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Approximation Solvability of a New System of Set-Valued Variational Inclusions Involving Generalized $H(\cdot,\cdot)$-Accretive Mapping in Real q-Uniformly Smooth Banach Spaces

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Abstract A new system of set-valued variational inclusions involving generalized $H(\cdot,\cdot)$-accretive mapping in real q-uniformly smooth Banach spaces is introduced, and then based on the generalized resolvent operator technique associated with $H(\cdot,\cdot)$-accretivity, the existence and approximation solvability of solutions using an iterative algorithm is investigated.

Keywords generalized $H(\cdot,\cdot)$-accretive mapping; system of set-valued variational inclusions; resolvent operator method; iterative algorithm.

MR(2010) Subject Classification 49J40; 47J20

1. Introduction

In 1994, Hassouni and Moudafi [1] introduced and studied a class of variational inclusions and developed a perturbed algorithm for finding approximate solutions of the variational inclusion. Since then, Adly [2], Ding [3], Ding and Luo [4], Huang [5, 6], Huang et al. [7], Ahmad and Ansari [8] have obtained some important extensions of the results in various different assumptions. For more details, we refer to [1–26] and the references therein.

In 2001, Huang and Fang [9] were the first to introduce the generalized m-accretive mapping and gave the definition of the resolvent operator for the generalized m-accretive mapping in Banach spaces. Since then a number of researchers investigated several classes of generalized m-accretive mappings such as H-accretive, (H,η)-accretive and (A,η)-accretive mappings, see for example [4, 10–22]. Recently, Zou and Huang [23, 24] and Kazmi et al. [20] introduced and studied a class of $H(\cdot,\cdot)$-accretive mappings in Banach spaces, a natural extension of M-monotone mapping and studied variational inclusions involving these mappings. Luo and Huang [25] introduced and studied a new class of B-monotone mappings in Banach spaces, an extension of H-monotone mapping [13]. They showed some properties of the proximal-point mapping associ-

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ated with B-monotone mapping and obtained some applications for solving variational inclusions in Banach spaces.

Recently, Kazmi et al. [26] introduced a class of accretive mappings called generalized $H(\cdot,\cdot)$-accretive mappings, a natural generalizition of accretive (monotone) mapping studied in [13–15, 22, 23, 25] in Banach spaces. They proved that the proximal-point mapping of the generalized $H(\cdot,\cdot)$-accretive mapping is single-valued and Lipschitz continuous and they also studied a system of generalized variational inclusions involving generalized $H(\cdot,\cdot)$-accretive mappings in real q-uniformly smooth Banach spaces.

Motivated and inspired by the research work going on in this field, we introduce and study a new system of set-valued variational inclusions involving generalized $H(\cdot,\cdot)$-accretive mapping in real q-uniformly smooth Banach spaces, which include many systems of variational inclusions studied by others in recent years. By using the properties of the resolvent operator associated with generalized $H(\cdot,\cdot)$-accretive mappings, we explore the approximation solvability of the above-mentioned system of set-valued variational inclusions. The results presented in this paper extend and improve the corresponding results in the literature.

2. Preliminaries

Let E be a real Banach space with its norm $\|\cdot\|$, E^* the topological dual of E, and d the metric induced by the norm $\|\cdot\|$. We denote by 2^E, $\langle\cdot,\cdot\rangle$ and $CB(E)$ the family of all nonempty subsets of E, the dual pair of E and E^*, and the family of all nonempty closed bounded subsets of E, respectively. Let $D(\cdot,\cdot)$ be the Hausdorff metric on $CB(E)$ defined by

$$D(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$ and $d(A,y) = \inf_{x\in A} d(x,y)$.

The following concepts and results are needed in the sequel.

Definition 2.1 ([27]) *For $q > 1$, a mapping $J_q : E \longrightarrow 2^{E^*}$ is said to be generalized duality mapping, if it is defined by*

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q, \|x\|^{q-1} = \|f\|\}, \quad \forall x \in E.$$

In particular, J_2 is the usual normalized duality mapping on E. It is well known that [2]

$$J_q(x) = \|x\|^{q-2} J_2(x), \quad \forall x (\neq 0) \in E.$$

Note that if $E \equiv H$, a real Hilbert space, then J_2 becomes the identity mapping on H.

Definition 2.2 ([27]) *A Banach space E is called smooth if, for every $x \in E$ with $\|x\| = 1$, there exists a unique $f \in E^*$ such that $\|f\| = f(x) = 1$.*

The modulus of smoothness of E is the function $\rho_E : [0,\infty) \longrightarrow [0,\infty)$, defined by

$$\rho_E(\tau) = \sup\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\}.$$

Definition 2.3 ([27]) *The Banach space E is said to be* *(i) Uniformly smooth if* $\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau}=0$;

(ii) q-uniformly smooth, for $q>1$, *if there exists a constant* $c>0$ *such that* $\rho_E(\tau)\le c\tau^q, \tau\in[0,\infty)$.

It is well known that L_q(or l_q) is [22]

$$\begin{cases} q\text{-uniformly smooth, if } 1<q\le 2,\\ 2\text{-uniformly smooth, if } q\ge 2.\end{cases}$$

Note that if E is uniformly smooth, J_q becomes single-valued. In the study of characteristic inequalities in q-uniformly smooth Banach space, Xu [27] established the following lemma.

Lemma 2.4 *Let* $q>1$ *be a real number and let* E *be a smooth Banach space. Then* E *is* q*-uniformly smooth if and only if there exists a constant* $c_q>0$ *such that for every* $x,y\in E$

$$\|x+y\|^q\le\|x\|^q+q\langle y,J_q(x)\rangle+c_q\|y\|^q.$$

Definition 2.5 ([4]) *Let* E *be a real uniformly smooth Banach space,* $A:E\to CB(E)$. *A mapping* $N:E\times E\to E$ *is said to be* α*-strongly accretive with respect to* A *in the first argument if there exists a constant* $\alpha>0$ *such that*

$$\langle N(w_1,\cdot)-N(w_2,\cdot),J_q(u-v)\rangle\ge\alpha\|u-v\|^q,\quad \forall u,v\in E, w_1\in A(u), w_2\in A(v).$$

Similarly, we can define β*-strongly accretive with respect to* A *in the second argument.*

Lemma 2.6 ([28]) *Suppose that* $q>1$. *Then the following inequality holds*

$$ab\le\frac{1}{q}a^q+\frac{q-1}{q}b^{\frac{q-1}{q}}$$

for arbitrary positive real numbers a,b.

Definition 2.7 *A set-valued mapping* $s:E\to 2^E$ *is said to be* ξ*-D-Lipschitz continuous if there exists* $\xi>0$ *such that*

$$D(s(x),s(y))\le\xi\|x-y\|,\quad\forall x,y\in E.$$

Throughout the rest of the paper unless otherwise stated, we assume that E is q-uniformly smooth Banach space.

Definition 2.8 *A mapping* $A:E\to E$ *is said to be*

(i) Accretive if $\langle Ax-Ay,J_q(x-y)\rangle\ge 0$, $\forall x,y\in E$;

(ii) Strictly accretive if $\langle Ax-Ay,J_q(x-y)\rangle>0$, $\forall x,y\in E$ *and equality holds if and only if* $x=y$;

(iii) δ*-strongly accretive if there exists a constant* $\delta>0$ *such that* $\langle Ax-Ay,J_q(x-y)\rangle\ge\delta\|x-y\|^q$, $\forall x,y\in E$.

