Journal of Mathematical Research with Applications
Jul., 2014, Vol. 34, No. 4, pp. 459–466
DOI:10.3770/j.issn:2095-2651.2014.04.008
Http://jmre.dlut.edu.cn

On the Structures of Hom-Lie Algebras

Shengxiang WANG[1,*], **Xiaohui ZHANG**[2]
1. *School of Mathematics Sciences, Chuzhou University, Anhui* 239000*, P. R. China;*
2. *Department of Mathematics, Southeast University, Jiangsu* 210096*, P. R. China*

Abstract Let A be a multiplicative Hom-associative algebra and L a multiplicative Hom-Lie algebra together with surjective twisting maps. We show that if A is a sum of two commutative Hom-associative subalgebras, then the commutator Hom-ideal is nilpotent. Furthermore, we obtain an analogous result for Hom-Lie algebra L extending Kegel's Theorem. Finally, we discuss the Hom-Lie ideal structure of a simple Hom-associative algebra A by showing that any non-commutative Hom-Lie ideal of A must contain $[A, A]$.

Keywords Hom-associative algebra; Hom-Lie algebra; Kegel's theorem.

MR(2010) Subject Classification 16G30; 17B05

1. Introduction

In recent years, Hom-structures including Hom-Lie algebras, Hom-algebras, Hom-coalgebras, Hom-modules, Hom-Hopf modules were widely studied. Hom-Lie algebras were firstly studied by Hartwig, Larsson and Silvestrov in [1], where the structure of Hom-Lie algebras was used to study the deformations of Witt and Virasoro algebras. Later, motivated by the new examples arising as an application of the general quasi-deformation construction and the desire to be able to treat within the same framework of the super and color Lie algebras, Larsson and Silvestrov extended the notion of Hom-Lie algebras to quasi-Hom Lie algebras and quasi-Lie algebras [2, 3]. Makhlouf and Silvestrov introduced the notions of Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras in [4–6].

Simultaneously, Yau [7–9] studied quasitriangular Hom-Hopf algebras and Hom-Yang Baxter equations. In [10], Caenepeel and Goyvaerts studied Hom-Hopf algebras and Hom-Lie algebras from a categorical view point and called them monoidal Hom-Hopf algebras and monoidal Hom-Lie algebras. Zhang et al. [11, 12] studied the antipode, integral and Drinfeld double of Hom-Hopf algebras. In [13], Sheng studied representations of Hom-Lie algebras, including derivations, deformations, central extensions and derivation extensions of Hom-Lie algebras. Later, Sheng and Bai [14] constructed solutions of the classical Yang-Baxter equations in terms of o-operators. In

Received December 1, 2012; Accepted April 16, 2014
Supported by the Excellent Young Talents Fund Project of Anhui Province (Grant No. 2013SQRL092ZD), the Natural Science Foundation of Anhui Province (Grant Nos. 1408085QA06; 1408085QA08), the Excellent Young Talents Fund Project of Chuzhou University (Grant No. 2013RC001) and the Research and Innovation Project for College Graduates of Jiangsu Province (Grant No. CXLX12-0071).
* Corresponding author
E-mail address: wangsx-math@163.com (Shengxiang WANG)

[15], Cheng and Su considered cohomology and universal central extension of Hom-Leibniz algebras, Cheng and Yang studied low-dimensional cohomology of q-deformed Heisenberg-Virasoro algebra of Hom-type in [16].

Kegel's Theorem is classical in the theory of the rings and algebras. Starting from the fact that a product of two commutative subrings is always a metabelian ring [17], Bahtuein and Kegel [18] proved that the sum of two abelian subalgebras is always a metabelian Lie algebra. Wang [19] studied the structure of a class of generalized Lie algebras and obtained an analogy of Kegel's Theorem for generalized Lie algebras. It is a natural question to ask whether we can obtain some results for the Hom-Lie algebras that are analogous to Kegel's result. This becomes our first motivation of the paper.

Meanwhile, Bahturin, Fishman and Montgomery studied the structure of the generalized Lie algebras in the category of comodules in [20]. Wang [19] extended the above work to generalized H-Lie algebras and obtained some extended results. As a natural generalization of generalized Lie algebras, we considered Hom-Lie algebras in Yetter-Drinfeld categories in [21], in which we mainly focused on the regular Hom-Lie algebras together with bijective twisting maps. Naturally, we hope to extend the above work to the case of Hom-Lie algebras under some weak conditions. This becomes our second motivation of the paper.

