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Homoclinic Solutions for a Class of Prescribed Mean Curvature Equation with a Deviating Argument

Shiping LU[1,*], **Zaitao LIANG**[2]

1. *College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Jiangsu* 210044*, P. R. China;*
2. *Department of Mathematics, Anhui Normal University, Anhui* 241000*, P. R. China*

Abstract In this paper, by using Mawhin's continuation theorem and some analysis methods, the existence of a set with $2kT$-periodic solutions for a kind of prescribed mean curvature equation with a deviating argument is studied, and then a homoclinic solution is obtained as a limit of a certain subsequence of the above set.

Keywords homoclinic solution; Mawhin's continuation theorem; prescribed mean curvature equation.

MR(2010) Subject Classification 34C37

1. Introduction

In this paper, we investigate the existence of homoclinic solutions for a class of prescribed mean curvature equation with a deviating argument

$$\Big(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\Big)'+cu'(t)+g(u(t))+h(u(t-\tau(t)))=p(t), \tag{1.1}$$

where $g,h\in C^1(R,R)$, $p,\tau\in C(R,R)$, τ is T-period, $c>0$ and $T>0$ are given constants.

As is well known, a solution $u(t)$ of Eq.(1.1) is named homoclinic (to 0) if $u(t)\to 0$ and $u'(t)\to 0$ as $|t|\to+\infty$. In addition, if $u\neq 0$, then u is called a nontrivial homoclinic solution.

Prescribed mean curvature equation and its modified forms which derived from differential geometry and physics have widely researched in many papers, and there exist many papers about the periodic solutions for prescribed mean curvature equation and its modified forms. For example, by using an approach based on the Leray-Schauder degree, Benevieri in [1] studied the periodic solutions for nonlinear equations with mean curvature-like operators. And in [2] Benevieri extended the results obtained in [1] to the N-dimensional case.

Recently, Feng in [3] studied the periodic solutions for prescribed mean curvature Liénard equation with a deviating argument as follows:

$$\Big(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\Big)'+f(u(t))u'(t)+g(t,u(t-\tau(t)))=e(t),$$

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* Corresponding author
E-mail address: lushiping88@sohu.com (Shiping LU); liangzaitao@sohu.com (Zaitao LIANG)

where $g \in C(R^2, R)$, f, e and τ are T-periodic. By using the continuation theorem established by Mawhin, the author obtained some sufficient conditions for the existence of periodic solution.

Obersnel in [4] studied the existence, regularity and stability properties of periodic solutions of a capillarity equation in the presence of lower and upper solutions

$$-(\frac{u'}{\sqrt{1+(u')^2}})' = f(t,u).$$

This equation, together with its N-dimensional counterpart

$$-\mathrm{div}(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = f(t,u),$$

plays an important role in various physical and geometrical questions: for example, capillarity-type problems in fluid mechanics, flux-limited diffusion phenomena and prescribed mean curvature problems [5–7]. There are also a class of prescribed mean curvature equations appearing in differential geometry and physics. Such problems have attracted the attention of Bonheure [8], López [9], Pan [10] and Obersnel and Omari [11].

In this paper, like in the work of Rabinowitz in [12], Lzydorek and Janczewska in [13], Tan and Li in [14] and Lu in [15], the existence of a homoclinic solution for Eq.(1.1) is obtained as a limit of a certain sequence of $2kT$-periodic solutions for the following equation:

$$(\frac{u'(t)}{\sqrt{1+(u'(t))^2}})' + cu'(t) + g(u(t)) + h(u(t-\tau(t))) = p_k(t), \tag{1.2}$$

where $k \in N$, $p_k : R \to R^n$ is a $2kT$-periodic function such that

$$p_k(t) = \begin{cases} p(t), & t \in [-kT, kT-\varepsilon_0), \\ p(kT-\varepsilon_0) + \frac{p(-kT)-p(kT-\varepsilon_0)}{\varepsilon_0}(t-kT+\varepsilon_0), & t \in [kT-\varepsilon_0, kT], \end{cases} \tag{1.3}$$

$\varepsilon_0 \in (0,T)$ is a constant independent of k. The existence of $2kT$-periodic solutions to Eq.(1.3) is obtained by using Mawhin's continuation theorem [16]. The rest of this paper is organized as follows. In Section 2, we provide some necessary background definitions and lemmas. In Section 3, we give the results we obtain.

2. Preliminary

In order to use Mawhin's continuation theorem [16], we first recall it.

Let X and Y be two Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively. A linear operator $L : D(L) \subset X \to Y$ is said to be a Fredholm operator of index zero provided that

(a) $\mathrm{Im}\, L$ is a closed subset of Y,

(b) $\dim \mathrm{Ker}\, L = \mathrm{codim}\, \mathrm{Im}\, L < \infty$.