Definition 2.9 ([23]) *Let* $A,B:E\to E$ *be single-valued mappings and* $H:E\times E\to E$ *be mapping.*

(i) $H(A,\cdot)$ *is said to be* α*-strongly accretive with respect to* A *if there exists a constant* $\alpha>0$ *such that* $\langle H(Ax,u)-H(Ay,u),J_q(x-y)\rangle\ge\alpha\|x-y\|^q$, $\forall x,y\in E$;

(ii) $H(\cdot,B)$ *is said to be* β*-relaxed accretive with respect to* B *if there exists a constant* $\beta > 0$ *such that* $\langle H(u,Bx) - H(u,By), J_q(x-y)\rangle \geq -\beta\|x-y\|^q$, $\forall x,y \in E$;

(iii) $H(\cdot,\cdot)$ *is said to be* $\alpha\beta$*-symmetric accretive with respect to* A *and* B, *if* $H(A,\cdot)$ *is* α*-strongly accretive with respect to* A *and* $H(\cdot,B)$ *is* β*-relaxed accretive with respect to* B *with* $\alpha \geq \beta$, *and* $\alpha = \beta$ *if and only if* $x = y, \forall x,y,u \in E$;

(iv) $H(\cdot,\cdot)$ *is said to* ξ*-Lipschitz continuous with respect to the first argument if there exists a constant* $\xi > 0$ *such that* $\|H(x,u) - H(y,u)\| \leq \xi\|x-y\|$, $\forall x,y,u \in E$;

(v) $H(\cdot,\cdot)$ *is said to* η*-Lipschitz continuous with respect to the second argument if there exists a constant* $\eta > 0$ *such that* $\|H(u,x) - H(u,y)\| \leq \eta\|x-y\|$, $\forall x,y,u \in E$.

Definition 2.10 ([25]) *Let* $T : E \to 2^E, M : E \times E \to 2^E$ *be set-valued mappings, and* $f,g : E \to E$ *be single-valued mappings.*

(i) T *is accretive if* $\langle u-v, J_q(x-y)\rangle \geq 0$, $\forall x,y \in E$, $\forall u \in Tx, v \in Ty$;

(ii) T *is strictly accretive if* $\langle u-v, J_q(x-y)\rangle > 0$, $\forall x,y \in E$, $\forall u \in Tx, v \in Ty$ *and equality holds if and only if* $x = y$;

(iii) T *is* r*-strongly accretive if there exists a constant* $r > 0$ *such that* $\langle u-v, J_q(x-y)\rangle \geq r\|x-y\|^q$, $\forall x,y \in E$, $\forall u \in Tx, v \in Ty$;

(iv) T *is* s*-relaxed accretive if there exists a constant* $s > 0$ *such that* $\langle u-v, J_q(x-y)\rangle \geq -s\|x-y\|^q$, $\forall x,y \in E$, $\forall u \in Tx, v \in Ty$;

(v) $M(f,\cdot)$ *is said to be* α*-strongly accretive with respect to* f *if there exists a constant* $\alpha > 0$ *such that* $\langle u-v, J_q(x-y)\rangle \geq \alpha\|x-y\|^q$, $\forall x,y,w \in E$, $\forall u \in M(f(x).w), v \in M(f(y),w)$;

(vi) $M(\cdot,g)$ *is said to be* β*-relaxed accretive with respect to* g *if there exists a constant* $\beta > 0$ *such that* $\langle u-v, J_q(x-y)\rangle \geq -\beta\|x-y\|^q$, $\forall x,y,w \in E$, $\forall u \in M(w,g(x)), v \in M(w,g(y))$;

(vii) $M(\cdot,\cdot)$ *is said to be* $\alpha\beta$*-symmetric accretive with respect to* f *and* g, *if* $M(f,\cdot)$ *is* α*-strongly accretive with respect to* f *and* $M(\cdot,g)$ *is* β*-relaxed accretive with respect to* g *with* $\alpha \geq \beta$, *and* $\alpha = \beta$ *if and only if* $x = y$.

Now, we define the following concept.

Definition 2.11 ([26]) *Let* $A,B,f,g : E \to E$ *and* $H : E \times E \to E$ *be single-valued mappings. Let* $M : E \times E \to 2^E$ *be a set-valued mapping. The mapping* M *is said to be generalized* $\alpha\beta$-$H(\cdot,\cdot)$*-accretive with respect to* A,B,f *and* g, *if* $M(f,g)$ *is* $\alpha\beta$*-symmetric accretive with respect to* f *and* g, *and* $(H(A,B) + \lambda M(f,g))(E) = E$ *for every* $\lambda > 0$.

Lemma 2.12 ([26]) *Let* $A,B,f,g : E \to E$; *let* $H : E \times E \to E$ *be* $\alpha'\beta'$*-symmetric accretive mapping with respect to* A *and* B *and* $\alpha' > \beta'$, *and let* $M : E \times E \to 2^E$ *be a generalized* $\alpha\beta - H(\cdot,\cdot)$*-accretive mapping with respect to mappings* A,B,f *and* g. *If the following inequality:* $\langle u-v, J_q(x-y)\rangle \geq 0$, *holds for all* $(v,y) \in \mathrm{Graph}(M(f,g))$, *then* $(u,x) \in \mathrm{Graph}(M(f,g))$, *where* $\mathrm{Graph}(M(f,g)) = \{(u,x) \in E \times E : (u,x) \in M(f(x),g(x))\}$.

Lemma 2.13 ([26]) *Let* $A,B,f,g : E \to E$ *and let* $H : E \times E \to E$ *be* $\alpha'\beta'$*-symmetric accretive mapping with respect to* A *and* B. *Let* $M : E \times E \to 2^E$ *be a generalized* $\alpha\beta$-$H(\cdot,\cdot)$*-accretive mapping with respect to mappings* A,B,f *and* g. *Then the mapping* $(H(A,B) + \lambda M(f,g))^{-1}$

is single-valued for all $\lambda > 0$.

Definition 2.14 ([26]) *Let $A, B, f, g : E \to E$ be single-valued mappings and let $H : E \times E \to E$ be $\alpha'\beta'$-symmetric accretive mapping with respect to A and B. Let $M : E \times E \to 2^E$ be generalized $\alpha\beta$-$H(\cdot,\cdot)$-accretive mapping with respect to mappings A, B, f and g. The proximal-point mapping $R^{H(\cdot,\cdot)}_{M(\cdot,\cdot),\lambda} : E \to E$ is defined by*

$$R^{H(\cdot,\cdot)}_{M(\cdot,\cdot),\lambda}(x) = (H(A,B) + \lambda M(f,g))^{-1}(x), \quad \forall x \in E.$$

Lemma 2.15 ([26]) *Let $A, B, f, g : E \to E$ and let $H : E \times E \to E$ be $\alpha'\beta'$-symmetric accretive mapping with respect to A and B. Suppose that $M : E \times E \to 2^E$ is a generalized $\alpha\beta$-$H(\cdot,\cdot)$-accretive mapping with respect to mappings A, B, f and g. Then the proximal-point mapping $R^{H(\cdot,\cdot)}_{M(\cdot,\cdot),\lambda} : E \to E$ is Lipschitz continuous with constant L, that is,*

$$\|R^{H(\cdot,\cdot)}_{M(\cdot,\cdot),\lambda}(x^*) - R^{H(\cdot,\cdot)}_{M(\cdot,\cdot),\lambda}(y^*)\| \le L\|x^* - y^*\|, \quad \forall x^*, y^* \in E,$$

where $L = \frac{1}{[\lambda(\alpha-\beta)+(\alpha'-\beta')]}$.