To give a positive answer to the questions above, we organize this paper as follows. In Section 2, we recall some basic definitions about Hom-associative algebras and Hom-Lie algebras. In Section 3, we obtain an analogue of Kegel's Theorem for Hom-Lie algebras (see Theorem 4.3). In Section 4, we determine the Hom-Lie ideal structures of the Hom-associative algebras (see Theorems 4.4 and 4.9).

2. Preliminaries

In this paper, k always denotes a fixed field, often omitted from the notation.

Definition 2.1 ([4]) *A Hom-associative algebra is a triple (A, m, α) consisting of a linear space A, a bilinear map $m : A \otimes A \to A$ and a linear map $\alpha : A \to A$ such that*

$$\alpha(a)(bc) = (ab)\alpha(c), \quad a, b, c \in A.$$

Definition 2.2 ([4]) *A Hom-Lie algebra is a triple $(L, [,], \alpha)$ consisting of a linear space L, a bilinear map $[,] : L \otimes L \to L$ and a linear map $\alpha : L \to L$ satisfying*

(1) Skew-symmetry: $[l, l'] = -[l', l],\ l, l' \in L$.

(2) Hom-Jacobi identity: $[\alpha(l), [l', l'']] + [\alpha(l'), [l'', l]] + [\alpha(l''), [l, l']] = 0,\ l, l', l'' \in L$.

A Hom-associative algebra (resp., A Hom-Lie algebra) is called multiplicative if α is an algebraic morphism. A Hom-associative algebra (resp., A Hom-Lie algebra) is called regular if α is an algebra automorphism.

Notice that monoidal Hom-algebras [10] may be viewed as regular Hom-associative algebras, monoidal Hom-Lie algebras may be viewed as regular Hom-Lie algebras, in which twisting maps are bijective. In the present paper, we always suppose that Hom-associative algebras and Hom-

Lie algebras are multiplicative together with surjective twisting maps.

3. Kegel's Theorem for Hom-Lie algebras

In this section, we will consider an analogue of Kegel's Theorem for Hom-Lie algebras, which extends the Kegel's Theorem for Lie algebras by setting $\alpha = id$.

Theorem 3.1 *Let α be an epimorphism and (A, α) a Hom-associative algebra with Hom-associative subalgebras X and Y which are commutative such that $A = X+Y$. Then $[A, A][A, A] = 0$.*

Proof It suffices to prove $[u, x][v, y] = 0$ holds for all $u, v \in X$ and $x, y \in Y$. Since α is an epimorphism and X, Y are Hom-associative subalgebras, there exist $u', v' \in X$ and $x', y' \in Y$ such that $\alpha(x') = x, \alpha(y') = y, \alpha(u') = u, \alpha(v') = v$. According to the Hom-associativity of A, we have $(ux)(vy) = (ux)(\alpha(v')\alpha(y')) = \alpha(u)(\alpha(x')(v'y')) = \alpha(u)((x'v')\alpha(y')) = (u(x'v'))\alpha(y)$. Let $x'v' = w + z, u'y' = w' + z'$, where $w, w' \in X, z, z' \in Y$. To prove our conclusion, we need the following calculations.

