Journal of Mathematical Research with Applications
Jul., 2014, Vol. 34, No. 4, pp. 435–445
DOI:10.3770/j.issn:2095-2651.2014.04.006
Http://jmre.dlut.edu.cn

Riemann-Hilbert Problem and Its Well-Posed-Ness for Elliptic Complex Equations of First Order in Multiply Connected Domains

Guochun WEN[1], **Liping WANG**[2,*]
1. *LMAM, School of Mathematical Sciences, Peking University, Beijing* 100871*, P. R. China;*
2. *College of Mathematics and Information Science, Hebei Normal University, Hebei* 050024*, P. R. China*

Abstract In this article, we first propose the Riemann-Hilbert problem for uniformly elliptic complex equations of first order and its well-posed-ness in multiply connected domains. Then we give the integral representation of solutions for modified Riemann-Hilbert problem of the complex equations. Moreover we shall obtain a priori estimates of solutions of the modified Riemann-Hilbert problem and verify its solvability. Finally the solvability results of the original boundary value problem can be obtained.

Keywords Riemann-Hilbert problems; quasilinear elliptic complex equations; multiply connected domains; priori estimates and existence of solutions.

MR(2010) Subject Classification 35J55; 35C15; 35B45

1. Formulation of Riemann-Hilbert problem for elliptic complex equations of first order and its well-posedness

Let D be an $N+1$ $(N\geq 1)$-connected domain in $\mathbb{C}$ with the boundary $\partial D=\cup_{j=0}^{N}\Gamma_j\in C_\mu^1$ $(0<\mu<1)$. Now we introduce the quasi-linear elliptic complex equation of first order

$$w_{\bar z}=Q_1w_z+Q_2\overline{w}_{\bar z}+A_1w+A_2\overline{w}+A_3,\quad z\in D,\tag{1.1}$$

where $z=x+iy$, $w_{\bar z}=[w_x+iw_y]/2$, $Q_j=Q_j(z,w), j=1,2$, $A_j=A_j(z,w)\,(j=1,2,3)$. We assume that equation (1.1) satisfies the following conditions.

Condition C (1) The functions $Q_j=Q_j(z,w)\,(j=1,2)$, $A_j=A_j(z,w)\,(j=1,2,3)$ are measurable in $z\in D$ for any continuous function $w(z)$ in $\overline{D}$, and satisfy

$$L_p[A_j(z,w),\overline{D}],\quad j=1,2,3,\tag{1.2}$$

where $p,p_0\,(2<p_0\leq p),k_0,k_1$ are non-negative constants.

(2) The above functions are continuous in $w\in\mathbb{C}$ for almost every point $z\in D$, and $Q_j(z,w)=0, j=1,2$, $A_j(z,w)=0, j=1,2,3$ for $z\notin D$.

Received April 22, 2013; Accepted February 24, 2014
Supported by the National Natural Science Foundation of China (Grant No. 11171349) and the Science Foundation of Hebei Province (Grant No. A2010000346).
* Corresponding author
E-mail address: wengc@math.pku.edu.cn (Guochun WEN); wlpxjj@163.com (Liping WANG)

(3) The complex equation (1.1) satisfies the uniform ellipticity condition

$$|Q_1(z,w)|+|Q_2(z,w)|\le q_0, \tag{1.3}$$

in which $q_0\,(<1)$ is a non-negative constant.

Let D be an $N+1\,(N\ge 1)$-connected bounded domain in $\mathbb{C}$ with the boundary $\partial D=\Gamma=\cup_{j=0}^{N}\Gamma_j\in C_\mu^1\,(0<\mu<1)$. Without loss of generality, we assume that D is a circular domain in $|z|<1$, bounded by the $(N+1)$-circles $\Gamma_j: |z-z_j|=r_j$, $j=0,1,\ldots,N$ and $\Gamma_0=\Gamma_{N+1}: |z|=1$, $z=0\in D$. In this article, the notations are the same as in [4–12]. Now we formulate the Riemann-Hilbert problem for equation (1.1) as follows.

Problem A The Riemann-Hilbert boundary value problem for (1.1) is to find a continuous solution $w(z)$ in $\overline{D}$ satisfying the boundary condition:

$$\mathrm{Re}[\overline{\lambda(z)}w(z)]=c(z),\quad z\in\Gamma, \tag{1.4}$$

where $\lambda(z),c(z)$ satisfy the conditions

$$C_\alpha[\lambda(z),\Gamma]\le k_0,\quad C_\alpha[c(z),\Gamma]\le k_2, \tag{1.5}$$

in which $\lambda(z)=a(z)+ib(z), |\lambda(z)|=1$ on Γ, and $\alpha\,(1/2<\alpha<1)$ is a positive constant. The index K of Problems A is defined as follows:

$$K=K_1+\cdots+K_m=\sum_{j=0}^{N}\frac{1}{2\pi}\Delta_{\Gamma_j}\arg\lambda(z),\quad j=0,1,\ldots,N, \tag{1.6}$$

in which the partial indexes $K_j=\Delta_{\Gamma_j}\arg\lambda(z)/2\pi\,(j=0,1,\ldots,N)$ of $\lambda(z)$ are integers.

When the index $K<0$, Problem A is not certainly solved, and when $K\ge 0$, the solution of Problem A is not surely unique. Hence we put forward a well-posed-ness of Problem A with modified boundary conditions.