3. System of set-valued variational inclusions and convergence analysis

Throughout the rest of the paper unless otherwise stated, we assume that, for each $i = 1, 2, E_i-$ is q_i-uniformly smooth Banach space with norm $\|\cdot\|_i$.

Let $A_1, B_1, f_1, g_1 : E_1 \to E_1, A_2, B_2, f_2, g_2 : E_2 \to E_2, m : E_2 \to E_1, n : E_1 \to E_2$ be nonlinear mappings, and $G : E_1 \to 2^{E_1}, Q : E_2 \to 2^{E_2}, U : E_1 \to 2^{E_1}$ and $V : E_2 \to 2^{E_2}$ any four set-valued mappings. Let $F_1, H_1 : E_1 \times E_2 \to E_1, F_2, H_2 : E_1 \times E_2 \to E_2$ be nonlinear mappings, and let $M_1 : E_1 \times E_1 \to 2^{E_1}$ and $M_2 : E_2 \times E_2 \to 2^{E_2}$ be generalized $\alpha_1\beta_1 - H_1(\cdot,\cdot)$-accretive and generalized $\alpha_2\beta_2 - H_2(\cdot,\cdot)$-accretive mappings, respectively. We consider the following system of set-valued variational inclusions (SSVI): find $(x, y) \in E_1 \times E_2, (s, v) \in G(x) \times Q(y), (u, t) \in U(x) \times V(y)$ such that

$$\begin{cases} m(y) \in F_1(s, v) + M_1(f_1(x), g_1(x)) \\ n(x) \in F_2(u, t) + M_2(f_2(y), g_2(y)) \end{cases}$$

We remark that for suitable choices of the mappings m, n, G, U, V, A_1, A_2, B_1, B_2, f_1, f_2, F_1, F_2, g_1, g_2, H_1, H_2, M_1, M_2 and the spaces E_1, E_2 reduce to various classes of system of variational inclusions and system of variational inequalities, see for example [11, 13–17, 19, 21, 22, 26] and the references therein.

Theorem 3.1 *$(x, y) \in E_1 \times E_2, (s, v) \in G(x) \times Q(y), (u, t) \in U(x) \times V(y)$ are solutions of (SSVI) if and only if (x, y, u, v, s, t) satisfies*

$$\begin{cases} x = R^{H_1(\cdot,\cdot)}_{M_1(\cdot,\cdot),\lambda_1}[H_1(A_1, B_1)(x) - \lambda_1 F_1(s, v) + \lambda_1 m(y)] \\ y = R^{H_2(\cdot,\cdot)}_{M_2(\cdot,\cdot),\lambda_2}[H_2(A_2, B_2)(y) - \lambda_2 F_2(u, t) + \lambda_2 n(x)] \end{cases} \tag{3.1}$$

where $\lambda_1, \lambda_2 > 0$ are constants;

$$R^{H_1(\cdot,\cdot)}_{M_1(\cdot,\cdot),\lambda_1}(x) = (H_1(A_1, B_1) + \lambda_1 M_1(f_1, g_1))^{-1}(x);$$

$$R^{H_2(\cdot,\cdot)}_{M_2(\cdot,\cdot),\lambda_2}(y) = (H_2(A_2, B_2) + \lambda_2 M_2(f_2, g_2))^{-1}(y), \forall x \in E_1, y \in E_2.$$

Proof This is an immediate consequence of the definitions of $R^{H_1(\cdot,\cdot)}_{M_1(\cdot,\cdot),\lambda_1}, R^{H_2(\cdot,\cdot)}_{M_2(\cdot,\cdot),\lambda_2}$, and hence, is omitted.

The relation (3.1) and Nadler [29] allow us to suggest the following iterative algorithm.

Algorithm 3.2 Let

$$z' = H_1(A_1, B_1)(x) - \lambda_1(F_1(s, v) - m(y))$$

and

$$z'' = H_2(A_2, B_2)(y) - \lambda_2(F_2(u, t) - n(x))$$

for convenience.

For given $(x_0, y_0) \in E_1 \times E_2, (s_0, v_0) \in G(x_0) \times Q(y_0), (u_0, t_0) \in U(x_0) \times V(y_0), (z'_0, z''_0) \in E_1 \times E_2$, compute

$$z'_1 = H_1(A_1, B_1)(x_0) - \lambda_1(F_1(s_0, v_0)) - m(y_0)),$$

$$z''_1 = H_2(A_2, B_2)(y_0) - \lambda_2(F_2(u_0, t_0) - n(x_0)).$$

For $(z'_1, z''_1) \in E_1 \times E_2$, we take $(x_1, y_1) \in E_1 \times E_2$ such that $x_1 = R^{H_1(\cdot,\cdot)}_{M_1(\cdot,\cdot),\lambda_1}(z'_1), y_1 = R^{H_2(\cdot,\cdot)}_{M_2(\cdot,\cdot),\lambda_2}(z''_1)$.

Then, by Nadler [29], there exist $(s_1, v_1) \in G(x_1) \times Q(y_1), (u_1, t_1) \in U(x_1) \times V(y_1)$ such that

$$\|u_1 - u_0\| \leq (1 + 1)D(U(x_1), U(x_0)),$$
$$\|v_1 - v_0\| \leq (1 + 1)D(Q(y_1), Q(y_0)),$$
$$\|s_1 - s_0\| \leq (1 + 1)D(G(x_1), G(x_0)),$$
$$\|t_1 - t_0\| \leq (1 + 1)D(V(y_1), V(y_0)),$$

where $D(\cdot,\cdot)$ is the Hausdorff metric on $CB(E_1)$ (for the sake of convenience we also denote by $D(\cdot,\cdot)$ the Hausdorff metric on $CB(E_2)$).