$$\begin{aligned}
[u, x][v, y] &= (ux - xu)(vy - yv) \\
&= (ux)(vy) - (xu)(vy) - (ux)(yv) + (xu)(yv) \\
&= (u(x'v'))\alpha(y) - (xu)(vy) - (ux)(yv) + (xu)(yv) \\
&= (uw)\alpha(y) + (uz)\alpha(y) + (xu)(yv) - (x(u'v'))\alpha(y) - (u(x'y'))\alpha(v) \\
&= (uw)\alpha(y) + (uz)\alpha(y) + (xu)(yv) - (x(v'u'))\alpha(y) - (u(y'x'))\alpha(v) \\
&= (uw)\alpha(y) + (uz)\alpha(y) + (xu)(yv) - ((x'v')u)\alpha(y) - ((u'y')x))\alpha(v) \\
&= (uw)\alpha(y) + \alpha(u)(zy) + (xu)(yv) - ((x'v')u)\alpha(y) - (uy)(xv) \\
&= (uw)\alpha(y) + \alpha(u)(zy) + (xu)(yv) - (wu)\alpha(y) - (zu)\alpha(y) - (uy)\alpha(w) - (uy)\alpha(z) \\
&= (uw)\alpha(y) + \alpha(u)(yz) + (xu)(yv) - (uw)\alpha(y) - (zu)\alpha(y) - (zu)\alpha(y) - \alpha(u)(yz) \\
&= (xu)(yv) - (zu)\alpha(y) - (uy)\alpha(w) \\
&= (xu)(yv) - \alpha(z)(uy) - (uy)\alpha(w) \\
&= (xu)(yv) - \alpha(z)\alpha(w') - \alpha(z)\alpha(z') - \alpha(w')\alpha(w) - \alpha(z')\alpha(w).
\end{aligned}$$

Now we compute the expression $(xu)(yv)$ in the last term as follows:

$$\begin{aligned}
(xu)(yv) &= (x(u'y'))\alpha(v) = (xw')\alpha(v) + (xz')\alpha(v) \\
&= \alpha(x)(w'v) + (z'x)\alpha(v) = \alpha(x)(vw') + (z'x)\alpha(v) \\
&= (xv)\alpha(w') + (z'x)\alpha(v) = (xv)\alpha(w') + z(xv) \\
&= \alpha(w)\alpha(w') + \alpha(z)\alpha(z') + z\alpha(w) + z\alpha(z) \\
&= \alpha(w')\alpha(w) + \alpha(z)\alpha(z') + \alpha(z')\alpha(w) + \alpha(z)\alpha(z').
\end{aligned}$$

Hence we have $[u, x][v, y] = 0$, as desired. □

Corollary 3.2 *Under the hypothesis of the theorem above, A is nilpotent. If A is also semiprime, then A is commutative.*

Theorem 3.3 *Let α be an epimorphism and $(L,[,],\alpha)$ a Hom-Lie algebra. Suppose $L = A + X$, where A and X are Hom-Lie subalgebras which are commutative. Then $[[L,L],[L,L]] = 0$.*

Proof It suffices to show that $[[a,x],[b,y]] = 0$ holds for all $a,b \in A$ and $x,y \in X$. Since α is an epimorphism and X, A are Hom-Lie subalgebras, there exist $x', y' \in X$ and $a', b' \in A$ such that $\alpha(x') = x, \alpha(y') = y, \alpha(a') = a, \alpha(b') = b$. By Hom-Jacobi identity, it is easy to see that $[\alpha(a),[b,c]] = -[\alpha(c),[a,b]] - [\alpha(b),[c,a]]$. Hence we have

$$\begin{aligned}
[[a,x],[b,y]] &= [\alpha[a',x'],[b,y]] \\
&= -[\alpha(b),[y,[a',x']]] - [\alpha(y),[[a',x'],b]] \\
&= -[\alpha(b),[y,[a',x']]] + [\alpha(y),[b,[a',x']]] \\
&= -[\alpha(b),[\alpha(y'),[a',x']]] + [\alpha(y),[b,[a',x']]] \\
&= [\alpha(b),[\alpha(a'),[x',y']]] + [\alpha(b),[\alpha(x'),[y',a']]] + [\alpha(y),[b,[a',x']]] \\
&= [\alpha(b),[\alpha(x'),[y',a']]] + [\alpha(y),[\alpha(b'),[a',x']]] \\
&= [\alpha(b),[x,[y',a']]] - [\alpha(y),[\alpha(a'),[x',b']]] - [\alpha(y),[\alpha(x'),[b',a']]] \\
&= -[\alpha(b),[x,[a',y']]] - [\alpha(y),[a,[x',b']]].
\end{aligned}$$