Problem B Find a continuous solution $w(z)$ of the complex equation (1.6) in D satisfying the boundary condition

$$\mathrm{Re}[\overline{\lambda(z)}w(z)]=r(z)+h(z),\quad z\in\Gamma, \tag{1.7}$$

where

$$h(z)=\begin{cases} 0,\ z\in\Gamma, & \text{if } K\ge N,\\ \left.\begin{array}{l} 0,\ z\in\Gamma_j,\ j=1,\ldots,K+1,\\ h_j,\ z\in\Gamma_j, j=K+2,\ldots,N+1, \end{array}\right\} & \text{if } 0\le K<N,\\ \left.\begin{array}{l} h_j,\ z\in\mathit{\Gamma}_j,\ j=1,\ldots,N,\ j=1,\ldots,N,\\ h_0+\mathrm{Re}\sum_{m=1}^{-K-1}(h_m^++\mathrm{i}h_m^-)z^m, z\in\Gamma_0, \end{array}\right\} & \text{if } K<0, \end{cases} \tag{1.8}$$

in which $h_j\,(j=0,1,\ldots,N+1)$, $h_m^\pm\,(m=1,\ldots,-K-1)$ are undetermined real constants; we must give the attention that the boundary circles $\Gamma_j\,(j=0,1,\ldots,N)$ of the domain D are moved round the positive direction. In addition, we may assume that the solution $w(z)$ satisfies the following point conditions

$$\mathrm{Im}[\overline{\lambda(a_j)}w(a_j)] = b_j,\ j \in J = \begin{cases} 1,\dots,2K-N+1,\ \text{if } K \ge N, \\ 1,\dots,K+1,\ \text{ if }\ 0 \le K < N, \end{cases} \tag{1.9}$$

where $a_j \in \Gamma_j\,(j = 1,\dots,N)$, $a_{j+N} \in \Gamma_0\,(j = 1,\dots,2K-N+1, K \ge N)$ and $a_j \in \Gamma_{j-1}\,(j = 1,\dots,K+1,\ 0 \le K < N)$ are distinct points, and $b_j\ (j \in J)$ are all real constants satisfying the conditions

$$|b_j| \le k_3, \quad j \in J, \tag{1.10}$$

herein k_3 is a non-negative constant. Problem B with $A_3(z,w) = 0$ in D, $r(z) = 0$ on Γ and $b_j = 0\,(j \in J)$ is called Problem B_0. The condition $0 < K < N$ is called the singular case, which only occurs in the case of multiply connected domains, and is not easy to handle.

In order to prove the solvability of Problem B for the complex equation (1.1), we need to give a representation theorem for Problem B.

2. Integral representation of solutions for modified Riemann-Hilbert problem of elliptic complex equations

Now we transform the boundary condition (1.7) into the standard form and first find a solution $S(z)$ of the modified Dirichlet problem with the boundary condition

$$\mathrm{Re}S(z) = S_1(z) - \theta(t),\ S_1(t) = \begin{cases} \arg\lambda(t) - K\arg t,\ t \in \Gamma_0, \\ \arg\lambda(t),\ t \in \Gamma_j, j = 1,\dots,N, \end{cases}$$

$$\theta(t) = \begin{cases} 0,\ t \in \Gamma_0, \\ \theta_j,\ t \in \Gamma_j,\ j = 1,\dots,N, \end{cases} \qquad \mathrm{Im}[S(1)] = 0, \tag{2.1}$$

where $\theta_j\ (j = 1,\dots,N)$ are real constants. Thus the boundary condition (1.7) can be transformed into the standard boundary condition

$$\mathrm{Re}[\overline{\lambda(t)}w(t)] = \mathrm{Re}[\overline{\Lambda(t)}\Psi(t)] = r(t) + h(t), \quad t \in \Gamma,$$

$$\Lambda(t) = \lambda(t)\overline{e^{iS(t)}} = \begin{cases} t^K,\ t \in \Gamma_0, \\ e^{i\theta_j}, t \in \Gamma_j, j=1,\dots,N, \end{cases} \qquad w(z) = e^{iS(z)}\Psi(z), \tag{2.2}$$

where the index of $\Lambda(z)$ is also equal to K, and the point constant (1.9) is also equal to

$$\mathrm{Im}[\overline{\Lambda(a_j)}\Psi(a_j)] = b_j, \quad j \in J, \tag{2.3}$$

and $\Psi(z)$ satisfies the complex equation

$$\begin{aligned}\Psi_{\bar z} =& Q_1(z)\Psi_z + e^{-2i\mathrm{Re}S(z)}Q_2(z)\overline{\Psi}_{\bar z} + [A_1(z) + e^{-iS(z)}(e^{iS(z)})'Q_1]\Psi + \\ & [A_2 e^{-2i\mathrm{Re}S(z)} + e^{-iS(z)}\overline{(e^{iS(z)})'}Q_2]\overline{\Psi} + e^{-iS(z)}A_3, \quad z \in D. \end{aligned} \tag{2.4}$$

The above boundary value problem will be called Problem B′. It is easy to see the equivalence of Problem B with the boundary conditions (1.7), (1.9) for (1.6) and Problem B′ with the boundary conditions (2.2), (2.3) for (2.4).