Let X and Y be two Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively, $\Omega \subset X$ be an open and bounded set, and $L : D(L) \subset X \to Y$ be a Fredholm operator of index zero. Continuous operator $N : \Omega \subset X \to Y$ is said to be L-compact in $\bar{\Omega}$ provided that

(c) $K_p(I-Q)N(\bar{\Omega})$ is a relative compact set of X,

(d) $QN(\bar{\Omega})$ is a bounded set of Y,

where we denote $X_1 = \mathrm{Ker}\, L$, $Y_2 = \mathrm{Im}\, L$, then we have the decompositions $X = X_1 \bigoplus X_2$,

$Y = Y_1 \bigoplus Y_2$. Let $P : X \to X_1$, $Q : Y \to Y_1$ be continuous linear projectors (meaning $P^2 = P$ and $Q^2 = Q$), and $K_p = L \mid^{-1}_{\operatorname{Ker} P \bigcap D(L)}$.

Lemma 2.1 ([16]) *Let X and Y be two Banach spaces with norms $\| \cdot \|_X$, $\| \cdot \|_Y$, respectively, Ω be an open and bounded set of X, and $L : D(L) \subset X \to Y$ be a Fredholm operator of index zero. The operator $N : \bar{\Omega} \subset X \to Y$ is said to be L-compact in $\bar{\Omega}$. In addition, if the following conditions hold:*

(H1) $Lv \neq \lambda N v, \forall (v, \lambda) \in \partial\Omega \times (0, 1)$;

(H2) $QNv \neq 0, \forall v \in \operatorname{Ker} L \cap \partial\Omega$;

(H3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, *where* $J : \operatorname{Im} Q \to \operatorname{Ker} L$ *is a homeomorphism,*

then Lv=Nv has at least one solution in $D(L) \cap \bar{\Omega}$.

Lemma 2.2 ([17]) *Let $0 \le \alpha \le T$ be constant, $\tau \in C(R, R)$ be T-periodic function, and $\max_{t\in[0,T]} |\tau(t)| \le \alpha$. Then, $\forall u \in C^1(R, R)$ which is T-periodic function, we have*

$$\int_0^T |u(t - \tau(t)) - u(t)|^2 \mathrm{d}t \le 2\alpha^2 \int_0^T |u'(t)|^2 \mathrm{d}t.$$

Lemma 2.3 *If $u : R \to R$ is continuously differentiable on R, $a > 0, \mu > 1$ and $p > 1$ are constants, then for every $t \in R$, the following inequality holds:*

$$|u(t)| \le (2a)^{-\frac{1}{\mu}} \Big(\int_{t-a}^{t+a} |u(s)|^\mu \mathrm{d}s \Big)^{\frac{1}{\mu}} + a(2a)^{-\frac{1}{p}} \Big(\int_{t-a}^{t+a} |u'(s)|^p \mathrm{d}s \Big)^{\frac{1}{p}}.$$

This lemma is a special case of Lemma 2.2 in [14].

Lemma 2.4 ([14]) *Let $u_k \in C^2_{2kT}$ be 2kT-periodic function for each $k \in \mathbf{N}$ with*

$$|u_k|_0 \le A_0, |u_k'|_0 \le A_1, |u_k''|_0 \le A_2,$$

where A_0, A_1 and A_2 are constants independent of $k \in \mathbf{N}$. Then there exists a function $u_0 \in C^1(R, R)$ such that for each interval $[c, d] \subset R$, there is a subsequence $\{u_{k_j}\}$ of $\{u_k\}_{k\in\mathbf{N}}$ with $u_{k_j}(t) \to u_0(t)$ and $u'_{k_j}(t) \to u'_0(t)$ uniformly on $[c, d]$.

Considering the following system

$$\begin{cases} x'(t) = \varphi(y(t)) = \frac{y(t)}{\sqrt{1-y^2(t)}}, \\ y'(t) = -c\varphi(y(t)) - g(x(t)) - h(x(t - \tau(t))) + p(t) \end{cases} \tag{2.1}$$

where $y(t) = \frac{x'(t)}{\sqrt{1+(x'(t))^2}}$. Obviously, if $(x(t), y(t))^T$ is a solution of (2.1), then $x(t)$ must be a solution of (1.1), and finding homoclinic solutions of (1.1) is equivalent to finding a solution $x(t)$ of (2.1) such that $(x(t), x'(t)) \to (0, 0)$ as $\mid t \mid\to +\infty$. Similarly, finding a $2kT-$periodic solution to (1.2) is equivalent to finding a $2kT-$periodic solution to the system

$$\begin{cases} x'(t) = \varphi(y(t)) = \frac{y(t)}{\sqrt{1-y^2(t)}}, \\ y'(t) = -c\varphi(y(t)) - g(x(t)) - h(x(t - \tau(t))) + p_k(t). \end{cases} \tag{2.2}$$

Let $X_k = Y_k = \{v = (x(t), y(t))^\top \in C(R, R^2), v(t) = v(t+2kT)\}$, where the normal defined by $\| v \| = \max\{|x|_0, |y|_0\}$, where $|x|_0 = \max_{t\in[0,2kT]} |x(t)|$, $|y|_0 = \max_{t\in[0,2kT]} |y(t)|$. It is obvious that X_k and Y_k are Banach spaces.