Let $[a',y'] = c + z, [x',b'] = d + w$, where $c,d \in A, z,w \in X$. Then we have

$$\begin{aligned}
[[a,x],[b,y]] &= -[\alpha(b),[x,c+z]] - [\alpha(y),[a,d+w]] \\
&= -[\alpha(b),[x,c]] - [\alpha(y),[a,w]] \\
&= [\alpha(x),[c,b]] + [\alpha(c),[b,x]] + [\alpha(a),[w,y]] + [\alpha(w),[y,a]] \\
&= -[\alpha(c),[b,x]] + [\alpha(w),[y,a]] \\
&= -[\alpha(c),[b,x]] - [\alpha(w),[a,y]] \\
&= -[\alpha(c),\alpha(d)+\alpha(w)] - [\alpha(w),\alpha(c)+\alpha(z)] \\
&= -[\alpha(c),\alpha(w)] - [\alpha(w),\alpha(c)] = 0,
\end{aligned}$$

as required. $\square$

4. On the Hom-ideal structure of A

In this section we consider some analogues of classical concepts of ring theory and Lie theory as follows. From now on, we always assume that the twisting map α is an epimorphism.

Recall from [10] that a subobject I of a Hom-associative algebra A is called a Hom-ideal if $(LI)L = L(IL) \subseteq I$. If $X \subseteq A$ is a subject of A, then

$$S(U) = \{\sum_i (a_i\alpha^{n_i}(x_i))b_i | a_i, b_i \in A, x_i \in X, n_i \in Z\}$$

is the Hom-ideal generated by X. Obviously, if $\alpha(X) = X$, then

$$S(U) = \{\sum_i (a_i x_i) b_i | a_i, b_i \in A, x_i \in X\}.$$

Recall from [10] that a Hom-Lie ideal U of A is a subobject of A such that $[A, U] \subseteq U$. Define the center of A as follows

$$Z(A) = \{a \in A | [a, A] = 0\}.$$

A is called prime if the product of any two non-zero Hom-ideals of A is non-zero. It is called semiprime if it has no non-zero nilpotent Hom-ideals, and is called simple if it has no nontrivial Hom-ideals.

Lemma 4.1 *Let A be a Hom-associative algebra and $a, b, c \in A$. Then*

(1) $[\alpha(a), bc] = [a, b]\alpha(c) + \alpha(b)[a, c]$,

(2) $[ab, \alpha(c)] = \alpha(a)[b, c] + [a, c]\alpha(b)$,

(3) $[ab, \alpha(c)] = [\alpha(a), bc] + [\alpha(b), ca]$.

Proof Straightforward. □

Lemma 4.2 *Let U be a Hom-Lie ideal of a Hom-associative algebra A. Then $S(U)$ is a Hom-Lie ideal of A.*

Proof Straightforward. □

Lemma 4.3 *Assume that A is a semiprime Hom-associative algebra and U is both a Hom-Lie ideal and a Hom-associative subalgebra of A. If $\alpha(U) \subseteq U$ and $[U, U] \neq 0$, then U contains a non-zero Hom-ideal of A.*

Proof Since $[U, U] \neq 0$, there exists $x, y \in U$ such that $[x, y] \neq 0$, and $[\alpha(x), ya] \in U$ for all $a \in A$. By Lemma 4.1 (2), $\alpha(a)[x, y] = [ax, \alpha(y)] - [a, y]\alpha(x)$, it is easy to see that $\alpha(a)[x, y] \in U$ since U is both a Hom-Lie ideal and a Hom-associative subalgebra of A. It follows that $I = \{\sum_i (a_i \alpha^{n_i}([x, y])) b_i | a_i, b_i \in A, n_i \in Z\} \subseteq U$. In fact, since α is an epimorphism, there exist $a_i', a_i'', b_i', b_i'' \in A$ such that where $\alpha^{n_i}(a_i') = a_i, \alpha(a_i'') = a_i', \alpha(b_i') = b_i, \alpha^{n_i+1}(b_i'') = b_i'$. Then we have $\sum_i (a_i \alpha^{n_i}([x, y])) b_i = \sum_i [a_i \alpha^{n_i}([x, y]), b_i] + b_i(a_i \alpha^{n_i}([x, y])) = \sum_i [\alpha^{n_i}(a_i'[x, y]), b_i] + \alpha^{n_i+1}((b_i'' a_i')[x, y])$. So $\sum_i (a_i \alpha^{n_i}([x, y])) b_i \in U + U \subseteq U$, as desired. Moreover, $I \neq 0$, for otherwise $[x, y]$, A will generate a nilpotent Hom-ideal of A. □