Compute

$$z'_2 = H_1(A_1, B_1)(x_1) - \lambda_1(F_1(s_1, v_1)) - m(y_1)),$$

$$z''_2 = H_2(A_2, B_2)(y_1) - \lambda_2(F_2(u_1, t_1) - n(x_1)).$$

By introduction, we can obtain sequences $(x_k, y_k) \in E_1 \times E_2, (s_k, v_k) \in G(x_k) \times Q(y_k), (u_k, t_k) \in U(x_k) \times V(y_k), (z'_k, z''_k) \in E_1 \times E_2$ by the iterative scheme:

$$x_k = R^{H_1(\cdot,\cdot)}_{M_1(\cdot,\cdot),\lambda_1}(z'_k), \quad y_k = R^{H_2(\cdot,\cdot)}_{M_2(\cdot,\cdot),\lambda_2}(z''_k).$$

$$u_k \in U(x_k), \|u_{k+1} - u_k\| \leq (1 + \frac{1}{k+1})D(U(x_{k+1}), U(x_k)),$$
$$v_k \in Q(y_k), \|v_{k+1} - v_k\| \leq (1 + \frac{1}{k+1})D(Q(y_{k+1}), Q(y_k)),$$
$$s_k \in G(x_k), \|s_{k+1} - s_k\| \leq (1 + \frac{1}{k+1})D(G(x_{k+1}), G(x_k)),$$

$$t_k \in V(y_k), \|t_{k+1} - t_k\| \le (1 + \frac{1}{k+1})D(V(y_{k+1}), V(y_k)),$$
$$z'_{k+1} = H_1(A_1, B_1)(x_k) - \lambda_1(F_1(s_k, v_k) - m(y_k)),$$
$$z''_{k+1} = H_2(A_2, B_2)(y_k) - \lambda_2(F_2(u_k, t_k) - n(x_k))$$

for $k = 0, 1, 2, \ldots$. We now study the convergence analysis of Algorithm 3.2.

Theorem 3.3 *For each $i = 1, 2$, let E_i be q_i-uniformly smooth Banach space; let $A_i, B_i, f_i, g_i : E_i \to E_i$ be single-valued mappings. Let $H_i : E_1 \times E_2 \to E_i$ be (α_i, δ_i) mixed Lipschitz continuous and $\alpha'_i\beta'_i$-symmetric accretive mappings with respect to A_i and B_i, which is ξ_i, η_i Lipschitz continuous.*

Let $M_1 : E_1 \times E_1 \to 2^{E_1}$ be generalized $\alpha_1\beta_1 - H_1(\cdot,\cdot)$-accretive mappings with respect to A_1, B_1, f_1 and g_1, and $M_2 : E_2 \times E_2 \to 2^{E_2}$ be generalized $\alpha_2\beta_2 - H_2(\cdot,\cdot)$-accretive mappings with respect to A_2, B_2, f_2 and g_2.

$F_i : E_1 \times E_2 \to E_i$ is Lipschitz continuous in both arguments with constants λ_{F_i} and $\overline{\lambda_{F_i}}$, respectively. $G : E_1 \to CB(E_1), Q : E_2 \to CB(E_2), U : E_1 \to CB(E_1), V : E_2 \to CB(E_2)$ be $D-$ Lipschitz continuous mappings with constants $\lambda_{DG}, \lambda_{DQ}, \lambda_{DU}$ and λ_{DV}, respectively. Let $m : E_2 \to E_1, n : E_1 \to E_2$ be Lipschitz continuous with constants λ_m and λ_n, respectively. F_1 is ω_1-strongly accretive with respect to G in the first argument and F_2 is ω_2-strongly accretive with respect to V in the second argument. If there exist $\lambda_1 > 0$ and $\lambda_2 > 0$, such that

$$\begin{cases} 0 < (s' + \sigma' + \sqrt[q_1]{\sigma'} + \tau' + \sqrt[q_2]{\tau'})L_1 < 1 \\ 0 < (s'' + \sigma' + \sqrt[q_1]{\sigma'} + \tau' + \sqrt[q_2]{\tau'})L_2 < 1 \end{cases} \tag{3.2}$$

where $s' = (1 - q_1(\alpha'_1 - \beta'_1) + c_{q_1}(\alpha_1\xi_1 + \delta_1\eta_1)^{q_1})^{\frac{1}{q_1}}$, $s'' = (1 - q_2(\alpha'_2 - \beta'_2) + c_{q_2}(\alpha_2\xi_2 + \delta_2\eta_2)^{q_2})^{\frac{1}{q_2}}$, $L_1 = \frac{1}{[\lambda_1(\alpha_1 - \beta_1) + (\alpha'_1 - \beta'_1)]}$, $L_2 = \frac{1}{[\lambda_2(\alpha_2 - \beta_2) + (\alpha'_2 - \beta'_2)]}$, $\sigma' = \max\{1 - \lambda_1 q_1\omega_1 + \lambda_1\lambda_m q_1 - \lambda_1\lambda_m, \lambda_1\lambda_m, \theta_1, \theta_2\}$, $\tau' = \max\{1 - \lambda_2 q_2\omega_2 + \lambda_2\lambda_n q_2 - \lambda_2\lambda_n, \lambda_2\lambda_n, \theta_3, \theta_4\}$, and $\theta_1 = \sqrt[q_1]{c_{q_1}}\lambda_1\lambda_{F_1}\lambda_{DG}$, $\theta_2 = \sqrt[q_1]{c_{q_1}}(\lambda_1\overline{\lambda}_{F_1}\lambda_{DQ} + \lambda_1\lambda_m)$, $\theta_3 = \sqrt[q_2]{c_{q_2}}(\lambda_2\lambda_{F_2}\lambda_{DU} + \lambda_n)$, $\theta_4 = \sqrt[q_2]{c_{q_2}}\lambda_2\overline{\lambda}_{F_2}\lambda_{DV}$.

Then the problem (SSVI) admits a solution (x, y, u, v, s, t) and the iterative sequences $\{x_k\}, \{y_k\}, \{u_k\}, \{v_k\}, \{s_k\}$ and $\{t_k\}$ generalized by Algorithm 3.2 converge to x, y, u, v, s and t, respectively.

Proof From Algorithm 3.2 we have

$$\begin{aligned}
&\|z'_{k+1} - z'_k\| \\
&\quad = \|H_1(A_1, B_1)(x_k) - \lambda_1(F_1(s_k, v_k)) - m(y_k)) - \\
&\qquad [H_1(A_1, B_1)(x_{k-1}) - \lambda_1(F_1(s_{k-1}, v_{k-1}) - m(y_{k-1}))]\| \\
&\quad \le \|H_1(A_1, B_1)(x_k) - H_1(A_1, B_1)(x_{k-1}) - (x_k - x_{k-1})\| + \\
&\qquad \|x_k - x_{k-1} - \lambda_1[F_1(s_k, v_k) - m(y_k) - (F_1(s_{k-1}, v_{k-1}) - m(y_{k-1}))]\|.
\end{aligned} \tag{3.3}$$

Since, for $i = 1, 2$, H_i is $\alpha'_i\beta'_i$-symmetric with respect to A_i and B_i, and (α_i, δ_i) mixed Lipschitz continuous, and A_i, B_i is ξ_i, η_i Lipschitz continuous, we have

$$\|H_1(A_1, B_1)(x_k) - H_1(A_1, B_1)(x_{k-1}) - (x_k - x_{k-1})\|^{q_1}$$

$$\begin{aligned}
&\leq \|x_k - x_{k-1}\|^{q_1} - q_1\langle H_1(A_1,B_1)(x_k) - H_1(A_1,B_1)(x_{k-1}), J_{q_1}(x_k - x_{k-1})\rangle + \\
&\quad c_{q_1}\|H_1(A_1,B_1)(x_k) - H_1(A_1,B_1)(x_{k-1})\|^{q_1} \\
&\leq (1 - q_1(\alpha_1' - \beta_1') + c_{q_1}(\alpha_1\xi_1 + \delta_1\eta_1)^{q_1})\|x_k - x_{k-1}\|^{q_1}. \qquad (3.4)
\end{aligned}$$