Now we define the operator

$$L : D(L) \subset X_k \to Y_k, Lv = v' = (x'(t), y'(t))^\top,$$

where $D(L) = \{v|v = (x(t), y(t))^T \in C^1(R, R^2),\ v(t) = v(t + 2kT)\}$.

Let $Z_k = \{v|v = (x(t), y(t))^\top \in C^1(R, R \times (-1, 1)), v(t) = v(t + 2kT)\}$. Define a nonlinear operator $N : \bar{\Omega} \subset (X_k \cap Z_k) \subset X_k \to Y_k$

$$Nv = (\frac{y(t)}{\sqrt{1 - y^2(t)}}, -c\varphi(y(t)) - g(x(t)) - h(x(t - \tau(t))) + p_k(t))^\top.$$

Then problem (2.2) can be written as $Lv = Nv$ in $\bar{\Omega}$.

We know $\operatorname{Ker} L = \{v|v \in X_k, v' = (x'(t), y'(t))^\top = (0, 0)^\top\}$, then $x'(t) \equiv 0, y'(t) \equiv 0$, obviously $x \in R$, $y \in R$, thus $\operatorname{Ker} L = R^2$, and it is also easy to prove that $\operatorname{Im} L = \{z \in Y_k, \int_0^{2kT} z(s)\mathrm{d}s = 0\}$, so L is a Fredholm operator of index zero.

Let $P : X_k \to \operatorname{Ker}L$, $Pv = \frac{1}{2kT}\int_0^{2kT} v(s)\mathrm{d}s$, $Q : Y_k \to \operatorname{Im} Q$, $Qz = \frac{1}{2kT}\int_0^{2kT} z(s)\mathrm{d}s$. If set $K_p = L|^{-1}_{\operatorname{Ker} p\cap D(L)}$, then

$$(K_p z)(t) = \int_0^{2kT} G_k(t, s)z(s)\mathrm{d}s,$$

where

$$G(t, s) = \begin{cases} \frac{s-2kT}{2kT}, & 0 \le t \le s; \\ \frac{s}{2kT}, & s \le t \le 2kT. \end{cases}$$

For all Ω such that $\overline{\Omega} \subset (X_k \cap Z_k) \subset X_k$, we have $K_p(I - Q)N(\bar{\Omega})$ is a relative compact set of X_k, $QN(\overline{\Omega})$ is a bounded set of Y_k, so the operator N is L-compact in $\overline{\Omega}$.

3. Main results

For the sake of convenience, we list the following conditions.

[A_1] There exist constants m_0, m_1 with $m_0 > m_1 > 0$ such that $xg(x) \le -m_0x^2$, $|xh(x)| \le m_1x^2$ and $g'(x) < 0$, $\forall x \in R$.

[A_2] There exists a constant $l > 0$ such that $|h(x_1) - h(x_2)| \le l|x_1 - x_2|$, and $h'(x) < 0$, $\forall x \in R$.

[A_3] $p \in C(R, R)$ is a bounded function with $p(t) \ne 0$ and $B := (\int_R |p(t)|^2\mathrm{d}t)^{\frac{1}{2}} + \varepsilon_0^{1/2}\sup_{t\in R} |p(t)| < +\infty$, where ε_0 is determined by (1.3).

Remark 3.1 From (1.3), we see that$|p_k(t)| \le \sup_{t\in R} |p(t)|$. If assumption [$A_3$] holds, then for each $k \in \mathbf{N}$, $(\int_{-kT}^{kT} |p_k(t)|^2\mathrm{d}t)^{\frac{1}{2}} < B$.

In order to study the existence of $2kT$-periodic solutions to system (2.2), we firstly study some properties of all possible $2kT$-periodic solutions to the following system:

$$\begin{cases} x'(t) = \lambda\varphi(y(t)) = \lambda\frac{y(t)}{\sqrt{1-y^2(t)}}, \\ y'(t) = -\lambda c\varphi(y(t)) - \lambda g(x(t)) - \lambda h(x(t - \tau(t))) + \lambda p_k(t),\ \lambda \in (0, 1], \end{cases}$$

where $(x_k, y_k)^\top \in Z_k \subset X_k$. For each $k \in \mathbf{N}$, let Σ represent the set of all the $2kT$-periodic solutions to the above system, and let $\alpha = \max_{t\in[0,T]} |\tau(t)|$.