Theorem 4.4 *Assume that L is a prime Hom-associative algebra and U is a Hom-Lie ideal of L such that $[U, U] \neq 0$. If $\alpha(U) \subseteq U$, then there exists a Hom-ideal I of L such that $0 \neq [I, L] \subseteq U$.*

Proof Define $N_L(U) = \{x \in L | [x, L] \subseteq U\}$. Note that $U \subseteq N_L(U)$. Thus $[N_L(U), L] \subseteq U \subseteq N_L(U)$ and also $N_L(U)$ is a Hom-Lie ideal. It is also a Hom-associative subalgebra. For any $x, y \in N_L(U)$ and $l \in L$, there exist $x' \in N_L(U), l' \in L$ such that $\alpha(x') = x, \alpha(l') = l$. By Lemma 4.1 (3), we have $[xy, l] = [xy, \alpha(l')] = [\alpha(x), yl'] + [\alpha(y), l'x] \in U$. This says that $xy \in N_L(U)$. It is easy to see that $[N_L(U), N_L(U)] \supseteq [U, U] \neq 0$. Applying Lemma 4.3 to $N_L(U)$, we may find a non-zero Hom-ideal I of L such that $I \subseteq N_L(U)$, i.e., $[I, L] \subseteq U$. Now we prove $[I, L] \neq 0$. If not, choose $x \in I$ and $l, m \in L$, then there exist $x' \in I, l' \in L$ such that $\alpha^2(x') = x, \alpha(l') = l$. By Lemma 4.1 (2), $x[l, m] = \alpha^2(x')[l, m] = [\alpha(x')l, \alpha(m)] - [\alpha(x'), m]\alpha(l)$, where $[\alpha(x')l, \alpha(m)] = [\alpha(x')\alpha(l'), \alpha(m)] = [1(x'l'), \alpha(m)] = 0$ and $[\alpha(x'), m]\alpha(l) = [x, m]\alpha(l) = 0$, since $1(x\alpha(l)) \in I$. So $x[l, m] = 0$, $[L, L] \subseteq Ann_L(I)$. It is not hard to show that $Ann_L(I)$ is a Hom-ideal. The simplicity of L gives $[L, L] = 0$, a contradiction. This completes the proof. $\square$

Corollary 4.5 *Let L be a simple Hom-associative algebra. If U is a Hom-Lie ideal with $[U, U] \neq 0$ and $\alpha(U) \subseteq U$, then $[L, L] \subseteq U$.*

As usual, we define a sequence of Hom-ideals (the derived series) by $L^0 = L, L^{(1)} = [L, L], L^{(2)} = [L^{(1)}, L^{(1)}], \ldots, L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$. L is called solvable if $L^{(n)} = 0$ for some n. For $U \in L$, $\langle U \rangle$ denotes the Hom-associative subalgebra of L generated by U.

Lemma 4.6 *Let U be a Hom-ideal of A and $\langle U \rangle$ be the Hom-associative subalgebra of L generated by U. Then*

(1) $\langle U \rangle$ is a Hom-Lie ideal of A,

(2) $[\langle U \rangle, A] \subseteq U$.

Proof Straightforward. $\square$

Lemma 4.7 *Let L be a simple Hom-Lie algebra and $\alpha(L) \subseteq L$. Then*

(1) If $L^{(2)} \neq 0$, then $L = \langle L^{(1)} \rangle$.

(2) If $L^{(3)} \neq 0$, then $L^{(2)} = L^{(1)}$.

Proof (1) Let $S = \langle L^{(1)} \rangle$. By Lemma 4.6 (1), S is a Hom-Lie ideal of L. By hypothesis, $[S, S] \supseteq L^{(2)} \neq 0$. $\alpha(S) \subseteq S$ since $\alpha(L) \subseteq L$. Then by Lemma 4.3, S contains a non-zero Hom-Lie ideal of L. So $S = L$ since L is simple.

