R_{12}+R_{12}(t)R_{22} & \cdots & \Sigma_{k=1}^m R_{1k}(t)R_{km} \\ 0 & R_{22}(t)R_{22} & \cdots & \Sigma_{k=2}^m R_{2k}(t)R_{km} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{mm}(t)R_{mm} \end{bmatrix},\quad m=1,2,\ldots,n. \end{aligned}$$

Noting that both the lower and the upper triangular matrices are nonsingular, we have that all even leading principal submatrices of $P\widetilde{A}_R^T(t)J\widetilde{A}_R(t)P^T$ are nonsingular. Then $\widetilde{A}_R(t)$ has the unique SR factorization

$$\widetilde{A}_R(t)=A(I_{2n}+tF)=SR(I_{2n}+tF)=(S+\Delta S(t))(R+\Delta R(t)), \tag{5.7}$$

which, with $\Delta S(1)=\Delta S$, $\Delta R(1)=\Delta R$, leads to (5.2).

In the following, we consider (5.3) and (5.4). Noting (1.2) and (5.7), we obtain

$$(I_{2n}+tF)^TR^TJR(I_{2n}+tF)=(R+\Delta R(t))^TJ(R+\Delta R(t)).$$

We can see that through expanding the above equation

$$tR^TJRF+tF^TR^TJR+t^2F^TR^TJRF-\Delta R^T(t)J\Delta R(t)=R^TJ\Delta R(t)+\Delta R^T(t)JR. \tag{5.8}$$

Left-multiplicating by R^{-T} and right-multiplicating by R^{-1} on the both sides of (5.8), we get

$$\begin{aligned}&tJRFR^{-1}+tR^{-T}F^TR^TJ+t^2R^{-T}F^TR^TJRFR^{-1}-R^{-T}\Delta R^T(t)J\Delta R(t)R^{-1}\\&\quad=J\Delta R(t)R^{-1}+R^{-T}\Delta R^T(t)J.\end{aligned}\tag{5.9}$$

According to the fact that the structure of $\Delta R(t)$ is the same as that of $\dot{R}(0)$ in [2], by the same method of [2], from (5.9), we get

$$\begin{aligned}J\Delta R(t)R^{-1}=&\mathrm{bup}\left[tJRFR^{-1}-t(JRFR^{-1})^T\right]+\mathrm{bup}(t^2R^{-T}F^TR^TJRFR^{-1})-\\&\mathrm{bup}\left[R^{-T}\Delta R^T(t)J\Delta R(t)R^{-1}\right].\end{aligned}\tag{5.10}$$

Taking the Frobenius norm on the both sides of (5.10), and using (3.1) with $D=I_{2n}$, (3.2), and (2.1) gives

$$\|\Delta R(t)R^{-1}\|_F\leqslant\sqrt{2}\|RFR^{-1}\|_Ft+(1/\sqrt{2})\|RFR^{-1}\|_F^2t^2+(1/\sqrt{2})\|\Delta R(t)R^{-1}\|_F^2.\tag{5.11}$$

Let $x(t)=\|\Delta R(t)R^{-1}\|_F$ and $c(t)=2\|RFR^{-1}\|_Ft+\|RFR^{-1}\|_F^2t^2$. Then (5.11) can be rewritten as

$$x^2(t)-\sqrt{2}x(t)+c(t)\geqslant 0.$$

Note that $c(t)$ is continuous with respect to t, and the condition (5.1) guarantees that $c(t)\leqslant 2\|RFR^{-1}\|_F+\|RFR^{-1}\|_F^2\leqslant 2k_2(R)\|F\|_F+k_2^2(R)\|F\|_F^2<1/2$ with $t\in[0,1]$, which implies $\Delta=2-4c(t)>2-4\times 1/2=0$. Meanwhile, $x(t)$ is continuous with respect to t and $c(0)=x(0)=0$. By Lemma 3.3, we obtain

$$\|\Delta RR^{-1}\|_F\leqslant\frac{1}{\sqrt{2}}\Big(1-\sqrt{1-4\|RFR^{-1}\|_F-2\|RFR^{-1}\|_F^2}\Big)\leqslant 1/\sqrt{2}.\tag{5.12}$$

From (5.10) with $t=1$, we have

$$\begin{aligned}J\Delta RR^{-1}=&\mathrm{bup}(JRFR^{-1}+R^{-T}F^TR^TJ+R^{-T}F^TR^TJRFR^{-1}-\\&R^{-T}\Delta R^TJ\Delta RR^{-1}).\end{aligned}\tag{5.13}$$

Let $R=D\widehat{R}$ with $D\in\mathbb{D}_{2n}$. Thus, noting (2.5), from (5.13), it follows that

$$\begin{aligned}J\Delta R\widehat{R}^{-1}=&\mathrm{bup}\big[JRF\widehat{R}^{-1}-D^{-1}(JRF\widehat{R}^{-1})^TD\big]+\mathrm{bup}(R^{-T}F^TR^TJRF\widehat{R}^{-1})-\\&\mathrm{bup}(R^{-T}\Delta R^TJ\Delta R\widehat{R}^{-1}).\end{aligned}\tag{5.14}$$