Theorem 3.1 *Assume conditions [A_1]–[A_3] hold, $c > \sqrt{2}\alpha l$ and $\frac{c+Tc\sqrt{m_0-m_1}}{\sqrt{2T(m_0-m_1)}(c-\sqrt{2}\alpha l)} < \frac{1}{B}$. For each $k \in \mathbf{N}$, if $(x,y)^\top \in \Sigma$, then there are positive constants ρ_0, ρ_1 ρ_2 and ρ_3 which are independent of k and λ, such that*

$$|x|_0 \le \rho_0, |y|_0 \le \rho_1 < 1, |x'|_0 \le \rho_2, |y'|_0 \le \rho_3.$$

Proof For each $k \in \mathbf{N}$, if $(x,y)^\top \in \Sigma$, then

$$\begin{cases} x'(t) = \lambda\varphi(y(t)) = \lambda\frac{y(t)}{\sqrt{1-y^2(t)}}, \\ y'(t) = -cx'(t) - \lambda g(x(t)) - \lambda h(x(t-\tau(t))) + \lambda p_k(t), \lambda \in (0,1]. \end{cases} \tag{3.1}$$

Multiplying the first equation of (3.1) by $y'(t)$ and integrating from $-kT$ to kT, we have

$$\int_{-kT}^{kT} y'(t)x'(t)\mathrm{d}t = \int_{-kT}^{kT} y'(t)\lambda\varphi(y(t))\mathrm{d}t = \int_{-kT}^{kT} \lambda\varphi(y(t))\mathrm{d}y(t) = 0.$$

It follows from the second equation of (3.1) that

$$\begin{aligned} c\int_{-kT}^{kT}(x'(t))^2\mathrm{d}t &= -\lambda\int_{-kT}^{kT} h(x(t-\tau(t)))x'(t)\mathrm{d}t + \lambda\int_{-kT}^{kT} p_k(t)x'(t)\mathrm{d}t \\ &= -\lambda\int_{-kT}^{kT}[h(x(t-\tau(t))) - h(x(t))]x'(t)\mathrm{d}t + \lambda\int_{-kT}^{kT} p_k(t)x'(t)\mathrm{d}t \\ &\le \int_{-kT}^{kT}|h(x(t-\tau(t))) - h(x(t))||x'(t)|\mathrm{d}t + \int_{-kT}^{kT}|p_k(t)||x'(t)|\mathrm{d}t. \end{aligned} \tag{3.2}$$

Combining (3.2) with (A_2) gives

$$c\int_{-kT}^{kT}(x'(t))^2\mathrm{d}t \le l\int_{-kT}^{kT}|x(t-\tau(t)) - x(t)||x'(t)|\mathrm{d}t + \int_{-kT}^{kT}|p_k(t)||x'(t)|\mathrm{d}t.$$

Applying Holder's inequality and Lemma 2.2 to the above inequality, we obtain

$$\begin{aligned} c \parallel x' \parallel_2^2 \le & l\Big(\int_{-kT}^{kT}|x(t-\tau(t)) - x(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\int_{-kT}^{kT}|x'(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}} + \\ & \Big(\int_{-kT}^{kT}|p_k(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\int_{-kT}^{kT}|x'(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}} \\ \le & \sqrt{2}\alpha l \parallel x' \parallel_2^2 + \parallel x' \parallel_2 \parallel p_k \parallel_2, \end{aligned}$$

which implies that

$$\|x'\|_2 \le \frac{B}{c-\sqrt{2}\alpha l} := d_0. \tag{3.3}$$

Multiplying the second equation of (3.1) by $x(t)$ and integrating from $-kT$ to kT, we have

$$\begin{aligned} \int_{-kT}^{kT} y'(t)x(t)\mathrm{d}t &= -\int_{-kT}^{kT} y(t)x'(t)\mathrm{d}t = -\lambda\int_{-kT}^{kT}\frac{y^2(t)}{\sqrt{1-y^2(t)}}\mathrm{d}t \\ &= \lambda\Big(-\int_{-kT}^{kT} x(t)g(x(t))\mathrm{d}t - \int_{-kT}^{kT} x(t)h(x(t-\tau(t)))\mathrm{d}t + \int_{-kT}^{kT} x(t)p_k(t)\mathrm{d}t\Big), \end{aligned}$$

i.e.,

$$\begin{aligned}
&\int_{-kT}^{kT}\frac{y^2(t)}{\sqrt{1-y^2(t)}}\mathrm{d}t-\int_{-kT}^{kT}x(t)g(x(t))\mathrm{d}t\\
&=\int_{-kT}^{kT}x(t)h(x(t-\tau(t)))\mathrm{d}t-\int_{-kT}^{kT}x(t)p_k(t)\mathrm{d}t\\
&\le\int_{-kT}^{kT}|x(t)||h(x(t-\tau(t)))-h(x(t))|\mathrm{d}t+\\
&\quad\int_{-kT}^{kT}|x(t)h(x(t)|\mathrm{d}t+\int_{-kT}^{kT}|x(t)||p_k(t)|\mathrm{d}t. \qquad (3.4)
\end{aligned}$$

Combining (3.4) with (A_1) and (A_2), we get

$$\|y\|_2^2+m_0\|x\|_2^2\le l\int_{-kT}^{kT}|x(t)||x(t-\tau(t))-x(t)|\mathrm{d}t+m_1\|x\|_2^2+\int_{-kT}^{kT}|x(t)||p_k(t)|\mathrm{d}t.$$