(2) Let $V = L^{(2)}$. The Hom-Jacobi identity and the hypothesis $\alpha(L) \subseteq L$ imply that V is a Hom-Lie ideal of $L^{(3)} = [V, V] \neq 0$ and that $\alpha(V) \subseteq V$. By Theorem 4.4, there is a Hom-ideal I of L, furthermore, $I = L$ since L is simple. Thus $V \supseteq [L, L]$. Clearly, $V \subseteq [L, L]$. This completes the proof. $\square$

Lemma 4.8 *Let V be a Hom-Lie ideal of $[L, L]$ and $T(V) = \{l \in L | [l, L] \subseteq V\}$. Assume that $\alpha(V) \subseteq V$, then*

(1) $T(V)$ is a Hom-Lie subalgebra of L,

(2) $[V, V] \subseteq T(V)$,

(3) $[V, T(V)] \subseteq T(V)$,

(4) $[[L, T(V)], T(V)] \subseteq T(V)$,

(5) $[[T(V), T(V)], L] \subseteq T(V)$,

(6) $L(([[T(V), T(V)], [T(V), T(V)]])L) \subseteq V + T(V)$.

Proof (1) Straightforward.

(2) Let $x, y \in V$ and $l \in L$. Then there exist $x', y' \in V, l' \in L$ such that $\alpha(x') = x, \alpha(y') =$

$y, \alpha(l') = l$. By Hom-Jacobi identity, we have

$$[l,[x,y]] = [\alpha(l'),[\alpha(x),\alpha(y)]] = -[\alpha(y),[l',x]] - [\alpha(x),[y,l']] \in V,$$

since $\alpha(V) \subseteq V$ and V is a Hom-Lie ideal of $[L,L]$.

(3) Similar to (2).

(4) Follows from (3).

(5) For any $x,y \in T(V)$ and $l,l' \in L$, there exist $x',y' \in T(V), m,n \in L$ such that $\alpha(x') = x, \alpha(y') = y, \alpha(m) = l', \alpha(n) = m$. By Hom-Jacobi identity, we have

$$[[[x,y],l],l'] = -[\alpha(m),[[x,y],l]] = [\alpha(l),[m,[x,y]]] + [\alpha[x,y],[l,m]].$$

It is easy to see that $[\alpha[x,y],[l,m]] \in V$ since $[x,y] \in T(V)$. So it is sufficent to show that $[\alpha(l),[m,[x,y]]] \in V$. In fact

$$\begin{aligned}[\alpha(l),[m,[x,y]]] &= [\alpha(l),[\alpha(n),[x,y]]] = -[\alpha(l),[\alpha(x),[y,n]]] - [\alpha(l),[\alpha(y),[n,x]]]\\ &= [\alpha^2(x),[[y,n],l]] + [\alpha[y,n],[l,\alpha(x)]] + [\alpha^2(y),[[n,x],l]] + [\alpha[n,x],[l,\alpha(y)]] \in V,\end{aligned}$$

since $x,y \in T(V)$ and V is a Hom-Lie ideal of $[L,L]$.

(6) From (5) and Lemma 4.1. $\square$

Theorem 4.9 *Let L be simple and $V \subseteq [L,L]$ be a Hom-Lie ideal of $[L,L]$ such that $V \neq [L,L]$. Assume that $\alpha(V) \subseteq V$. Then V is a solvable Hom-Lie subalgebra of $[L,L]$. Moreover $[V,V]$ is nilpotent.*

Proof We will consider two cases:

(i) $L^{(3)} = 0$. In this case, it is easy to see that $V^{(3)} = 0$, and so V is a solvable Hom-Lie subalgebra of $[L,L]$.

(ii) $L^{(3)} \neq 0$. If $[L,L] \subseteq T(V)$, then $[L,[L,L]] \subseteq [L,T(V)] \subseteq V$. But then $L^{(2)} = [[L,L],[L,L]] \subseteq [T(V),T(V)] \subseteq V$. By Lemma 4.7 (2), $L^{(2)} = [L,L]$, and so $[L,L] \subseteq V$, which contradicts the hypothesis $V \neq [L,L]$. Thus we may assume that $[l,m] \bar{\in} T(V)$ for some $l,m \in L$. By Lemma 4.8 (1,2,3), we have $[V+T(V),V+T(V)] \subseteq [V,V]+[V,T(V)]+[T(V),T(V)] \subseteq T(V)$. Hence $V+T(V) \neq L$. By Lemma 4.8 (6), $L([[T(V),T(V)],[T(V),T(V)]]L) \subseteq V+T(V)$. This contradicts that L is H-simple, unless $[[T(V),T(V)],[T(V),T(V)]] = 0$, i.e., $T(V)^{(2)} = 0$. Also, since $[V,V] \subseteq T(V)$, it follows that $V^{(3)} = 0$. Thus V is a solvable Hom-Lie subalgebra of $[L,L]$. Moreover $[V,V]$ is nilpotent. $\square$

Acknowledgements We thank the referees for their time and comments.