Theorem 2.1 *Under the above conditions, Problem B with the index $K \geq 0$ for analytic functions has a solution.*

Proof We first find the solution of Problem B$'$, and then obtain the solution of Problem B. By using the formula (2.54), Chapter II, [6], we can introduce

$$P_0(z,t) = P_{N+1}(z,t) = \begin{cases} \dfrac{z^K e^{iS(z)}\lambda(t)(t+z)r(t)}{t^K e^{iS(t)}(t-z)\,t},\ t\in\Gamma_0, \\ 0,\ t\in\Gamma_j,\ j=1,\ldots,N, \end{cases}$$

$$P_j(z,t) = \begin{cases} \dfrac{e^{i\theta_j}e^{iS(z)}\lambda(t)r(t)(t+z-2z_j)}{e^{iS(t)}(t-z)(t-z_j)},\ t\in\Gamma_j, \\ 0,\ t\in\Gamma\setminus\Gamma_j,\ j=1,\ldots,N, \end{cases} \tag{2.5}$$

and find a solution of the boundary value problem with the boundary conditions

$$\begin{aligned} &\mathrm{Re}[\overline{\Lambda(z)}P_*(z,t)] = -\mathrm{Re}[\overline{\Lambda(z)}Q(z,t)] + h(z,t), \quad z\in\Gamma, \\ &Q(z,t) = \sum_{m=1,m\neq j}^{N+1} P_m(z,t), \quad z\in\Gamma_j,\ j=1,\ldots,N+1, \\ &\mathrm{Im}[\overline{\Lambda(a_j)}P_*(a_j,t)] = -\mathrm{Im}[\overline{\Lambda(a_j)}Q(a_j,t)], \quad a_j\in\Gamma, \\ &j\in J = \begin{cases} 1,\ldots,2K-N+1,\ \text{if}\ K>N-1, \\ 1,\ldots,K+1,\ \text{if}\ 0\leq K\leq N-1. \end{cases} \end{aligned} \tag{2.6}$$

$$P(z,t) = \sum_{j=1}^{N+1} P_j(z,t) + P_*(z,t),\ t\in\Gamma \tag{2.7}$$

is the Schwarz kernel of Problem B$'$. Thus we get the representation of solutions of Problem B$'$ as follows:

$$\Psi(z) = \frac{1}{2\pi i}\int_\Gamma P(z,t)r(t)\mathrm{d}t + \Psi_0(z), \tag{2.8}$$

in which $\Psi_0(z)$ is the solution of corresponding homogeneous problem, which can be determined by some point conditions

$$\mathrm{Im}[\overline{\Lambda(a_j)}\Psi_0(a_j)] = b_j - \mathrm{Im}\Big[\frac{\overline{\Lambda(a_j)}}{2\pi i}\int_\Gamma P(a_j,t)r(t)\mathrm{d}t\Big], \quad j\in J. \tag{2.9}$$

Thus the solution of original boundary value problem (Problem B) can be expressed as

$$w(z) = \Phi(z) = \Psi(z)e^{iS(z)} = \frac{1}{2\pi i}\int_\Gamma T(z,t)r(t)\mathrm{d}t + \Phi_0(z), \tag{2.10}$$

where $T(z,t) = P(z,t)e^{iS(z)}$, $T(z,t)$ is the Schwarz kernel, and $w_0(z) = \Phi_0(z) = \Psi_0(z)e^{iS(z)}$ is a solution of Problem B with the point conditions

$$\mathrm{Im}[\overline{\lambda(a_j)}\Phi_0(a_j)] = b_j - \mathrm{Im}\Big[\frac{\overline{\lambda(a_j)}}{2\pi i}\int_\Gamma T(a_j,t)r(t)\mathrm{d}t\Big], \quad j\in J.\ \square \tag{2.11}$$

Theorem 2.2 *Under the above conditions, Problem B with the index $K < 0$ for analytic functions has a solution.*

Proof Similarly to the proof of Theorem 2.1, we first find the solution of Problem B$'$. If $K < 0$, similarly to Theorem 2.1, we introduce

$$P_0(z,t) = P_{N+1}(z,t) = \begin{cases} \dfrac{2z^{|K|}e^{iS(z)}\lambda(t)r(t)}{e^{iS(t)}(t-z)t^{|K|}}, & t \in \Gamma_0, \\ 0, \quad t \in \Gamma_j, \ j = 1, \ldots, N, \end{cases}$$

$$P_j(z,t) = \begin{cases} \dfrac{e^{i\theta_j}e^{iS(z)}\lambda(t)r(t)(t+z-2z_j)}{e^{iS(t)}(t-z)(t-z_j)}, & t \in \Gamma_j, \\ 0, \quad t \in \Gamma\backslash\Gamma_j, \ j = 1, \ldots, N. \end{cases} \tag{2.12}$$