Applying Holder's inequality and Lemma 2.2 to the above inequality, we obtain

$$\|y\|_2^2+(m_0-m_1)\|x\|_2^2\le\sqrt{2}\alpha l\|x'\|_2\|x\|_2+\|p_k\|_2\|x\|_2,$$

which implies that

$$(m_0-m_1)\|x\|_2^2\le\sqrt{2}\alpha l\|x'\|_2\|x\|_2+\|p_k\|_2\|x\|_2, \qquad (3.5)$$

and

$$\|y\|_2^2\le\sqrt{2}l\|x'\|_2\|x\|_2+\|p_k\|_2\|x\|_2. \qquad (3.6)$$

So from (3.3), (3.5) and (A_3), we can conclude that

$$\|x\|_2\le\frac{cB}{(m_0-m_1)(c-\sqrt{2}\alpha l)}:=d_1. \qquad (3.7)$$

Thus by using Lemma 2.3 for all $t\in[-kT,kT]$, we get

$$\begin{aligned}
|x(t)|&\le(2T)^{-\frac12}\Big(\int_{t-T}^{t+T}|x(s)|^2\mathrm{d}s\Big)^{\frac12}+T(2T)^{-\frac1p}\Big(\int_{t-T}^{t+T}|x'(s)|^p\mathrm{d}s\Big)^{\frac1p}\\
&\le(2T)^{-\frac12}\Big(\int_{t-kT}^{t+kT}|x(s)|^2\mathrm{d}s\Big)^{\frac12}+T(2T)^{-\frac12}\Big(\int_{t-kT}^{t+kT}|x'(s)|^2\mathrm{d}s\Big)^{\frac12}\\
&=(2T)^{-\frac12}\Big(\int_{-kT}^{kT}|x(s)|^2\mathrm{d}s\Big)^{\frac12}+T(2T)^{-\frac12}\Big(\int_{-kT}^{kT}|x'(s)|^2\mathrm{d}s\Big)^{\frac12}. \qquad (3.8)
\end{aligned}$$

From (3.3), (3.7) and (3.8), we obtain

$$|x|_0=\max_{t\in[-kT,kT]}|x(t)|\le(2T)^{-\frac12}d_1+\sqrt{\frac{T}{2}}d_0:=\rho_0. \qquad (3.9)$$

From (3.3), (3.6) and (3.7) it follows

$$\|y\|_2\le\frac{cB}{\sqrt{m_0-m_1}(c-\sqrt{2}\alpha l)}:=d_2. \qquad (3.10)$$

Multiplying the second equation of (3.1) by $y'(t)$ and integrating from $-kT$ to kT, we have

$$\int_{-kT}^{kT}(y'(t))^2\mathrm{d}t=-\int_{-kT}^{kT}\lambda y'(t)g(x(t))\mathrm{d}t-\int_{-kT}^{kT}\lambda y'(t)h(x(t-\tau(t)))\mathrm{d}t+\int_{-kT}^{kT}\lambda y'(t)p_k(t)\mathrm{d}t$$

$$= \int_{-kT}^{kT} \lambda^2 g'(x(t)) \frac{y^2(t)}{\sqrt{1-y^2(t)}} \mathrm{d}t - \int_{-kT}^{kT} \lambda y'(t)[h(x(t-\tau(t))) - h(x(t))] \mathrm{d}t +$$
$$\int_{-kT}^{kT} \lambda^2 h'(x(t)) \frac{y^2(t)}{\sqrt{1-y^2(t)}} \mathrm{d}t + \int_{-kT}^{kT} \lambda y'(t) p_k(t) \mathrm{d}t.$$

From (A_1) and (A_2), we know that

$$\int_{-kT}^{kT} (y'(t))^2 \mathrm{d}t \le l \int_{-kT}^{kT} |y'(t)||x(t-\tau(t)) - x(t)| \mathrm{d}t + \int_{-kT}^{kT} |y'(t)||p_k(t)| \mathrm{d}t.$$

Applying Holder's inequality and Lemma 2.2 to the above inequality gives

$$\|y'\|_2^2 \le \sqrt{2}\alpha l \|x'\|_2 \|y'\|_2 + \|p_k\|_2 \|y'\|_2.$$

From (3.3) and (A_3), we can conclude that

$$\|y'\|_2 \le \frac{cB}{c - \sqrt{2}\alpha l} := d_3. \tag{3.11}$$

In the similar way to (3.9), we get

$$|y|_0 = \max_{t \in [-kT, kT]} |y(t)| \le (2T)^{-\frac{1}{2}} d_2 + \sqrt{\frac{T}{2}} d_3 = \frac{cB + TcB\sqrt{m_0 - m_1}}{\sqrt{2T(m_0 - m_1)}(c - \sqrt{2}\alpha l)}.$$

Since $\frac{c + Tc\sqrt{m_0 - m_1}}{\sqrt{2T(m_0 - m_1)}(c - \sqrt{2}\alpha l)} < \frac{1}{B}$, we have

$$|y|_0 \le \frac{cB + TcB\sqrt{m_0 - m_1}}{\sqrt{2T(m_0 - m_1)}(c - \sqrt{2}\alpha l)} := \rho_1 < 1. \tag{3.12}$$