Since F_1 is Lipschitz continuous in both arguments, G, Q are D-Lipschitz continuous, and m is Lipschitz continuous, we have

$$\begin{aligned}
&\|F_1(s_k,v_k) - m(y_k) - (F_1(s_{k-1},v_{k-1}) - m(y_{k-1}))\| \\
&\quad = \|F_1(s_k,v_k) - F_1(s_{k-1},v_k) + F_1(s_{k-1},v_k) - F_1(s_{k-1},v_{k-1}) - (m(y_k) - m(y_{k-1}))\| \\
&\quad \leq \|F_1(s_k,v_k) - F_1(s_{k-1},v_k)\| + \|F_1(s_{k-1},v_k) - F_1(s_{k-1},v_{k-1})\| + \|m(y_k) - m(y_{k-1})\| \\
&\quad \leq \lambda_{F_1}\|s_k - s_{k-1}\| + \overline{\lambda}_{F_1}\|v_k - v_{k-1}\| + \lambda_m\|y_k - y_{k-1}\| \\
&\quad \leq \lambda_{F_1}(1+\frac{1}{k})D(G(x_k),G(x_{k-1})) + \overline{\lambda}_{F_1}(1+\frac{1}{k})D(Q(y_k),Q(y_{k-1})) + \lambda_m\|y_k - y_{k-1}\| \\
&\quad \leq (1+\frac{1}{k})\lambda_{F_1}\lambda_{DG}\|x_k - x_{k-1}\| + (1+\frac{1}{k})\overline{\lambda}_{F_1}\lambda_{DQ}\|y_k - y_{k-1}\| + \lambda_m\|y_k - y_{k-1}\| \\
&\quad = (1+\frac{1}{k})\lambda_{F_1}\lambda_{DG}\|x_k - x_{k-1}\| + [(1+\frac{1}{k})\overline{\lambda}_{F_1}\lambda_{DQ} + \lambda_m]\|y_k - y_{k-1}\|. \qquad (3.5)
\end{aligned}$$

Again, since F_1 is ω_1-strongly accretive with respect to G in the first argument, utilizing (3.3) and Lemmas 2.4, 2.6, we have

$$\begin{aligned}
&\|x_k - x_{k-1} - \lambda_1[F_1(s_k,v_k) - m(y_k) - (F_1(s_{k-1},v_{k-1}) - m(y_{k-1}))]\|^{q_1} \\
&\leq \|x_k - x_{k-1}\|^{q_1} - \lambda_1 q_1\langle F_1(s_k,v_k) - m(y_k) - (F_1(s_{k-1},v_{k-1}) - m(y_{k-1})), J_{q_1}(x_k - x_{k-1})\rangle + \\
&\quad c_{q_1}\lambda_1^{q_1}\|F_1(s_k,v_k) - m(y_k) - (F_1(s_{k-1},v_{k-1}) - m(y_{k-1}))\|^{q_1} \\
&= \|x_k - x_{k-1}\|^{q_1} - \lambda_1 q_1\langle F_1(s_k,v_k) - F_1(s_{k-1},v_{k-1}), J_{q_1}(x_k - x_{k-1})\rangle + \\
&\quad \lambda_1 q_1\langle m(y_k) - m(y_{k-1}), J_{q_1}(x_k - x_{k-1})\rangle + \\
&\quad c_{q_1}\lambda_1^{q_1}\|F_1(s_k,v_k) - m(y_k) - (F_1(s_{k-1},v_{k-1}) - m(y_{k-1}))\|^{q_1} \\
&\leq \|x_k - x_{k-1}\|^{q_1} - \lambda_1 q_1\omega_1\|x_k - x_{k-1}\|^{q_1} + \lambda_1 q_1\|m(y_k) - m(y_{k-1})\|\cdot\|x_k - x_{k-1}\|^{q_1} + \\
&\quad c_{q_1}\lambda_1^{q_1}\|F_1(s_k,v_k) - m(y_k) - (F_1(s_{k-1},v_{k-1}) - m(y_{k-1}))\|^{q_1} \\
&\leq (1 - \lambda_1 q_1\omega_1)\|x_k - x_{k-1}\|^{q_1} + \lambda_1 q_1\lambda_m\|y_k - y_{k-1}\|\cdot\|x_k - x_{k-1}\|^{q_1} + \\
&\quad c_{q_1}\lambda_1^{q_1}\|F_1(s_k,v_k) - m(y_k) - (F_1(s_{k-1},v_{k-1}) - m(y_{k-1}))\|^{q_1} \\
&\leq (1 - \lambda_1 q_1\omega_1)\|x_k - x_{k-1}\|^{q_1} + \lambda_1 q_1\lambda_m(\frac{1}{q_1}\|y_k - y_{k-1}\|^{q_1} + \frac{q_1-1}{q_1}\|x_k - x_{k-1}\|^{q_1}) + \\
&\quad c_{q_1}\lambda_1^{q_1}\{(1+\frac{1}{k})\lambda_{F_1}\lambda_{DG}\|x_k - x_{k-1}\| + [(1+\frac{1}{k})\overline{\lambda}_{F_1}\lambda_{DQ} + \lambda_m]\|y_k - y_{k-1}\|\}^{q_1} \\
&= (1 - \lambda_1 q_1\omega_1 + \lambda_1\lambda_m(q_1 - 1))\|x_k - x_{k-1}\|^{q_1} + \lambda_1\lambda_m\|y_k - y_{k-1}\|^{q_1} + \\
&\quad \{\sqrt[q_1]{c_{q_1}}\lambda_1(1+\frac{1}{k})\lambda_{F_1}\lambda_{DG}\|x_k - x_{k-1}\| + \sqrt[q_1]{c_{q_1}}\lambda_1[(1+\frac{1}{k})\overline{\lambda}_{F_1}\lambda_{DQ} + \lambda_m]\|y_k - y_{k-1}\|\}^{q_1} \\
&\leq [(\sigma + \sqrt[q_1]{\sigma})\|x_k - x_{k-1}\| + (\sigma + \sqrt[q_1]{\sigma})\|y_k - y_{k-1}\|]^{q_1}
\end{aligned}$$

which implies that

$$\|x_k - x_{k-1} - \lambda_1[F_1(s_k,v_k) - m(y_k) - (F_1(s_{k-1},v_{k-1}) - m(y_{k-1}))]\|$$

$$\leq (\sigma + \sqrt[q_1]{\sigma})\|x_k - x_{k-1}\| + (\sigma + \sqrt[q_1]{\sigma})\|y_k - y_{k-1}\|, \tag{3.6}$$

where

$$\sigma = \max\{1 - \lambda_1 q_1 \omega_1 + \lambda_1 \lambda_m q_1 - \lambda_1 \lambda_m, \lambda_1 \lambda_m, \sqrt[q_1]{c_{q_1}} \lambda_1 (1 + \frac{1}{k}) \lambda_{F_1} \lambda_{DG},$$
$$\sqrt[q_1]{c_{q_1}} \lambda_1 [(1 + \frac{1}{k}) \overline{\lambda}_{F_1} \lambda_{DQ} + \lambda_m]\}.$$