Similarly to the proof of Theorem 2.1, we can find a solution of the boundary value problem with the boundary conditions

$$\begin{aligned} &\mathrm{Re}[\overline{\Lambda(z)}P_*(z,t)] = -\mathrm{Re}[\overline{\Lambda(z)}Q(z,t)] + h(z,t), \quad z \in \Gamma, \\ &Q(z,t) = \sum_{m=1, m\neq j}^{N+1} P_m(z,t), \quad z \in \Gamma_j, \ j = 1, \ldots, N+1, \end{aligned} \tag{2.13}$$

and

$$P(z,t) = \sum_{j=1}^{N+1} P_j(z,t) + P_*(z,t), \quad t \in \Gamma \tag{2.14}$$

is the Schwarz kernel of Problem B$'$. Thus we get the representation of solutions of Problem B$'$ as follows:

$$\Psi(z) = \frac{1}{2\pi i}\int_\Gamma P(z,t)r(t)\mathrm{d}t. \tag{2.15}$$

Thus the solution of original boundary value problem (Problem B) can be expressed as

$$w(z) = \Phi(z) = \Psi(z)e^{iS(z)} = \frac{1}{2\pi i}\int_\Gamma T(z,t)r(t)\mathrm{d}t, \tag{2.16}$$

in which $T(z,t) = P(z,t)e^{iS(z)}$, $T(z,t)$ is the Schwarz kernel. In the above discussion, we have to use the $N - 2K - 1$ solvability conditions of Problem B, if $K < 0$.

We first consider the homogeneous modified Riemann-Hilbert problem (Problem B_0) for the complex equation (1.6), and give the integral representation of solutions of Problem B_0 for (1.1).

According to (2.65)–(2.74), Chapter I, [6], we introduce the two double integral operator of homogeneous modified Riemann-Hilbert problem (Problem B_0) for $w_{\bar z} = F$ in the domain D as follows

$$\begin{aligned} &\tilde{T}F = -\frac{2}{\pi}\iint_D [P(z,\zeta)F(\zeta) + Q(z,\zeta)\overline{F(\zeta)}]\mathrm{d}\sigma_\zeta = TF + \sum_{j=1}^{N+1} T_jF + T_*F, \\ &P(z,\zeta) = \frac{1}{2}[G_1(z,\zeta) + G_2(z,\zeta) + H_1(z,\zeta) - H_2(z,\zeta)], \quad z,\zeta \in \overline{D}, \\ &Q(z,\zeta) = \frac{1}{2}[G_1(z,\zeta) - G_2(z,\zeta) + H_1(z,\zeta) + H_2(z,\zeta)], \quad z,\zeta \in \overline{D}, \end{aligned}$$

$$G_1(z,\zeta)=\frac{1}{\zeta-z}+\sum_{j=1}^{N+1}g_j(z,\zeta),\ G_2(z,\zeta)=\frac{1}{\zeta-z}-\sum_{j=1}^{N+1}g_j(z,\zeta),\quad z,\zeta\in D,$$

$$g_0(z,\zeta)=g_{N+1}(z,\zeta)=\frac{z}{1-\bar{\zeta}z},\ g_j(z,\zeta)=\frac{e^{2i\theta_j}(z-z_j)}{r_j^2-(\overline{\zeta-z_j})(z-z_j)},\quad j=1,\ldots,N,\quad (2.17)$$

where $H_1(t,\zeta)$, $H_2(t,\zeta)$ are the solution with some boundary conditions. □

Theorem 2.3 *Let the complex equation (1.6) satisfy Condition C. Then any solution $w(z)$ ($w_{\bar{z}}\in L_{p_0}(\overline{D}), 2<p_0\le p$) of Problem B with the index $K=0$ for (1.6) possesses the representation*

$$w(z)=\Phi(z)+\tilde{T}F,\tag{2.18}$$

where $F(z)=w_{\bar{z}}$, $\Phi(z)$ is an analytic function as stated in (2.10) with $K=0$ in D, and $\tilde{T}F$ is as stated in (2.17), and $\Phi(z)$ satisfies the estimates

$$C_\beta[\Phi(z),\overline{D}]\le M_1,\ L_{p_0}[\Phi'(z),\overline{D}]\le M_2,\tag{2.19}$$

in which $\beta=\min(1-2/p_0,\alpha)$, $M_j=M_j(p_0,\beta,k,D), j=1,2,\ k=k(k_0,k_1,k_2,k_3)$. Moreover $\tilde{T}F$ satisfies the homogeneous boundary condition of Problem B, and $\tilde{S}F=(\tilde{F})_z$ possesses the properties

$$\|\tilde{S}F\|_{L_{p_0}(\overline{D})}\le\tilde{\Lambda}\|F\|_{L_{p_0}(\overline{D})},\quad \tilde{\Lambda}\le 1,\ \text{if } K=0.\tag{2.20}$$

and for a positive number $q_0<1$ there exists a constant $2<p_0\le p$ such that

$$q_0\tilde{\Lambda}_{p_0}<1.\tag{2.21}$$

Proof By using (3.6), Chapter I, [1], Theorems 2.1 and 2.2, we can get (2.19), and (2.20), (2.21) can be obtained by the method of Theorem 3.5, Chapter I, [4], Lemma 2.7, Chapter II, [6] and Theorem 3.1, [12]. □

3. Estimates of solutions for modified Riemann-Hilbert problem of elliptic complex equation in multiply connected domains

First of all, we give the estimates of solutions of Problem B for the equation (1.6).

Theorem 3.1 *Suppose that the first order complex equation (1.6) satisfies Condition C. Then any solution $w(z)$ of Problem B for the complex equation (1.6) satisfies the conditions*

$$C_\beta[w(z),\overline{D}]<M_3,\quad L_{p_0}[|w_{\bar{z}}|+|w_z|,\overline{D}]\le M_4,\tag{3.1}$$

in which $\beta=\min(1-2/p_0,\alpha)$, $k=k(k_0,k_1,k_2,k_3)$, $M_j=M_j(q_0,p_0,\beta,k,D)$ ($j=3,4$) are positive constants.