Let $f_\rho = \max_{|x| \le \rho_0} |f(x)|$ and $h_\rho = \max_{|x| \le \rho_0} |h(x)|$. From (3.1), we have

$$|x'(t)|_0 \le \lambda \frac{|y(t)|}{\sqrt{1-y^2(t)}} \le \frac{\rho_1}{1-\rho_1^2} := \rho_2, \tag{3.13}$$

and

$$|y'(t)|_0 \le c|\varphi(y(t))| + |g(x(t))| + |h(x(t-\tau(t)))| + |p_k(t)| \le c\rho_3 + g_\rho + h_\rho + B := \rho_3. \tag{3.14}$$

From (3.9), (3.12), (3.13) and (3.14), we know $\rho_0, \rho_1, \rho_2, \rho_3$ are the constants independent of k and λ. Hence the conclusion of Theorem 3.1 holds.

Theorem 3.2 *Assume that the conditions of Theorem 3.1 are satisfied. Then for each $k \in N$, system (2.2) has at least one $2kT$-periodic solution $(x_k(t), y_k(t))^T$ in $\Sigma \subset X_k$ such that*

$$|x_k|_0 \le \rho_0, |y_k|_0 \le \rho_1 < 1, |x'_k|_0 \le \rho_2, |y'_k|_0 \le \rho_3,$$

where $\rho_0, \rho_1, \rho_2, \rho_3$ are the constants defined in Theorem 3.1.

Proof In order to use Lemma 2.1, for each $k \in N$, we consider the following system:

$$\begin{cases} x'(t) = \lambda\varphi(y(t)) = \lambda \frac{y(t)}{\sqrt{1-y^2(t)}}, \\ y'(t) = -\lambda c\varphi(y(t)) - \lambda g(x(t)) - \lambda h(x(t-\tau(t))) + \lambda p_k(t), \lambda \in (0,1) \end{cases} \tag{3.15}$$

where $y(t) = \frac{\frac{1}{\lambda}x'(t)}{\sqrt{1+(\frac{1}{\lambda}x'(t))^2}}$. Let $\Omega_1 \subset X_k$ represent the set of all the $2kT$-periodic of system (3.15). Since $(0,1) \subset (0,1]$, we have $\Omega_1 \subset \Sigma$, where Σ is defined in Theorem 3.1. If $(x,y)^T \in \Omega_1$, by using Theorem 3.1, we get

$$|x|_0 \le \rho_0, \quad |y|_0 \le \rho_1 < 1.$$

Let $\Omega_2 = \{v = (x,y)^T \in \operatorname{Ker} L, QNv = 0\}$. If $(x,y)^T \in \Omega_2$, then $(x,y)^T = (a_1,a_2)^T \in R^2$ (constant vector), we see that

$$\begin{cases} \int_{-kT}^{kT} \frac{a_2}{\sqrt{1-a_2^2}} \mathrm{d}t = 0, \\ \int_{-kT}^{kT} -c\frac{a_2}{\sqrt{1-a_2^2}} - g(a_1) - h(a_1) + p_k(t)\mathrm{d}t = 0, \end{cases}$$

i.e.,

$$\begin{cases} a_2 = 0, \\ \int_{-kT}^{kT} -g(a_1) - h(a_1) + p_k(t)\mathrm{d}t = 0. \end{cases} \tag{3.16}$$

Multiplying the second equation of (3.16) by a_1, we have

$$2kTm_0a_1^2 \le 2kTm_1a_1^2 + \int_{-kT}^{kT} a_1p_k(t)\mathrm{d}t \le 2kTm_1a_1^2 + \sqrt{2kT}|a_1|B,$$

thus

$$|a_1| \le \frac{B}{\sqrt{2kT}(m_0-m_1)} \le \frac{B}{\sqrt{2T}(m_0-m_1)} := \beta.$$

Now, if we set $\Omega = \{v = (x,y)^T \in X_k, |x|_0 < \rho_0 + \beta, |y|_0 < \rho^* < 1\}$, where $\rho_1 < \rho^* < 1$, then $\Omega \supset \Omega_1 \cup \Omega_2$. So condition [H$_1$] and condition [H$_2$] of Lemma 2.1 are satisfied. What remains is verifying condition [H$_3$] of Lemma 2.2. In order to do this, let

$$H(v,\mu) : (\Omega \cap KerL) \times [0,1] \longrightarrow R : H(v,\mu) = \mu(x,y)^T + (1-\mu)JQN(v),$$

where $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is a linear isomorphism, $J(x,y) = (y,x)^T$. From assumption [A$_1$], we have $v^T H(v,\mu) \ne 0, \forall (v,\mu) \in \partial\Omega \cap \operatorname{Ker} L \times [0,1]$. Hence

$$\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} = \deg\{H(v,0), \Omega \cap \operatorname{Ker} L, 0\} = \deg\{H(v,1), \Omega \cap \operatorname{Ker} L, 0\} \ne 0.$$