Note that $\lim_{k\to\infty} \sqrt[q_1]{c_{q_1}}\lambda_1(1+\frac{1}{k})\lambda_{F_1}\lambda_{DG} = \theta_1$, $\lim_{k\to\infty} \sqrt[q_1]{c_{q_1}}\lambda_1[(1+\frac{1}{k})\overline{\lambda}_{F_1}\lambda_{DQ}+\lambda_m] = \theta_2$, where $\theta_1 = \sqrt[q_1]{c_{q_1}}\lambda_1\lambda_{F_1}\lambda_{DG}$ and $\theta_2 = \sqrt[q_1]{c_{q_1}}(\lambda_1\overline{\lambda}_{F_1}\lambda_{DQ} + \lambda_1\lambda_m)$. Utilizing (3.4) and (3.6), we deduce from (3.3) that

$$\|z'_{k+1} - z'_k\| \leq (s' + \sigma + \sqrt[q_1]{\sigma})\|x_k - x_{k-1}\| + (\sigma + \sqrt[q_1]{\sigma})\|y_k - y_{k-1}\|, \tag{3.7}$$

where $s' = (1 - q_1(\alpha'_1 - \beta'_1) + c_{q_1}(\alpha_1\xi_1 + \delta_1\eta_1)^{q_1})^{\frac{1}{q_1}}$.

On the other hand, again from Algorithm 3.2, we have

$$\begin{aligned}
&\|z''_{k+1} - z''_k\| \\
&\quad = \|H_2(A_2, B_2)(y_k) - \lambda_2(F_2(u_k, t_k)) - n(x_k)) - \\
&\qquad [H_2(A_2, B_2)(y_{k-1}) - \lambda_2(F_2(u_{k-1}, t_{k-1}) - n(x_{k-1}))]\| \\
&\quad \leq \|H_2(A_2, B_2)(y_k) - H_2(A_2, B_2)(y_{k-1}) - (y_k - y_{k-1})\| + \\
&\qquad \|y_k - y_{k-1} - \lambda_2[F_2(u_k, t_k) - n(x_k) - (F_2(u_{k-1}, t_{k-1}) - n(x_{k-1}))]\|.
\end{aligned} \tag{3.8}$$

Utilizing the same arguments as those for (3.5), we have

$$\begin{aligned}
&\|H_2(A_2, B_2)(y_k) - H_2(A_2, B_2)(y_{k-1}) - (y_k - y_{k-1})\|^{q_2} \\
&\quad \leq \|y_k - y_{k-1}\|^{q_2} - q_2\langle H_2(A_2, B_2)(y_k) - H_2(A_2, B_2)(y_{k-1}), J_{q_2}(y_k - y_{k-1})\rangle + \\
&\qquad c_{q_2}\|H_2(A_2, B_2)(y_k) - H_2(A_2, B_2)(y_{k-1})\|^{q_2} \\
&\quad \leq (1 - q_2(\alpha'_2 - \beta'_2) + c_{q_2}(\alpha_2\xi_2 + \delta_2\eta_2)^{q_2})\|y_k - y_{k-1}\|^{q_2}.
\end{aligned} \tag{3.9}$$

Since F_2 is Lipschitz continuous in both arguments, U, V are D-Lipschitz continuous, and n is Lipschitz continuous, we have

$$\begin{aligned}
&\|F_2(u_k, t_k) - n(x_k) - (F_2(u_{k-1}, t_{k-1}) - n(x_{k-1}))\| \\
&\quad = \|F_2(u_k, t_k) - F_2(u_{k-1}, t_k) + F_2(u_{k-1}, t_k) - F_2(u_{k-1}, t_{k-1}) - (n(x_k) - n(x_{k-1}))\| \\
&\quad \leq \|F_2(u_k, t_k) - F_2(u_{k-1}, t_k)\| + \|F_2(u_{k-1}, t_k) - F_2(u_{k-1}, t_{k-1})\| + \|n(x_k) - n(x_{k-1})\| \\
&\quad \leq \lambda_{F_2}\|u_k - u_{k-1}\| + \overline{\lambda}_{F_2}\|t_k - t_{k-1}\| + \lambda_n\|x_k - x_{k-1}\| \\
&\quad \leq \lambda_{F_2}(1 + \frac{1}{k})D(U(x_k), U(x_{k-1})) + \overline{\lambda}_{F_2}(1 + \frac{1}{k})D(V(y_k), V(y_{k-1})) + \lambda_n\|x_k - x_{k-1}\| \\
&\quad \leq (1 + \frac{1}{k})\lambda_{F_2}\lambda_{DU}\|x_k - x_{k-1}\| + (1 + \frac{1}{k})\overline{\lambda}_{F_2}\lambda_{DV}\|y_k - y_{k-1}\| + \lambda_n\|x_k - x_{k-1}\| \\
&\quad = [(1 + \frac{1}{k})\lambda_{F_2}\lambda_{DU} + \lambda_n]\|x_k - x_{k-1}\| + (1 + \frac{1}{k})\overline{\lambda}_{F_2}\lambda_{DV}\|y_k - y_{k-1}\|.
\end{aligned} \tag{3.10}$$

It follows that F_2 is ω_2-strongly accretive with respect to the first argument. Utilizing (3.10) and Lemmas 2.4, 2.6 gives

$$\|y_k - y_{k-1} - \lambda_2[F_2(u_k, t_k) - n(x_k) - (F_2(u_{k-1}, t_{k-1}) - n(x_{k-1}))]\|^{q_2}$$