Proof Since the solution $w(z)$ of Problem B for the complex equation (1.6) can be expressed as (2.18), and the analytic function $\Phi(z)$ possesses the properties in (2.19), it is necessary to consider any solution $W(z)=\tilde{T}\omega$ of the complex equation:

$$\left.\begin{aligned}&W_{\bar{z}}=Q_1(z)W_z+Q_2(z)\overline{W}_{\bar{z}}+A_1(z)W+A_2(z)\bar{W}+A(z),\\&A(z)=Q_1(z)\Phi'(z)+Q_2(z)\overline{\Phi'(z)}+A_1(z)\Phi(z)+A_2(z)\overline{\Phi(z)}+A_3(z),\end{aligned}\right\}z\in D,\tag{3.2}$$

where $A(z) \in L_{p_0}(\overline{D})$.

We first verify the uniqueness of solutions of the homogeneous problem B_0 with the index $K \geq 0$, i.e., the solution $W(z) \equiv 0$ of the homogeneous problem B_0 for the homogeneous equation

$$W_{\bar{z}} = Q_1(z)W_z + Q_2(z)\overline{W}_{\bar{z}} + A_1(z)W + A_2(z)\overline{W} \text{ in } D \tag{3.3}$$

with the index $K \geq 0$. The solution $W(z)$ of (3.3) can be expressed as

$$W(z) = \Psi[\zeta(z)]e^{\phi(z)} \text{ in } D, \tag{3.4}$$

where $\zeta(z) = \eta(\chi(z))$ is a homeomorphism in $\overline{D}$, which quasiconformally maps D onto the $N+1$-connected circular domain G with boundary $L = \zeta(\Gamma)$ in $\{|\zeta| < 1\}$, such that three points on Γ are mapped onto three points on L respectively, $\Psi(\zeta)$ is an analytic function in G, $\phi(z) = i\tilde{T}_1 g(z)$, $\chi(z) = z + Th$ are the solutions of the complex equations

$$\phi(z) = i\tilde{T}_1 g, \quad \chi(z) = z + Th \tag{3.5}$$

of the complex equations

$$\begin{aligned} \phi_{\bar{z}} &= [Q_1 + Q_2\overline{W_z}/W_z]\phi_z + A_1 + A_2\overline{W}/W, \text{ in } D, \\ \chi_{\bar{z}} &= [Q_1 + Q_2\overline{W_z}/W_z]\chi_z \text{ in } D, \end{aligned} \tag{3.6}$$

respectively, $\tilde{T}_1 g$ is a double integral satisfying the modified Dirichlet boundary condition in D, $\chi(z)$ is a homeomorphism in $\overline{D}$, $\zeta = \eta(\chi)$ is a univalent analytic function, which conformally maps $E = \chi(D)$ onto the domain G, $\zeta(z) = \eta[\chi(z)]$ in D, and $\Psi(\zeta)$ is an analytic function in G. Since $\tilde{S}h = [\tilde{T}h]_z$ possesses the properties in (2.20) and (2.21), and $\tilde{S}h$ has the similar properties, we can get

$$L_{p_0}[g(z), \overline{D}] \leq L_{p_0}[|A_1| + |A_2|, \overline{D}]/(1 - q_0\tilde{\Lambda}_{p_0}),$$

$$L_{p_0}[h(z), \overline{D}] \leq L_{p_0}[|A_1| + |A_2|, \overline{D}]/(1 - q_0\Lambda_{p_0}).$$

By the principle of contract mapping, we can obtain that $\psi(z)$, $\chi(z)$ of the equations in (3.6), and $\psi(z)$, $\chi(z), \zeta(z)$ satisfy the estimates

$$\begin{aligned} &C_\beta[\phi, \overline{D}] \leq k_4,\ L_{p_0}[|\phi_{\bar{z}}| + |\phi_z|, \overline{D}] \leq k_4,\ L_{p_0}[|\chi_{\bar{z}}| + |\chi_z|, \overline{D}] \leq k_5, \\ &C_\beta[\zeta(z), \overline{D}] \leq k_4,\ C_\beta[z(\zeta), \overline{G}] \leq k_4, \end{aligned} \tag{3.7}$$

in which $\beta = \min(\alpha, 1 - 2/p_0)$, $p_0\,(2 < p_0 \leq p)$, $k_j = k_j(q_0, p_0, \beta, k_0, k_1, D)\,(j = 4, 5)$ are non-negative constants dependent on $q_0, p_0, \beta, k_0, k_1, D$, and $\Psi[\zeta(z)] = \tilde{T}\omega$ satisfies the the boundary condition

$$\text{Re}[\overline{\lambda(z(\zeta))}\Psi(\zeta)] = h(z(\zeta)) \text{ in } L \tag{3.8}$$

of homogeneous Problem B for analytic functions. According to Theorem 6.2, Chapter V, [6], it follows $\Psi(\zeta) \equiv 0$ in G.

If the index $K < 0$, we can use the method of Theorem 4.1, Chapter II, [4] to verify Problem B_0 at most has a solution.