So condition H$_3$ of Lemma 2.1 is satisfied. Therefore, by using Lemma 2.1, we see that Eq.(2.2) has a $2kT$-periodic solution $(x_k,y_k)^T \in \bar{\Omega}$. Obviously, $(x_k,y_k)^T$ is a $2kT$-periodic solution to Eq.(3.1) for the case of $\lambda = 1$, so $(x_k,y_k)^T \in \Sigma$. Thus, by using Theorem 3.1, we get

$$|x_k|_0 \le \rho_0, |y_k|_0 \le \rho_1 < 1, |x_k'|_0 \le \rho_2, |y_k'|_0 \le \rho_3.$$

Hence the conclusion of Theorem 3.2 holds. □

Theorem 3.3 *Suppose that the conditions in Theorem 3.1 hold, then Eq.(1.1) has a nontrivial homoclinic solution.*

Proof From Theorem 3.2, we see that for each $k \in \mathbf{N}$, there exists a $2kT$-periodic solution $(x_k,y_k)^\top$ to Eq.(2.2) with

$$|x_k|_0 \le \rho_0, |y_k|_0 \le \rho_1 < 1, |x_k'|_0 \le \rho_2, |y_k'|_0 \le \rho_3, \tag{3.17}$$

where $\rho_0, \rho_1, \rho_2, \rho_3$ are constants independent of $k \in \mathbf{N}$. And $x_k(t)$ is a solution of (1.2), so

$$\Big(\frac{x_k'(t)}{\sqrt{1+(x_k'(t))^2}}\Big)' + cx_k'(t) + g(x_k(t)) + h(x_k(t-\tau(t))) = p_k(t), \tag{3.18}$$

which together with $y_k(t) = \frac{x_k'(t)}{\sqrt{1+(x_k'(t))^2}}$ implies that $y_k(t)$ is continuously differentiable for $t \in R$. Also, from (3.17), we have $|y_k|_0 \le \rho_1 < 1$. It follows that $x_k'(t) = \varphi(y_k(t)) = \frac{y_k(t)}{\sqrt{1-y_k^2(t)}}$ is continuously differentiable for $t \in R$, i.e.,

$$x_k''(t) = \frac{y_k'(t)}{(1-y_k^2(t))^{3/2}}.$$

By using (3.17) again, we have

$$|x_k''|_0 \le \frac{\rho_3}{\sqrt{1-\rho_1^2}} := \rho_4. \tag{3.19}$$

Clearly, ρ_4 is a constant independent of $k \in \mathbf{N}$. By using Lemma 2.4, we see that there is a function $x_0 \in C^1(R,R)$ such that for each interval $[a,b] \subset R$, there is a subsequence $\{x_{k_j}\}$ of $\{x_k\}_{k\in\mathbf{N}}$ with $x_{k_j}(t) \to x_0(t)$ and $x_{k_j}'(t) \to x_0'(t)$ uniformly on $[a,b]$. Below, we will show that $x_0(t)$ is just a homoclinic solution to Eq.(1.1).

For all $a,b \in R$ with $a < b$, there must be a positive integer j_0 such that for $j > j_0, [-k_jT, k_jT - \varepsilon_0] \supset [a-\alpha, b+\alpha]$. So for $j > j_0$, from (1.3) and (3.18) we see that

$$\Big(\frac{x_{k_j}'(t)}{\sqrt{1+(x_{k_j}'(t))^2}}\Big)' + cx_{k_j}'(t) + g(x_{k_j}(t)) + h(x_{k_j}(t-\tau(t))) = p(t), t \in [a,b], \tag{3.20}$$

which results in

$$\begin{aligned}\Big(\frac{x_{k_j}'(t)}{\sqrt{1+(x_{k_j}'(t))^2}}\Big)' &= -cx_{k_j}'(t) - g(x_{k_j}(t)) - h(x_{k_j}(t-\tau(t))) + p(t) \\ &\to -cx_0'(t) - g(x_0(t)) - h(x_0(t-\tau(t))) + p(t), \text{ uniformly on } [a,b].\end{aligned}$$

Since $\frac{x_{k_j}'(t)}{\sqrt{1+(x_{k_j}(t))^2}} \to \frac{x_0'(t)}{\sqrt{1+(x_0'(t))^2}}$ uniformly for $t \in [a,b]$ and $\frac{x_{k_j}'(t)}{\sqrt{1+(x_{k_j}(t))^2}}$ is continuously differentiable for $t \in (a,b)$, we have

$$\Big(\frac{x_0'(t)}{\sqrt{1+(x_0'(t))^2}}\Big)' = -cx_0'(t) - g(x_0(t)) - h(x_0(t-\tau(t))) + p(t), \quad t \in (a,b).$$

Considering a,b are two arbitrary constants with $a < b$, it is easy to see that $x_0(t), t \in R$, is a solution to system (1.1).

Now, we will prove $x_0(t) \to 0$ and $x_0'(t) \to 0$ as $|t| \to \infty$.