Taking the Frobenius norm on the both sides of (5.14), and using (3.1), (2.3) and (2.1), we get

$$\|\Delta R\widehat{R}^{-1}\|_F\leqslant\sqrt{1+\zeta_D^2}\|RF\widehat{R}^{-1}\|_F+\|RFR^{-1}\|_F\|RF\widehat{R}^{-1}\|_F+\|\Delta RR^{-1}\|_F\|\Delta R\widehat{R}^{-1}\|_F,$$

which, combined with the first inequality of (5.12), (5.1), the fact $\sqrt{1+\zeta_D^2}>1$, and (2.1), implies

$$\begin{aligned}\|\Delta R\widehat{R}^{-1}\|_F\leqslant&\frac{\sqrt{2}\left(\sqrt{1+\zeta_D^2}+\|RFR^{-1}\|_F\right)\|RF\widehat{R}^{-1}\|_F}{\sqrt{2}-1+\sqrt{1-4\|RFR^{-1}\|_F-2\|RFR^{-1}\|_F^2}}&(5.15)\\\leqslant&(2+\sqrt{2})\Big(\sqrt{1+\zeta_D^2}+k_2(R)\|F\|_F\Big)\|RF\widehat{R}^{-1}\|_F&(5.16)\\\leqslant&(\sqrt{3}+\sqrt{6})\sqrt{1+\zeta_D^2}\|RF\widehat{R}^{-1}\|_F.&(5.17)\end{aligned}$$

Therefore, using (2.1), and combining (5.17) with

$$\|\Delta R\|_F \leqslant \|\Delta R\widehat{R}^{-1}\|_F\|\widehat{R}\|_2 \tag{5.18}$$

and

$$F = D_R - I_{2n} \tag{5.19}$$

leads to (5.4).

Expanding (5.2) and using (1.1), we get

$$SRF = S\Delta R + \Delta SR + \Delta S\Delta R. \tag{5.20}$$

Left-multiplicating by S^TJ and right-multiplicating by R^{-1} on the both sides of (5.20) yields

$$S^TJ\Delta S = JRFR^{-1} - J\Delta RR^{-1} - S^TJ\Delta S\Delta RR^{-1}. \tag{5.21}$$

Putting (5.13) into (5.21) leads to

$$\begin{aligned} S^TJ\Delta S =&JRFR^{-1} - \mathrm{bup}(JRFR^{-1} + R^{-T}F^TR^TJ) - \mathrm{bup}(R^{-T}F^TR^TJRFR^{-1})+ \\ &\mathrm{bup}(R^{-T}\Delta RJ\Delta RR^{-1}) - S^TJ\Delta S\Delta RR^{-1}. \end{aligned} \tag{5.22}$$

Noting (2.2), we have

$$JRFR^{-1} - \mathrm{bup}(JRFR^{-1} + R^{-T}F^TR^TJ) = \mathrm{blow}(JRFR^{-1}) + \mathrm{bup}((JRFR^{-1})^T). \tag{5.23}$$

By Lemma 3.2,

$$\|\mathrm{blow}(JRFR^{-1}) + \mathrm{bup}((JRFR^{-1})^T)\|_F \leqslant \sqrt{2}\|JRFR^{-1}\|_F = \sqrt{2}\|RFR^{-1}\|_F. \tag{5.24}$$

Thus, taking the Frobenius norm on the both sides of (5.22), and noting (5.23), (5.24), (3.2), (2.3) and (2.1), we obtain

$$\begin{aligned} \|S^TJ\Delta S\|_F \leqslant&\sqrt{2}\|RFR^{-1}\|_F + (1/\sqrt{2})\|RFR^{-1}\|_F^2 + (1/\sqrt{2})\|\Delta RR^{-1}\|_F^2+ \\ &\|S^TJ\Delta S\|_F\|\Delta RR^{-1}\|_F. \end{aligned} \tag{5.25}$$

Using the first inequality of (5.12) and (2.1), we have

$$\begin{aligned} \|\Delta RR^{-1}\|_F \leqslant&\frac{1}{\sqrt{2}}\frac{4\|RFR^{-1}\|_F + 2\|RFR^{-1}\|_F^2}{1+\sqrt{1-4\|RFR^{-1}\|_F - 2\|RFR^{-1}\|_F^2}} \\ \leqslant&(1/\sqrt{2})\left(4\|RFR^{-1}\|_F + 2\|RFR^{-1}\|_F^2\right) \\ \leqslant&\sqrt{2}k_2(R)\|F\|_F\left(2 + k_2(R)\|F\|_F\right). \end{aligned} \tag{5.26}$$