Denote $4d = \min_{z\in\Gamma}|z|$ and $D_1 = \{|z| \le d\}$, $D_2 = \{d < |z| \le 2d\}$, $D_3 = \{2d < |z| \le 3d\}$, $D_4 = \{3d < |z| \le 4d\}$, and construct two continuously differential functions

$$\tau_1(z) = \begin{cases} 0 \text{ in } D_1, \\ 1 \text{ in } \overline{D}\backslash\{D_1 \cup D_2\}, \\ \tau_1(z) \text{ in } D_2, \end{cases} \quad \tau_2(z) = \begin{cases} 1 \text{ in } D_1 \cup D_2, \\ 0 \text{ in } \overline{D}\backslash\{D_1 \cup D_2 \cup D_3\}, \\ \tau_2(z) \text{ in } D_3, \end{cases}$$

where $0 \le \tau_1(z) \le 1$ in D_2 and $0 \le \tau_2(z) \le 1$ in D_3. From (3.2), we see that two functions $\tilde{W}(z) = \tau_1(z)z^{-K}W(z)$ and $\hat{W}(z) = \tau_2(z)W(z)$ are the solutions of following complex equations

$$\begin{aligned} &\tilde{W}_{\bar{z}} = Q_1\tilde{W}_z + Q_2\overline{\tilde{W}}_{\bar{z}} + A_1(z)\tilde{W} + [A_2(z)\tau_1 z^{-K}/\overline{\tau_1 z^{-K}}]\overline{\tilde{W}} + \tilde{A}, \\ &\tilde{A} = [(\tau_1 z^{-K})_{\bar{z}} - Q_1(\tau_1 z^{-K})_z]W - Q_2\overline{(\tau_1 z^{-K})_z}\overline{W} + \tau_1 z^{-K}A(z) \text{ in } D, \\ &\hat{W}_{\bar{z}} = Q_1\hat{W}_z + Q_2\overline{\hat{W}}_{\bar{z}} + A_1(z)\hat{W} + [A_2(z)\tau_2(z)/\overline{\tau_2(z)}]\overline{\hat{W}} + \hat{A}, \\ &\hat{A} = [\tau_{1\bar{z}} - Q_1\tau_{1z}]W - Q_2\overline{\tau_{1z}}\overline{W} + \tau_2(z)A(z) \text{ in } D, \end{aligned} \tag{3.9}$$

and satisfy the boundary conditions

$$\mathrm{Re}[\overline{\Lambda(z)}\tilde{W}(z)] = h(z) \text{ on } \Gamma, \ \mathrm{Re}[\overline{\Lambda(z)}\hat{W}(z)] = 0 \text{ on } \Gamma, \tag{3.10}$$

respectively. The indexes of above boundary value problems are equal to $K = 0$, and the function $W(z)$ is bounded in $\overline{D}$ from (2.19), (2.20), (3.4) and (3.7). Moreover by using Theorem 4.3, Chapter II, [4], we can obtain the estimates

$$\begin{aligned} &C_\beta[\tilde{W}(z), \overline{D}] \le M_5, \ L_{p_0}[|\tilde{W}_{\bar{z}}| + |\tilde{W}_z|, \overline{D}] \le M_6, \\ &C_\beta[\hat{W}(z), \overline{D}] \le M_7, \ L_{p_0}[|\hat{W}_{\bar{z}}| + |\hat{W}_z|, \overline{D}] \le M_8, \end{aligned}$$

where $M_j = M_j(q_0, p, \beta, k, D)\,(j = 5, 6, 7, 8)$ are positive constants. In particular we have

$$\begin{aligned} &C_\beta[W(z), \overline{D}\backslash\{D_1 \cup D_2\}] \le M_5, \ L_{p_0}[|W_{\bar{z}}| + |W_z|, \overline{D}\backslash\{D_1 \cup D_2\}] \le M_6, \\ &C_\beta[W(z), D_1 \cup D_2] \le M_7, \ L_{p_0}[|W_{\bar{z}}| + |W_z|, D_1 \cup D_2] \le M_8. \end{aligned}$$

Combining the above estimates, we get

$$\begin{aligned} &C_\beta[W(z), \overline{D}] \le M_9 = M_9(M_5, M_7, \tau_1, \tau_2, D), \\ &L_{p_0}[|W_{\bar{z}}| + |W_z|, \overline{D}] \le M_{10} = M_{10}(M_6, M_8, \tau_1, \tau_2, D). \end{aligned} \tag{3.11}$$

□

4. The solvability of modified Riemann-Hilbert problem for elliptic complex equations in multiply connected domains

Finally we prove the solvability of Problem B for the equation (1.1).