Since

$$\begin{aligned}\int_{-\infty}^{+\infty} (|x_0(t)|^2 + |x_0'(t)|^2)\mathrm{d}t &= \lim_{i\to+\infty} \int_{-iT}^{iT} (|x_0(t)|^2 + |x_0'(t)|^2)\mathrm{d}t \\ &= \lim_{i\to+\infty}\lim_{j\to+\infty} \int_{-iT}^{iT} (|x_{k_j}(t)|^2 + |x_{k_j}'(t)|^2)\mathrm{d}t,\end{aligned}$$

clearly, for every $i \in N$, if $k_j > i$, then by (3.3) and (3.7),

$$\int_{-iT}^{iT} (|x_{k_j}(t)|^2 + |x_{k_j}'(t)|^2)\mathrm{d}t \le \int_{-k_jT}^{k_jT} (|x_{k_j}(t)|^2 + |x_{k_j}'(t)|^2)\mathrm{d}t \le d_0^2 + d_1^2.$$

Let $i \to +\infty$ and $j \to +\infty$. We have

$$\int_{-\infty}^{+\infty} (|x_0(t)|^2 + |x_0'(t)|^2)\mathrm{d}t \le d_0^2 + d_1^2, \tag{3.21}$$

and then

$$\int_{|t|\geq r}(|x_0(t)|^2+|x_0'(t)|^2)\mathrm{d}t\to 0, \tag{3.22}$$

as $r\to+\infty$. So by using Lemma 2.2, we obtain

$$\begin{aligned}|x_0(t)|&\leq (2T)^{-\frac{1}{2}}\Big(\int_{t-T}^{t+T}|x(s)|^2\mathrm{d}s\Big)^{\frac{1}{2}}+T(2T)^{-\frac{1}{2}}\Big(\int_{t-T}^{t+T}|x'(s)|^2\mathrm{d}s\Big)^{\frac{1}{2}}\\&\leq [(2T)^{-\frac{1}{2}}+T(2T)^{-\frac{1}{2}}]\Big(\int_{t-T}^{t+T}|x(s)|^2\mathrm{d}s+\int_{t-T}^{t+T}|x'(s)|^2\mathrm{d}s\Big)^{\frac{1}{2}}\to 0,\ \text{as } |t|\to+\infty.\end{aligned}$$

Finally, we will show that

$$x_0'(t)\to 0 \text{ as } |t|\to\infty. \tag{3.23}$$

From (3.17), we know

$$|x_0(t)|\leq\rho_0, |x_0'(t)|\leq\rho_2 \text{ for } t\in R.$$

From (1.1), (A_1) (A_2) and (A_3), we have

$$\begin{aligned}\Big|\Big(\frac{x_0'(t)}{\sqrt{1+(x_0'(t))^2}}\Big)'\Big|&\leq c|x_0'(t)|+|f(x_0(t))|+|h(x_0(t-\tau(t)))|+|p(t)|\\&\leq c\rho_2+\sup_{x\in[-\rho_0,\rho_0]}(f(x)+h(x))+B:=M_2,\ \text{for } t\in R.\end{aligned}$$

If (3.23) does not hold, then there exist $\varepsilon_1\in(0,\frac{1}{4})$ and a sequence $\{t_k\}$ such that

$$|t_1|<|t_2|<|t_3|<\cdots,|t_k|+1<|t_{k+1}|,\quad k=1,2,\ldots,$$

and

$$|x_0'(t_k)|\geq\frac{2\varepsilon_1}{\sqrt{1-(2\varepsilon_1)^2}},\quad k=1,2,\ldots.$$

From this we have for $t\in[t_k,t_k+\varepsilon_1/(1+M_2)]$,

$$\begin{aligned}|x_0'(t)|&\geq\Big|\frac{x_0'(t)}{\sqrt{1+(x_0'(t))^2}}\Big|=\Big|\frac{x_0'(t_k)}{\sqrt{1+(x_0'(t_k))^2}}+\int_{t_k}^{t}\Big(\frac{x_0'(s)}{\sqrt{1+(x_0'(s))^2}}\Big)'\mathrm{d}s\Big|\\&\geq\Big|\frac{x_0'(t_k)}{\sqrt{1+(x_0'(t_k))^2}}\Big|-\int_{t_k}^{t}\Big|\Big(\frac{x_0'(s)}{\sqrt{1+(x_0'(s))^2}}\Big)'\Big|\mathrm{d}s\\&\geq\varepsilon_1.\end{aligned}$$

It follows that

$$\int_{-\infty}^{+\infty}|x_0'(t)|^2\mathrm{d}t\geq\sum_{k=1}^{\infty}\int_{t_k}^{t_k+\frac{\varepsilon_1}{1+M_2}}|x_0'(t)|^2\mathrm{d}t=\infty,$$

which contradicts (3.21), and so (3.23) holds. Clearly, $x_0(t)\not\equiv 0$, otherwise, $p(t)\equiv 0$, which contradicts assumption $[A_3]$.

Hence the conclusion of Theorem 3.3 holds. □

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