Squaring (5.26) and using (5.1) gives

$$\|\Delta RR^{-1}\|_F^2 \leqslant (5+2\sqrt{6})k_2^2(R)\|F\|_F^2. \tag{5.27}$$

Putting (5.27) into (5.25) and using the second inequality of (5.12), (5.1), and (2.1), we get

$$\begin{aligned} \|S^TJ\Delta S\|_F \leqslant&(2+2\sqrt{2})k_2(R)\|F\|_F + (6+2\sqrt{6})(\sqrt{2}+1)k_2^2(R)\|F\|_F^2 \\ \leqslant&(2+2\sqrt{2}+2\sqrt{3}+\sqrt{6})k_2(R)\|F\|_F. \end{aligned} \tag{5.28}$$

Thus, combining (5.28) with $\|\Delta S\|_F = \|SJS^T\Delta S\|_F \leqslant \|S\|_2\|S^TJ\Delta S\|_F$ and (5.19) leads to (5.3). $\Box$

Remark 5.2 We can also get the following rigorous perturbation bounds from (5.15), (5.16), (5.18), and (5.19) when A suffers from right multiplicative perturbation:

$$\frac{\|\Delta R\|_F}{\|R\|_2} \leqslant \frac{\sqrt{2} \inf_{D\in\mathbb{D}_{2n}} \left[\sqrt{1+\zeta_D^2} + k_2(R)\|D_R - I_{2n}\|_F\right] k_2(D^{-1}R)\|D_R - I_{2n}\|_F}{\sqrt{2}-1+\sqrt{1-4k_2(R)\|D_R - I_{2n}\|_F - 2k_2^2(R)\|D_R - I_{2n}\|_F^2}} \tag{5.29}$$

$$\leqslant (2+\sqrt{2}) \inf_{D\in\mathbb{D}_{2n}} \left[\sqrt{1+\zeta_D^2} + k_2(R)\|D_R - I_{2n}\|_F\right] k_2(D^{-1}R)\|D_R - I_{2n}\|_F. \tag{5.30}$$

Remark 5.3 The following first order perturbation bounds of factor R with A suffering from right multiplicative perturbation can be got from the proof of Theorem 5.1 by omitting the higher order perturbation terms:

$$\frac{\|\Delta R\|_F}{\|R\|_2} \lesssim \inf_{D\in\mathbb{D}_{2n}} \left[\sqrt{1+\zeta_D^2} k_2(D^{-1}R)\right] \|D_R - I_{2n}\|_F. \tag{5.31}$$

Clearly, (5.4) is a small constant times of (5.31).

Remark 5.4 By omitting the higher order perturbation terms, we can also get the first order perturbation bound of factor S from the proof of Theorem 5.1 when A suffers from right multiplicative perturbation:

$$\frac{\|\Delta S\|_F}{\|S\|_2} \lesssim \sqrt{2} k_2(R)\|D_R - I_{2n}\|_F. \tag{5.32}$$

As we did for the left multiplicative perturbation, we can also better the bounds (5.3) and (5.32). The refined bounds are presented in the following theorem.

Theorem 5.5 *Suppose that the conditions of Theorem 5.1 hold. Then*

$$\frac{\|\Delta S\|_F}{\|S\|_2} \leqslant (2+2\sqrt{2}+2\sqrt{3}+\sqrt{6})k_2(R)\|\widetilde{\alpha}D_R - I_{2n}\|_F, \tag{5.33}$$

$$\frac{\|\Delta S\|_F}{\|S\|_2} \lesssim \sqrt{2} k_2(R)\|\widetilde{\alpha}D_R - I_{2n}\|_F, \tag{5.34}$$

where $\widetilde{\alpha} = \frac{\mathrm{trace}(D_R)}{\|D_R\|_F^2}$.

6. Example

A simple example is given below to show that our first order multiplicative bounds are better than the bounds (3.4) and (3.5), respectively.

Let $S = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$, $R = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}$ and $D_L = D_R = \begin{pmatrix} 1+\delta & 0 \\ 0 & 1+\delta \end{pmatrix}$ with $\gamma \gg 1$ and $\delta > 0$. Then we have

$$\frac{\|R^{-1}\|_2\|\Delta A\|_F}{\|S\|_2} = \sqrt{2+\gamma^2}\delta$$

and

$$\|D_L - I_2\|_F = \sqrt{2}\delta,$$

which show that the bounds (4.3) and (4.4) are much sharper than (3.4) and (3.5), respectively.

Moreover, let $D = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}$. Then

$$\frac{\|S^{-1}\|_2\|\Delta A\|_F}{\|R\|_2} = \sqrt{2+\gamma^2}\delta, \quad \|D_R - I_2\|_F = \sqrt{2}\delta,$$

and

$$\sqrt{2}\|R^{-1}\|_2\|S^{-1}|_2\|\Delta A\|_F = \sqrt{2(2+\gamma^2)}\delta, \quad \sqrt{1+\zeta_D^2}k_2(D^{-1}R)\|D_R - I_2\|_F = \frac{\sqrt{2(1+\gamma^2)}}{\gamma}\delta.$$

Thus, the bounds (5.32) and (5.31) are much sharper than (3.4) and (3.5), respectively.

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