References

[1] J. T. HARTWIG, D. LARSSON, S. D. SILVESTOV. *Deformations of Lie algebras using σ-derivations*. J. Algebra, 2006, **295**(2): 314–361.

[2] D. LARSSON, S. D. SILVESTOV. *Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities*. J. Algebra, 2007, **288**(2): 321–344.

[3] D. LARSSON, S. D. SILVESTOV. *Quasi-Lie algebras*. Contemp. Math., 2005, **391**: 241–248.

[4] A. MAKHLOUF, S. D. SILVESTOV. *Hom-algebra structures*. J. Gen. Lie Theory, 2008, **3**(2): 51–64.

[5] A. MAKHLOUF, S. D. SILVESTOV. *Hom-Lie Admissible Hom-Coalgebras and Hom-Hopf Algebras*. Springer, Berlin, 2009.

[6] A. MAKHLOUF, S. D. SILVESTOV. *Hom-algebras and Hom-coalgebras*. J. Algebra Appl., 2010, **9**(4): 553–589.

[7] D. YAU. *Hom-Yang-Baxter equation, Hom-Lie algebras, and quasitriangular bialgebras*. J. Physics A: Math Theory, 2009, **42**(16): 165–202.

[8] D. YAU. *Hom-bialgebras and comodule algebras*. Int. Electron J. Algebra, 2010, **8**(1): 45–64.

[9] D. YAU. *Enveloping algebras of Hom-Lie algebras*. J. Gen. Lie Theory Appl., 2008, **2**(2): 95–108.

[10] S. CAENEPEEL, I. GOYVAERTS. *Monoidal Hom-Hopf algebras*. Comm. Algebra, 2011, **39**(6): 2216–2240.

[11] Yuanyuan CHEN, Zongwei WANG, Liangyun ZHANG. *Integrals for monoidal Hom-Hopf algebras and their applications*. J. Math. Phys., 2013, **54**(7): 1–22.

[12] Zongwei WANG, Yuanyuan CHEN, Liangyun ZHANG. *The antipode and Drinfel'd double of Hom-Hopf algebras*. Scientia Sinica Mathematica, 2012, **42**(11): 1079–1093.

[13] Yunhe SHENG. *Representations of Hom-Lie Algebras*. Algebr. Represent. Theory, 2012, **15**(6): 1081–1098.

[14] Yunhe SHENG, Chengming BAI. *A new approach to hom-Lie bialgebras*. J. Algebra, 2014, **399**(1): 232–250.

[15] Yongsheng CHENG, Yucai SU. *(Co)Homology and universal central extension of Hom-Leibniz Algebras*. Acta Math. Sin. (Engl. Ser.), 2011, **27**(5): 813–830.

[16] Yongsheng CHENG, Hengyun YANG. *Low-dimensional cohomology of q-deformed Heisenberg-Virasoro algebra of Hom-type*. Front. Math. China, 2010, **5**(4): 607–622.

[17] O. H. KEGEL. *On rings that are sums of two subrings*. J. Algebra, 1964, **1**: 103–109.

[18] Y. BAHTURIN, O. H. KEGEL. *Universal sums of abelian subalgebras*. Comm. Algebra, 1995, **23**(8): 2975–2990.

[19] Shuanhong WANG. *On the generalized H-Lie structure of associative algebras in Yetter-Drinfeld categories*. Comm. Algebra, 2002, **30**(1): 307–325.

[20] Y. BAHTURIN, D. FISHMAN, S. MONTGOMERY. *On the generalized Lie structure of associative algebras*. Israel J. Math., 1996, **96**(1): 27–48.

[21] Shengxiang WANG, Shuanhong WANG. *Hom-Lie algebras in Yetter-Drinfeld categories*. Comm. Algebra, 2014, **42**(10): 4540–4561.