$$
\begin{aligned}
&\leq \|y_k - y_{k-1}\|^{q_2} - \lambda_2 q_2 \langle F_2(u_k,t_k) - n(x_k) - (F_2(u_{k-1},t_{k-1}) - n(x_{k-1})), J_{q_2}(y_k - y_{k-1})\rangle + \\
&\quad c_{q_2}\lambda_2^{q_2}\|F_2(u_k,t_k) - n(x_k) - (F_2(u_{k-1},t_{k-1}) - n(x_{k-1}))\|^{q_2} \\
&\leq \|y_k - y_{k-1}\|^{q_2} - \lambda_2 q_2 \langle F_2(u_k,t_k) - F_2(u_{k-1},t_{k-1}), J_{q_2}(y_k - y_{k-1})\rangle + \\
&\quad \lambda_2 q_2 \langle n(x_k) - n(x_{k-1}), J_{q_2}(y_k - y_{k-1})\rangle + \\
&\quad c_{q_2}\lambda_2^{q_2}\|F_2(u_k,t_k) - n(x_k) - (F_2(u_{k-1},t_{k-1}) - n(x_{k-1}))\|^{q_2} \\
&\leq \|y_k - y_{k-1}\|^{q_2} - \lambda_2 q_2 \omega_2 \|y_k - y_{k-1}\|^{q_2} + \lambda_2 q_2 \|n(x_k) - n(x_{k-1})\| \cdot \|y_k - y_{k-1}\|^{q_2-1} + \\
&\quad c_{q_2}\lambda_2^{q_2}\|F_2(u_k,t_k) - n(x_k) - (F_2(u_{k-1},t_{k-1}) - n(x_{k-1}))\|^{q_2} \\
&\leq (1 - \lambda_2 q_2 \omega_2)\|y_k - y_{k-1}\|^{q_2} + \lambda_2 q_2 \lambda_n (\frac{1}{q_2}\|x_k - x_{k-1}\|^{q_2} + \frac{q_2 - 1}{q_2}\|y_k - y_{k-1}\|^{q_2}) + \\
&\quad c_{q_2}\lambda_2^{q_2}\{[(1+\frac{1}{k})\lambda_{F_2}\lambda_{DU} + \lambda_n]\|x_k - x_{k-1}\| + (1+\frac{1}{k})\overline{\lambda}_{F_2}\lambda_{DV}\|y_k - y_{k-1}\|\}^{q_2} \\
&= (1 - \lambda_2 q_2 \omega_2 + \lambda_2 \lambda_n (q_2 - 1))\|y_k - y_{k-1}\|^{q_2} + \lambda_2 \lambda_n \|x_k - x_{k-1}\|^{q_2} + \\
&\quad \{\sqrt[q_2]{c_{q_2}}\lambda_2[(1+\frac{1}{k})\lambda_{F_2}\lambda_{DU} + \lambda_n]\|x_k - x_{k-1}\| + \sqrt[q_2]{c_{q_2}}\lambda_2(1+\frac{1}{k})\overline{\lambda}_{F_2}\lambda_{DV}\|y_k - y_{k-1}\|\}^{q_2} \\
&\leq [(\tau + \sqrt[q_2]{\tau})\|x_k - x_{k-1}\| + (\tau + \sqrt[q_2]{\tau})\|y_k - y_{k-1}\|]^{q_2}
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\|y_k - y_{k-1} - \lambda_2[F_2(u_k,t_k) - n(x_k) - (F_2(u_{k-1},t_{k-1}) - n(x_{k-1}))]\| \\
&\quad \leq (\tau + \sqrt[q_2]{\tau})\|x_k - x_{k-1}\| + (\tau + \sqrt[q_2]{\tau})\|y_k - y_{k-1}\|, \qquad (3.11)
\end{aligned}
$$

where

$$
\begin{aligned}
\tau = \max\{&1 - \lambda_2 q_2 \omega_2 + \lambda_2 \lambda_n q_2 - \lambda_2 \lambda_n, \lambda_2 \lambda_n, \sqrt[q_2]{c_{q_2}}\lambda_2[(1+\frac{1}{k})\lambda_{F_2}\lambda_{DU} + \lambda_n], \\
&\sqrt[q_2]{c_{q_2}}\lambda_2(1+\frac{1}{k})\overline{\lambda}_{F_2}\lambda_{DV}\}.
\end{aligned}
$$

Note that $\lim_{k\to\infty} \sqrt[q_2]{c_{q_2}}\lambda_2[(1+\frac{1}{k})\lambda_{F_2}\lambda_{DU} + \lambda_n] = \theta_3$, $\lim_{k\to\infty} \sqrt[q_2]{c_{q_2}}\lambda_2(1+\frac{1}{k})\overline{\lambda}_{F_2}\lambda_{DV} = \theta_4$, where $\theta_3 = \sqrt[q_2]{c_{q_2}}\lambda_2(\lambda_{F_2}\lambda_{DU} + \lambda_n)$ and $\theta_4 = \sqrt[q_2]{c_{q_2}}\lambda_2\overline{\lambda}_{F_2}\lambda_{DV}$. Utilizing (3.9) and (3.11), we deduce from (3.8) that

$$
\|z''_{k+1} - z''_k\| \leq (s'' + \tau + \sqrt[q_2]{\tau})\|y_k - y_{k-1}\| + (\tau + \sqrt[q_2]{\tau})\|x_k - x_{k-1}\|, \qquad (3.12)
$$

where $s'' = (1 - q_2(\alpha'_2 - \beta'_2) + c_{q_2}(\alpha_2\xi_2 + \delta_2\eta_2)^{q_2})^{\frac{1}{q_2}}$.

Adding (3.7) and (3.12), we have

$$
\begin{aligned}
&\|z'_{k+1} - z'_k\| + \|z''_{k+1} - z''_k\| \\
&\quad \leq (s' + \sigma + \sqrt[q_1]{\sigma})\|x_k - x_{k-1}\| + (\sigma + \sqrt[q_1]{\sigma})\|y_k - y_{k-1}\| + \\
&\qquad (s'' + \tau + \sqrt[q_2]{\tau})\|y_k - y_{k-1}\| + (\tau + \sqrt[q_2]{\tau})\|x_k - x_{k-1}\| \\
&\quad \leq (s' + \sigma + \sqrt[q_1]{\sigma} + \tau + \sqrt[q_2]{\tau})\|x_k - x_{k-1}\| + \\
&\qquad (s'' + \tau + \sqrt[q_2]{\tau} + \sigma + \sqrt[q_1]{\sigma})\|y_k - y_{k-1}\|. \qquad (3.13)
\end{aligned}
$$

Also from the iterative scheme, we have

$$
\|x_k - x_{k-1}\| = \|R^{H_1(\cdot,\cdot)}_{M_1(\cdot,\cdot),\lambda_1}(z'_k) - R^{H_2(\cdot,\cdot)}_{M_2(\cdot,\cdot),\lambda_2}(z'_{k-1})\| \leq L_1\|z'_k - z'_{k-1}\| \qquad (3.14)
$$

where $L_1 = \frac{1}{[\lambda_1(\alpha_1-\beta_1)+(\alpha_1'-\beta_1')]}$, and

$$\|y_k - y_{k-1}\| = \|R^{H_2(\cdot,\cdot)}_{M_2(\cdot,\cdot),\lambda_2}(z_k'') - R^{H_2(\cdot,\cdot)}_{M_2(\cdot,\cdot),\lambda_2}(z_{k-1}'')\| \le L_2\|z_k'' - z_{k-1}''\| \tag{3.15}$$

where $L_2 = \frac{1}{[\lambda_2(\alpha_2-\beta_2)+(\alpha_2'-\beta_2')]}$.