Theorem 4.1 *Under the conditions in Theorem 3.1, Problem B for (1.6) is solvable.*

Proof We introduce the quasi-linear elliptic complex equation with the parameter $t \in [0, 1]$:

$$w_{\bar{z}} - tF(z, w, w_z) = A(z), \tag{4.1}$$

where $A(z)$ is any measurable function in D and $A(z) \in L_{p_0}(\overline{D})\,(2 < p_0 \le p)$. When $t = 0$, it is not difficult to see that there exists a unique solution $w(z)$ of Problem B for the complex equation (4.1), which possesses the form

$$w(z) = \Phi(z) + \psi(z),\ \psi(z) = \tilde{T}A, \tag{4.2}$$

where $\tilde{T}A$ is stated as in (2.20), $\Phi(z)$ is an analytic function in D and satisfies the boundary conditions

$$\begin{aligned}&\mathrm{Re}[\overline{\lambda(z)}\Phi(z)] = r(z) - \mathrm{Re}[\overline{\lambda(z)}\psi(z)] + h(z),\ z \in \Gamma,\\ &\mathrm{Im}[\overline{\lambda(a_j)}\Phi(a_j)] = b_j - \mathrm{Im}[\overline{\lambda(a_j)}\psi(a_j)],\ j \in J.\end{aligned} \tag{4.3}$$

From Theorem 3.1, we see that $w(z) \in B = C_\beta(\overline{D}) \cap W^1_{p_0}(D)$. Suppose that when $t = t_0 (0 \le t_0 < 1)$, Problem B for the complex equation (4.1) has a unique solution, we shall prove that there exists a neighborhood of t_0 : $E = \{|t - t_0| \le \delta, 0 \le t \le 1\}$, $\delta\,(> 0)$ is a small positive constant, so that for every $t \in E$ and any function $A(z) \in L_{p_0}(\overline{D})$, Problem B for (4.1) is solvable. In fact, the complex equation (4.1) can be written in the form

$$w_{\bar z} - t_0 F(z, w, w_z) = (t - t_0)F(z, w, w_z) + A(z). \tag{4.4}$$

We arbitrarily select a function $w_0(z) \in B = C_\beta(\overline{D}) \cap W^1_{p_0}(D)$, in particular $w_0(z) = 0$ on $\overline{D}$. Let $w_0(z)$ be replaced into the position of $w(z)$ in the right hand side of (4.4). By Condition C, it is obvious that

$$B_0(z) = (t - t_0)F(z, w_0, w_{0z}) + A(z) \in L_{p_0}(\overline{D}). \tag{4.5}$$

Note that (4.4) has a solution $w_1(z) \in B$. Applying the successive iteration, we can find out a sequence of functions: $w_n(z) \in B$, $n = 1, 2, \dots$, which satisfy the complex equations

$$w_{n+1\bar z} - t_0 F(z, w_{n+1}, w_{n+1z}) = (t - t_0)F(z, w_n, w_{nz}) + A(z),\ n = 1, 2, \dots. \tag{4.6}$$

The difference of the above equations for $n + 1$ and n is as follows:

$$\begin{aligned}&(w_{n+1} - w_n)_{\bar z} - t_0[F(z, w_{n+1}, w_{n+1z}) - F(z, w_n, w_{nz})]\\ &\quad = (t - t_0)[F(z, w_n, w_{nz}) - F(z, w_{n-1}, w_{n-1z})],\quad n = 1, 2, \dots.\end{aligned} \tag{4.7}$$

From Condition C, it can be seen that

$$\begin{aligned}&F(z, w_{n+1}, w_{n+1z}) - F(z, w_n, w_{nz})\\ &\quad = F(z, w_{n+1}, w_{n+1z}) - F(z, w_{n+1}, w_{nz}) + [F(z, w_{n+1}, w_{nz}) - F(z, w_n, w_{nz})]\\ &\quad = \tilde{Q}_{n+1}(w_{n+1} - w_n)_z + \tilde{A}_{n+1}(w_{n+1} - w_n),\\ &\quad |\tilde{Q}_{n+1}| \le q_0 < 1,\ \tilde{A}_{n+1} \in L_{p_0}(\overline{D}),\ n = 1, 2, \dots,\\ &L_{p_0}[F(z, w_n, w_{nz}) - F(z, w_{n-1}, w_{n-1z}), \overline{D}]\\ &\quad \le q_0 L_{p_0}[(w_n - w_{n-1})_z, \overline{D}] + k_0 C[w_n - w_{n-1}, \overline{D}]\\ &\quad \le (q_0 + k_0)[C_\beta[w_n - w_{n-1}, \overline{D}] + L_{p_0}[|(w_n - w_{n-1})_{\bar z}| +\\ &\qquad |(w_n - w_{n-1})_z|, \overline{D}] = (q_0 + k_0)L_n.\end{aligned}$$

Moreover, $w_{n+1}(z)-w_n(z)$ satisfies the homogeneous boundary conditions

$$\begin{aligned}&\mathrm{Re}[\overline{\lambda(z)}(w_{n+1}(z)-w_n(z))]=h(z),\ z\in\Gamma,\\&\mathrm{Im}[\overline{\lambda(a_j)}(w_{n+1}(a_j)-w_n(a_j))]=0,\ j\in J.\end{aligned}\tag{4.8}$$

On the basis of Theorem 3.1, we have

$$L_{n+1}=C_\beta[w_{n+1}-w_n,\overline{D}]+L_{p_0}[|(w_{n+1}-w_n)_{\bar z}|+|(w_{n+1}-w_n)_z|,\overline{D}]\le M_{11}|t-t_0|(q_0+k_0)L_n,$$

where $M_{11}=M_3+M_4$, M_3, M_4 is as stated in (3.1). Provided $\delta\,(>0)$ is small enough, so that $\eta=\delta M_{11}(q_0+k_0)<1$, it can be obtained that

$$L_{n+1}\le\eta L_n\le\eta^nL_1=\eta^n[C_\beta(w_1,\overline{D})+L_{p_0}(|w_{1\bar z}|+|w_{1z}|,\overline{D})]\tag{4.9}$$

for every $t\in E$. Thus

$$\begin{aligned}S(w_n-w_m)&=C_\beta[w_n-w_m,\overline{D}]+L_{p_0}[|(w_n-w_m)_{\bar z}|+|(w_n-w_m)_z|,\overline{D}]\\&\le L_n+L_{n-1}+\cdots+L_{m+1}\le(\eta^{n-1}+\eta^{n-2}+\cdots+\eta^m)L_1\\&=\eta^m(1+\eta+\cdots+\eta^{n-m-1})L_1\le\eta^{N+1}\frac{1-\eta^{n-m}}{1-\eta}L_1\le\frac{\eta^{N+1}}{1-\eta}L_1\end{aligned}\tag{4.10}$$

for $n\ge m>N$, where N is a positive integer. This shows that $S(w_n-w_m)\to0$ as $n,m\to\infty$. Following the completeness of the Banach space $B=C_\beta(\overline{D})\cap W^1_{p_0}(D)$, there is a function $w_*(z)\in B$, such that when $n\to\infty$,

$$S(w-w_*)=C_\beta[w_n-w_*,\overline{D}]+L_{p_0}[(w_n-w_*)_{\bar z}|+|(w_n-w_*)_z|,\overline{D}]\to0.$$

By Condition C, from (3.1) it follows that $w_*(z)$ is a solution of Problem B for (4.4), i.e., (4.1) for $t\in E$. It is easy to see that the positive constant δ is independent of $t_0\,(0\le t_0<1)$. Hence from that Problem B for the complex equation (4.4) with $t=t_0=0$ is solvable, we can derive that when $t=\delta,2\delta,\dots,[1/\delta]\delta,1$, Problem B for (4.4) are solvable, especially Problem B for (4.4) with $t=1$ and $A(z)=0$, namely Problem B for (1.6) has a unique solution. □

Theorem 4.2 *Let the complex equation (1.1) satisfy Condition C. Then Problem A for (1.1) possesses the following solvability results.*

(1) If the index $K>N-1$, then Problem A for (1.1) is solvable, and its general solution includes $2K-N+1$ arbitrary real constants.

(2) If $0\le K\le N-1$, then Problem A has $N-K$ solvability conditions, and its general solution is dependent on $K+1$ arbitrary real constants.

(3) If $K<0$, then Problem A has $N-2K-1$ solvability conditions.

Proof Let the solution $w(z)$ of Problem B for (1.6) be substituted into the boundary condition (1.7). If the function $h(z)=0,\ z\in\Gamma$, i.e.,

$$\begin{cases}h_j=0,\ j=1,\dots,N,\ \text{ if }\ K\ge0,\\ h_j=0,j=0,1,\dots,N,\ h_m^\pm=0,\ m=1,\dots,-K-1,\ \text{if }\ K<0,\end{cases}$$

then the function $w(z)$ is just a solution of Problem A for (1.1). Hence the total number of above equalities is the total of solvability conditions as stated in this theorem. Also note that the real constants b_j $(j \in J)$ in (1.9) are arbitrarily chosen. This shows that the general solution of Problem A for (1.1) includes the number of arbitrary real constants as stated in the theorem. $\square$

References

[1] I. N. VEKUA. *Generalized Analytic Functions*. Pergamon, Oxford, 1962.

[2] M. A. LAVRENT'EV, B. V. SHABAT. *Methods of Function Theory of a Complex Variable*. GITTL, Moscow, 1958. (in Russian)

[3] A. V. BITSADZE. *Some Classes of Partial Differential Equations*. Gordon and Breach, New York, 1988.

[4] Guochun WEN, H. BEGEHR. *Boundary Value Problems for Elliptic Equations and Systems*. Longman Scientific and Technical Company, Harlow, 1990.

[5] Guochun WEN. *Conformal Mappings and Boundary Value Problems, Translations of Mathematics Monographs*. Amer. Math. Soc., Providence, RI, 1992.

[6] Guochun WEN. *Linear and Nonlinear Elliptic Complex Equations*. Shanghai Scientific and Technical Publishers, Shanghai, 1986. (in Chinese)

[7] Guochun WEN, Zhongwei DAI, Maoying TIAN. *Function Theoretic Methods of Free Boundary Problems and Their Applications to Mechanics*. Higher Education Press, Beijing, 1996. (in Chinese)

[8] Sha HUANG, Yuying QIAO, Guochun WEN. *Real and Complex Clifford Analysis*. Springer, 2005.

[9] Guochun WEN. *Approximate Methods and Numerical Analysis for Elliptic Complex Equations*. Gordon and Breach, Amsterdam, 1999.

[10] Guochun WEN. *Recent Progress in Theory and Applications of Modern Complex Analysis*. Science Press, Beijing, 2010.

[11] Guochun WEN, Dechang CHEN, Zuoliang XU. *Nonlinear Complex Analysis and Its Applications*. Mathematics Monograph Series 12, Science Press, Beijing, 2008.

[12] Guochun WEN. *The Poincaré problem with negative index for linear elliptic systems of second order in a multiply connected domain*. J. Math. Res. Exposition, 1982, **1**(1): 61–76. (in Chinese)