Utilizing (3.14) and (3.15), we conclude from (3.13) that

$$\begin{aligned}&\|z_{k+1}' - z_k'\| + \|z_{k+1}'' - z_k''\| \\ &\le [(s' + \sigma + \sqrt[q_1]{\sigma} + \tau + \sqrt[q_2]{\tau})L_1]\|z_k' - z_{k-1}'\| + \\ &\quad [(s'' + \tau + \sqrt[q_2]{\tau} + \sigma + \sqrt[q_1]{\sigma})L_2]\|z_k'' - z_{k-1}''\|.\end{aligned} \tag{3.16}$$

Observe that

$$\lim_{k\to\infty} \sigma = \sigma' = \max\{1 - \lambda_1 q_1\omega_1 + \lambda_1\lambda_m q_1 - \lambda_1\lambda_m, \lambda_1\lambda_m, \theta_1, \theta_2\}, \tag{3.17}$$

$$\lim_{k\to\infty} \tau = \tau' = \max\{1 - \lambda_2 q_2\omega_2 + \lambda_2\lambda_n q_2 - \lambda_2\lambda_n, \lambda_2\lambda_n, \theta_3, \theta_4\}. \tag{3.18}$$

By (3.2), we know that $0 < s < 1$, where

$$s = \max\{(s' + \sigma' + \sqrt[q_1]{\sigma'} + \tau' + \sqrt[q_2]{\tau'})L_1, (s'' + \sigma' + \sqrt[q_1]{\sigma'} + \tau' + \sqrt[q_2]{\tau'})L_2\}.$$

Now we take a fixed $s_0 \in (0,1)$ arbitrarily. Then from (3.17) and (3.18) it follows that there exists an integer $\overline{k} \ge 1$ such that for all $k \ge \overline{k}$,

$$(s' + \sigma + \sqrt[q_1]{\sigma} + \tau + \sqrt[q_2]{\tau})L_1 < s_0, \quad (s'' + \sigma + \sqrt[q_1]{\sigma} + \tau + \sqrt[q_2]{\tau})L_2 < s_0, \tag{3.19}$$

so, we obtain from (3.16) that

$$\|z_{k+1}' - z_k'\| + \|z_{k+1}'' - z_k''\| \le s_0(\|z_k' - z_{k-1}'\| + \|z_k'' - z_{k-1}''\|), \quad \forall k \ge \overline{k}, \tag{3.20}$$

which implies that $\{z_k'\}$ and $\{z_k''\}$ are both Cauchy sequences. Thus, there exist $z' \in E_1$ and $z'' \in E_2$ such that $z_k' \to z'$ and $z_k'' \to z''$ as $k \to \infty$.

From (3.14) and (3.15) it follows that $\{x_k\}$ and $\{y_k\}$ are also Cauchy sequences in E_1 and E_2, respectively, that is, there exist $x \in E_1, y \in E_2$, such that $x_k \to x$ and $y_k \to y$ as $k \to \infty$. Also from the iterative scheme, we have

$$\begin{aligned}\|u_{k+1} - u_k\| &\le (1 + \frac{1}{k+1})D(U(x_{k+1}), U(x_k)) \le (1 + \frac{1}{k+1})\lambda_{DU}\|x_{k+1} - x_k\|,\\ \|v_{k+1} - v_k\| &\le (1 + \frac{1}{k+1})D(Q(y_{k+1}), Q(y_k)) \le (1 + \frac{1}{k+1})\lambda_{DQ}\|y_{k+1} - y_k\|,\\ \|s_{k+1} - s_k\| &\le (1 + \frac{1}{k+1})D(G(x_{k+1}), G(x_k)) \le (1 + \frac{1}{k+1})\lambda_{DG}\|x_{k+1} - x_k\|,\\ \|t_{k+1} - t_k\| &\le (1 + \frac{1}{k+1})D(V(y_{k+1}), V(y_k)) \le (1 + \frac{1}{k+1})\lambda_{DV}\|y_{k+1} - y_k\|,\end{aligned}$$

and hence $\{u_k\}, \{v_k\}, \{s_k\}$ and $\{t_k\}$ are also Cauchy sequences. Accordingly, there exist $u, s \in E_1$ and $v, t \in E_2$, such that $u_k \to u, v_k \to v, s_k \to s$ and $t_k \to t$, respectively.

Now, we will show that $u \in U(x), v \in Q(y), s \in G(x)$ and $t \in V(y)$. Indeed, since $u_k \in U(x_k)$ and

$$d(u_k, U(x)) \le \max\{d(u_k, U(x)), \sup_{\omega_1 \in U(x)} d(U(x_k), \omega_1)\}$$

$$\begin{aligned}&\le \max\{\sup_{\omega_2\in U(x_k)} d(\omega_2, U(x)), \sup_{\omega_1\in U(x)} d(U(x_k),\omega_1)\}\\ &= D(U(x_k),U(x)),\end{aligned}$$

we have

$$\begin{aligned}d(u,U(x)) &\le \|u-u_k\| + d(u_k,U(x)) \le \|u-u_k\| + D(U(x_k),U(x))\\ &\le \|u-u_k\| + \lambda_{DU}\|x_k - x\| \to 0, \text{ as } k\to\infty,\end{aligned}$$

which implies that $d(u,U(x))=0$.

Taking into account that $U(x)\in CB(E_1)$, we deduce that $u\in U(x)$. Similarly, we can show that $v\in Q(y), s\in G(x)$ and $t\in V(y)$.

By the continuity of $H_1,H_2,A_1,A_2,B_1,B_2,m,n,R^{H_1(\cdot,\cdot)}_{M_1(\cdot,\cdot),\lambda_1}, R^{H_2(\cdot,\cdot)}_{M_2(\cdot,\cdot),\lambda_2}, G,Q,U,V,F_1,F_2$ and Algorithm 3.2, we know that x,y,u,v,s,t satisfy the following relation:

$$\begin{cases} x = R^{H_1(\cdot,\cdot)}_{M_1(\cdot,\cdot),\lambda_1}[H_1(A_1,B_1)(x) - \lambda_1 F_1(s,v) + \lambda_1 m(y)]\\ y = R^{H_2(\cdot,\cdot)}_{M_2(\cdot,\cdot),\lambda_2}[H_2(A_2,B_2)(y) - \lambda_2 F_2(u,t) + \lambda_2 n(x)]\end{cases}$$

By Theorem 3.1, (x,y,u,v,s,t) is a solution of problem (SSVI). This completes the proof. □

4. An application

Condition (3.2) in Theorem 3.3 holds for some suitable value of constants, for example, we now apply the results of Theorem 3.3 to L^p spaces. Assume $p=3$ and t_p is the unique solution of the equation $(p-2)t^{p-1}+(p-1)t^{p-2}-1=0, 0<t<1$, then $C_p=(1+t_p^{p-1})(1+t_p)^{1-p}$. Let $q_1=q_2=3, C_{q_1}=C_{q_2}=2-\sqrt{2}, \alpha_1=\alpha_2=\alpha'_1=\alpha'_2=0.4, \beta_1=\beta_2=\beta'_1=\beta'_2=0.1$, $\xi_1=\delta_1=\eta_1=0.1, \xi_2=\delta_2=\eta_2=0.1, \lambda_1=\lambda_2=10, \omega_1=\omega_2=0.03, \lambda_m=\lambda_n=0.01$, $\lambda_{F_1}=\lambda_{DG}=\overline{\lambda}_{F_1}=\lambda_{DQ}=0.01, \lambda_{F_2}=\lambda_{DU}=\overline{\lambda}_{F_2}=\lambda_{DV}=0.01$. Thus, if all the conditions for Theorem 3.3 are satisfied, one can apply Theorem 3.3 to the approximation-solvability of the following system of set-valued variational inclusion problem:

find $(x,y)\in L^3\times L^3, (s,v)\in G(x)\times Q(y), (u,t)\in U(x)\times V(y)$ such that

$$\begin{cases} m(y)\in F_1(s,v) + M_1(f_1(x),g_1(x))\\ n(x)\in F_2(u,t)+M_2(f_2(y),g_2(y))\end{cases}$$

where corresponding mappings are above mentioned.

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